DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 3, 2023
Electronic Journal,
reg. N ФC77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru

Dynamical systems

Methods of nonhyperbolic shadowing

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Abstract. This paper is a survey of some recent results giving sufficient conditions of shadowing for dynamical systems in the absence of hyperbolicity. The main topics of the survey are as follows: method of pairs of Lyapunov type functions, shadowing in a neighborhood of a nonisolated fixed point, conditional multiscale shadowing for sequences of mappings of a Banach space, conditional shadowing for dynamical systems on so-called simple time scales. The paper contains a new result on conditional multiscale shadowing in the case of an infinite family of projections of the phase space.

Keywords: Dynamical system, pseudotrajectory, shadowing, hyperbolicity

1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) is now an intensively developing branch of the global theory of dynamical systems. One can find basic information concerning shadowing in the monographs [1-3]; the survey [4] is devoted to some recent results of the theory.

The main property of dynamical systems studied by the shadowing theory can be stated as follows. Consider a homeomorphism f of a metric space (X, dist). Let $d > 0$. A sequence $\{y_n \in X\}$ is called a *d-pseudotrajectory* of f if the inequalities

$$\mathrm{dist}(f(y_n), y_{n+1}) < d \tag{1}$$

hold.

One says that f has the (standard) *shadowing property* if for any $\varepsilon > 0$ there is a $d > 0$ such that for any d-pseudotrajectory $\{y_n\}$ of f there is a point $x \in X$ for which

$$\mathrm{dist}(f^n(x), y_n) < \varepsilon.$$

In this case, we say that the exact trajectory $\{f^n(x)\}$ ε-*shadows* the pseudotrajectory $\{y_n\}$.

The first sufficient conditions under which a dynamical system has the shadowing property were obtained by D.V. Anosov [5] and R. Bowen [6]. Applying principally different approaches, they showed that, for a diffeomorphism, shadowing is a corollary of hyperbolicity (and, as easilly seen from the proofs in [5] and [6], in a neighborhood of a hyperbolic set, a diffeomorphism has the Lipschitz shadowing property, i.e., there exists a constant L such that, for small enough d, any d-pseudotrajectory is Ld-shadowed by an exact trajectory).

At present, relations between shadowing and hyperbolicity are well studied (see the book [3] for details).

In the present paper, we give a survey of some recent results giving sufficient conditions of shadowing in the absence of hyperbolicity. This survey is motivated by the talk given by the author at the Joint PDMI – MIRAN Session "Differential Equations and Dynamical Systems" (St. Petersburg, May 12 – 14, 2023).

The paper also contains a new result – we generalize the main theorem of the author's paper [7] to the case of an infinite family of projections (see Section 4 below). This generalization is not straightforward; in fact, we have to work with different Banach spaces etc.

The structure of the paper is as follows. In Section 2, we describe the method of pairs of Lyapunov type functions developed by A.A. Petrov and the author in the paper [8]. This method is applied to a perturbation of the hyperbolic automorphism of the 2-torus studied by J. Lewowicz in [9]. It is noted that, using the described method, one can obtain sufficient conditions of shadowing for nonsmooth systems.

In Section 3, it is shown that the method described in Section 2 can be applied to study shadowing in a neighborhood of a fixed point belonging to a "critical" manifold consisting of fixed points. In this case, not every pseudotrajectory in such a neighborhood can be shadowed, but it is possible to shadow pseudotrajectories for which "one step errors" are small enough compared to the distances of points of the pseudotrajectory to the critical manifold.

In Section 4, we study conditional shadowing for sequences of mappings of a Banach space X. It is assumed that there exists a countable family of projections that commute with linear terms of the considered mappings. Conditions of shadowing are obtained in terms of the norms of projections of one step errors of pseudotrajectories and of estimates of Lipschitz constants of projections of the "small nonlinear" terms. As mentioned above, the main theorem of this section generalizes the result of the author's paper [7] in which the case of a finite family of projections has been studied.

Finally, in Section 5, we describe conditions of conditional shadowing for dynamical systems on so-called simple time scales consisting of a family of isolated segments of the positive ray of $\mathbb{R}$; these results are obtained in the author's paper [10].

2 Method of a pair of Lyapunov type functions

Let f be a homeomorphism of a compact metric space (X, dist).

We assume that there exist two continuous nonnegative functions V and W defined in a closed neighborhood of the diagonal of $X \times X$ such that

$$V(p,p) = W(p,p) = 0 \text{ for any } p \in X$$

and conditions (C1)-(C9) below are satisfied. In what follows, arguments of the functions V and W are assumed to be close enough, so that the functions are defined.

Our conditions are formulated not directly in terms of the functions V and W but in terms of some geometric objects defined via these functions. Our main reasoning for the choice of these form of conditions is as follows.

(1) Precisely these conditions are used in the proofs.

(2) It is relatively easy to check conditions of that kind for particular functions V and W.

Let us introduce the main objects which we work with.

Fix positive numbers a and b and a point $p \in X$. Set

$$P(a,b,p) = \{q \in X : \ V(q,p) \le a, W(q,p) \le b\},$$

$$Q(a,b,p) = \{q \in P(a,b,p) : \ V(q,p) = a\},$$

$$T(a,b,p) = \{q \in P(a,b,p) : \ V(q,p) = 0\}.$$

Denote by $B(\varepsilon, p)$ the open ε-ball centered at p and set

$$\mathrm{Int}^0\, P(a,b,p) = \{q \in P(a,b,p):\ V(q,p) < a, W(q,p) < b\},$$

$$\partial^0\, P(a,b,p) = Q(a,b,p) \cup \{q \in P(a,b,p):\ W(q,p) = b\},$$

$$\mathrm{Int}^0\, Q(a,b,p) = \{q \in P(a,b,p):\ V(q,p) = a, W(q,p) < b\}.$$

Conditions (C1)-(C4) contain our assumptions on the geometry of the sets introduced above.

(C1) For any $\varepsilon > 0$ there exists a $\Delta_0 = \Delta_0(\varepsilon) > 0$ such that

$$P(\Delta_0, \Delta_0, p) \subset B(\varepsilon, p) \text{ for } p \in X.$$

There exists a $\Delta_1 > 0$ such that if $p \in X$, $\delta_1, \delta_2, \Delta < \Delta_1$ and $\delta_2 < \Delta$, then there exists a number $\alpha = \alpha(\delta_1, \delta_2, \Delta) > 0$ such that

(C2) $Q(\delta_1, \delta_2, p)$ is not a retract of $P(\delta_1, \delta_2, p)$;

(C3) $Q(\delta_1, \delta_2, p)$ is a retract of $P(\delta_1, \delta_2, p) \setminus T(\delta_1, \delta_2, p)$;

(C4) there exists a retraction $\sigma :\ P(\delta_1, \Delta, p) \to P(\delta_1, \delta_2, p)$ such that

$$V(\sigma(q), p) \geq \alpha V(q,p) \text{ for } q \in P(\delta_1, \Delta, p).$$

In the next group of conditions, we state our assumptions on the behavior of the introduced objects and their images under the homeomorphism f.

We assume that for any $\Delta < \Delta_1$ there exist positive numbers $\delta_1, \delta_2 < \Delta$ such that the following relations hold for any $p \in X$:

(C5)

$$f(P(\delta_1, \delta_2, p)) \in \mathrm{Int}^0\, P(\Delta, \Delta, f(p)), \quad f^{-1}(P(\delta_1, \delta_2, f(p))) \in \mathrm{Int}^0\, P(\Delta, \Delta, p);$$

(C6) $f(T(\delta_1, \delta_2, p)) \subset \mathrm{Int}^0\, P(\delta_1, \delta_2, f(p))$;

(C7) $f(T(\delta_1, \Delta, p)) \cap Q(\delta_1, \delta_2, f(p)) = \emptyset$;

(C8) $f(P(\delta_1, \delta_2, p)) \cap \partial^0 P(\delta_1, \delta_2, f(p)) \subset \mathrm{Int}^0\, Q(\delta_1, \delta_2, f(p))$;

(C9) $f(S(\delta_1, \Delta, p)) \cap P(\delta_1, \delta_2, f(p)) = \emptyset$, where

$$S(\delta_1, \Delta, p) = \{q \in P(\Delta, \Delta, p):\ V(q,p) \geq \delta_1\}.$$

Theorem 1 [8]. *Under conditions* (C1)-(C9), f *has the finite shadowing property* (*and hence, the standard shadowing property*) *on the space* X.

Let us give an example of application of Theorem 1 to a diffeomorphism with nonhyperbolic behavior.

Example 1. Consider a diffeomorphism studied by Lewowicz in [9]. This diffeomorphism is a perturbation of a hyperbolic automorphism of the 2-torus T^2.

Fix numbers $0 < \alpha < 1 < \beta$ and a small $r > 0$ and define a map $F:\ \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = (\alpha x + \lambda(x)\mu(y), \beta y),$$

where

$$\lambda(x) = \int_0^x ((1-\alpha) - h(s))\, ds,$$

$h : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $h(0) = 0$, $0 \le h(x) < 1$, and $\lambda(x) = 0$ for $|x| \ge r$, and $\mu : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\mu(0) = 1$, $\mu(y) = \mu(-y)$, μ is not increasing for $y \ge 0$, and $\mu(y) = 0$ for $|y| \ge r$.

Let A be an integer hyperbolic 2×2 matrix with $\det A = 1$. If $0 < \alpha < 1 < \beta$ are the eigenvalues of A and u_1 and u_2 are the corresponding eigenvectors, then

$$A(x, y) = (\alpha x, \beta y)$$

in coordinates whose axes are parallel to u_1 and u_2.

The lattice Ξ with vertices

$$\{(n + 1/2)u_1, (m + 1/2)u_2 :\ m, n \in \mathbb{Z}\}$$

is invariant with respect to the action of the map $v \mapsto Av$. Let $\pi :\ \mathbb{R}^2 \to \mathbb{R}^2/\Xi$ be the corresponding projection of the plane to the 2-torus.

Define $f :\ T^2 \to T^2$ by $f(\pi(\xi, \eta)) = \pi \circ F(x, y)$ (of course, we extend F periodically with repect to the above-mentioned lattice).

It is shown in [9] that if r is small enough, then f is an expansive diffeomorphism of the 2-torus (see the definition below).

Note that the defined diffeomorphism f is not structurally stable since the eigenvalues of Df at the zero fixed point are 1 and β.

At the same time, it is shown in [8] that conditions (C1)-(C9) are satisfied for functions V and W defined as follows: If $p = (p_x, p_y)$ and $q = (q_x, q_y)$, then $V(p, q) = |p_y - q_y|$ and $W(p, q) = |p_x - q_x|$ (such functions are properly defined if p and q are close enough). Hence, Theorem 1 is applicable to f, and f has the shadowing property.

Let us relate the obtained result to two classical properties of dynamical systems, namely, to topological stability and expansivity.

Recall the standard definitions.

Denote by $H(X)$ the space of homeomorphisms of a compact metric space (X, dist) endowed with the metric

$$\rho(f,g) = \max_{p\in X} \max(\mathrm{dist}(f(p), g(p)), \mathrm{dist}(f^{-1}(p), g^{-1}(p))).$$

A homeomorphism f is called *topologically stable* if for any $\varepsilon > 0$ there exists a neighborhood Y of f in $H(X)$ such that if $g \in Y$, then there exists a continuous map $h: \ X \to X$ such that $f \circ h = h \circ g$ and

$$\mathrm{dist}(h(p), p) < \varepsilon, \quad p \in X.$$

A homeomorphism f is called *expansive* if there exists a positive α such that if

$$\mathrm{dist}(f^k(p), f^k(q)) \le \alpha, \quad k \in \mathbb{Z},$$

then $p = q$.

Walters in [11] proved that if a homeomorphism f is expansive and has the shadowing property, then f is topologically stable.

Thus, we can apply Theorem 1 to establish the topological stability of the diffeomorphism f in Example 1.

In his proof of topological stability of f, Lewowicz reduced the problem to the study of suspension flows, which does not seem natural.

In its turn, Theorem 1 can be applied to nonsmooth homeomorphisms, for example, to perturbations f of a hyperbolic automorphism of the 2-torus T^2 corresponding to the map

$$F(x, y) = (\mu_1(x), \mu_2(y)),$$

where μ_1 and μ_2 are increasing continuous functions for which there exist numbers $r, \lambda \in (0, 1)$ such that

(1) $|\mu_1(x+\nu) - \mu_1(x)| \le \lambda|\nu|$ and $\lambda^{-1}|\nu| \le |\mu_2(y+\nu) - \mu_2(y)|$ for $|\nu| < r$;

(2) $\mu_1(x) = \alpha x, \quad |x| \ge r$;

(3) $\mu_2(y) = \beta y, \quad |y| \ge r$.

In this case, one can apply the same Lyapunov type functions V and W as in Example 1.

3 Shadowing near a nonisolated fixed point

The method of Lyapunov type functions can be applied in the case of nonisolated fixed points. Of course, if a fixed point p of a homeomorphism f belongs to a submanifold M consisting of fixed points (as in the case studied below), then f does not have the standard shadowing property in any neighborhood of p. Nevertheless, sometimes it is possible to establish a "conditional" shadowing property for pseudotrajectories $\{x_k\}$ whose points do not belong to M assuming that the size of "one step errors"

$$\mathrm{dist}(x_{k+1}, f(x_k))$$

is small compared to the distances from the points x_k to the manifold M. Such an approach (in the case of a nontransverse homoclinic point) had been suggested by S. Tikhomirov.

Let us restrict our consideration to a simple (but nontrivial) example of the 2-dimensional diffeomorphism

$$f(x,y) = \left(\frac{x}{2}, y(1+x^2)\right). \tag{2}$$

The origin is a nonisolated fixed point of f (every point of the line $M = \{(0,y):\ y \in \mathbb{R}\}$ is a fixed point).

Represent $p \in \mathbb{R}^2$ in the form $p = (p_x, p_y)$ and consider a finite pseudotrajectory $p_0, \dots, p_n$ of f such that $(p_k)_x \neq 0$ and

$$|f(p_k) - p_{k+1}| \leq d(p_k)_x^2, \quad k = 0, \dots, n-1, \tag{3}$$

for some $d > 0$. Set

$$K_0 = \{(x,y):\ 0 < |x| < 1\}.$$

In [12], the following result is proved.

Theorem 2. *There exists a neighborhood K of the origin and a number $c > 0$ such that, for any $\varepsilon > 0$ and for any pseudotrajectory $p_0, \dots, p_n$ of f in $K \cap K_0$ that satisfies conditions* (3) *with $d = c\varepsilon$, there exists a point p satisfying the inequalities*

$$|f^k(p) - p_k| \leq \varepsilon, \quad 0 \leq k \leq n.$$

In the proof of this result, the method of the previous section is applied; the corresponding Lyapunov type functions are

$$V(q,p) = |p_y - q_y| \text{ and } W(q,p) = \frac{|p_x - q_x|}{|p_x((1 - |p_x|)}.$$

Analyzing the proof, one can see that the method is applicable not only to diffeomorphism (2) but in more general situations as well.

4 Multiscale conditional shadowing

Let X be a Banach space with norm $|.|$. Consider a sequence $f_n,\ n \geq 0$, of mappings of the space X having the form

$$f_n(x) = A_n x + g_n(x), \tag{4}$$

where A_n are linear mappings.

A sequence $x_n \in X,\ n \geq 0$, is called a trajectory of (4) if

$$x_{n+1} = f_n(x_n), \quad n \geq 0.$$

Let $n \geq m \geq 0$; set

$$\Phi(n,n) = \mathrm{Id}$$

and

$$\Phi(n,m) = A_{n-1} \cdots A_m \text{ if } n > m.$$

We fix a family of projections $P_{k,n}$ of the space X, where $k \in K$ and $n \geq 0$; here K is a countable index set having the form $K = KS \cup KU$.

It is assumed that the projections $P_{k,n}$ have the following properties:

$$P_{i,n} P_{j,n} = 0 \text{ if } i \neq j;$$

$$\sum_{k \in K} P_{k,n} = \mathrm{Id} \text{ for } n \geq 0;$$

and

$$A_n P_{k,n} = P_{k,n+1} A_n \text{ for } k \in K \text{ and } n \geq 0. \tag{5}$$

Of course, property (5) implies that

$$\Phi(n,m) P_{k,m} = P_{k,n} \Phi(n,m) \text{ for } k \in K \text{ and } n \geq m. \tag{6}$$

Let $k \in K$ and $n \geq 0$; denote by $X_{k,n}$ the image $P_{k,n} X$.

We assume, in addition, that the restrictions of A_n to $X_{k,n}$ are invertible for $k \in KU$; this allows us to define for $k \in KU$ the operators

$$\Psi(n,m) = A_n^{-1} \cdots A_{m-1}^{-1}|_{X_{k,m}}$$

acting from $X_{k,m}$ to $X_{k,n}$ with $n < m$. We agree that

$$\Psi(n, n+1) = A_n^{-1}|_{X_{k,n+1}}.$$

Formally, we have to write $\Psi_k(n,m)$ instead of $\Psi(n,m)$, but the present short form will lead to no misunderstanding.

Of course, an analog of property (6) is valid; if $k \in KU$ and $n < m$, then

$$\Psi(n,m)P_{k,m} = P_{k,n}\Psi(n,m).$$

We make the following

Main assumption. *There exists a number $M > 0$ and two sequences of positive numbers $\alpha_{k,n}$ and $\beta_{k,n}$ with the following properties.*

The inequalities

$$\sum_{k\in KS}\sum_{l=1}^{n}\|\Phi(n,l)P_{k,l}\|\alpha_{k,l} \le M, \quad n \ge 0, \tag{7}$$

and

$$\sum_{k\in KS}\sum_{l=1}^{n}\|\Phi(n,l)P_{k,l}\|\beta_{k,l} \le \frac{1}{4}, \quad n \ge 0, \tag{8}$$

hold.

The inequalities

$$\sum_{k\in KU}\sum_{l=n+1}^{\infty}\|\Psi(n,l)P_{k,l}\|\alpha_{k,l} \le M, \quad n \ge 0, \tag{9}$$

and

$$\sum_{k\in KU}\sum_{l=n+1}^{\infty}\|\Psi(n,l)P_{k,l}\|\beta_{k,l} \le \frac{1}{4}, \quad n \ge 0, \tag{10}$$

hold.

Of course, the choice of sequences $\alpha_{k,n}$ and $\beta_{k,n}$ is arbitrary to a large extent; our future estimates essentially depend on this choice.

Fix a sequence

$$V = \{v_n \in X:\ n \ge 0\};$$

set $w_{k,n} = P_{k,n}v_n$ for $k \in K$,

$$\|v_n\| = \sum_{k\in K}|w_{k,n}|, \quad n \ge 0,$$

and

$$\|V\|_\infty = \sup_{n\geq 0} \|v_n\|.$$

(Of course, we work with sequences V for which the above values are finite). We emphasize that the value $\|v_n\|$ depends on the index n.

Let y_n be a sequence of points of X with known "errors"

$$\delta_{n+1} = f_n(y_n) - y_{n+1}, \quad n \geq 0.$$

Introduce functions

$$\gamma_n(v) = g_n(y_n + v) - g_n(y_n).$$

Note that $\gamma_n(0) = 0$.

Our goal is to find a trajectory x_n of (4) for which we can estimate the values

$$|x_n - y_n|, \quad n \geq 0,$$

in terms of the values δ_n (to be exact, in terms of the values $|P_{k,n}\delta_n|$ and of Lipschitz constants of the functions $P_{k,n}\gamma_{n-1}$).

Represent x_n in the form

$$x_n = y_n + v_n;$$

then it follows from the equalities

$$x_{n+1} = f_n(x_n) = A_n x_n + g_n(x_n)$$

that the sequence v_n must satisfy the nonlinear difference equation

$$v_{n+1} = A_n v_n + \delta_{n+1} + \gamma_n(v_n). \tag{11}$$

Our main result in this section is the following statement.

Theorem 3. *Let conditions* (7)–(10) *be satisfied. Fix a positive* d *and assume that*

$$|P_{k,l}\delta_l| \leq \alpha_{k,l} d, \quad k \in K,\ l \geq 0. \tag{12}$$

Assume, in addition, that Lipschitz constants of the projections $P_{k,l}\gamma_{l-1}(v)$ *satisfy the estimates*

$$\text{Lip}\ P_{k,l}\gamma_{l-1}(v) \leq \beta_{k,l}, \quad k \in K,\ l \geq 0. \tag{13}$$

Then there exists a solution V of Eq. (11) *such that*

$$\|V\|_\infty \le 4Md.$$

Proof. The set

$$\mathcal{V} = \{V: \ \|V\|_\infty \le 4Md\}$$

is a subset of the Banach space of sequences V with norm $\|V\|_\infty$.

Define on $\mathcal{V}$ the operator T which takes a sequence

$$V = \{v_0, \dots, v_n, \dots\} \in \mathcal{V} \tag{14}$$

to a sequence

$$Z = T(V) = \{z_n \in X: \ n \ge 0\},$$

where z_n are determined by their projections

$$\zeta_{k,n} = P_{k,n} z_n, \quad k \in K, \, n \ge 0.$$

If $k \in KS$, set $\zeta_{k,0} = 0$ and

$$\zeta_{k,n} = \sum_{l=1}^{n} \Phi(n,l) P_{k,l}(\delta_l + \gamma_{l-1}(v_{l-1})), \quad n > 0.$$

If $k \in KU$, set

$$\zeta_{k,n} = -\sum_{l=n+1}^{\infty} \Psi(n,l) P_{k,l}(\delta_l + \gamma_{l-1}(v_{l-1})), \quad n \ge 0.$$

Recall that $w_{k,l} = P_{k,l} v_l$ and note that the inclusion $v \in \mathcal{V}$ implies the estimates

$$|v_l| = \left| \sum_{k \in K} P_{k,l} v_l \right| = \left| \sum_{k \in K} w_{k,l} \right| \le \|v_l\| \le 4Md, \quad l \ge 0. \tag{15}$$

Fix an index $k \in KS$ and represent

$$\zeta_{k,n} = P_{k,n} \sum_{l=1}^{n} \Phi(n,l)(\delta_l + \gamma_{l-1}(v_{l-1})) =$$

$$= \sum_{l=1}^{n} P_{k,n} \Phi(n,l)(\delta_l + \gamma_{l-1}(v_{l-1})) = \sum_{l=1}^{n} \Phi(n,l) P_{k,l}(\delta_l + \gamma_{l-1}(v_{l-1})), \quad n > 0.$$

Here we refer to property (6) of the projections.

Conditions (7) and (12) imply that

$$\sum_{k\in KS}\left|\sum_{l=1}^{n}\Phi(n,l)P_{k,l}\delta_l\right| = \sum_{k\in KS}\left|\sum_{l=1}^{n}\Phi(n,l)P_{k,l}P_{k,l}\delta_l\right| \le$$

$$\le \sum_{k\in KS}\sum_{l=1}^{n}\|\Phi(n,l)P_{k,l}\|\alpha_{k,l}d \le Md. \tag{16}$$

Condition (13) combined with inequalities (15) implies that

$$|P_{k,l}\gamma_{l-1}(v_{l-1})| \le \beta_{k,l}|v_{l-1}| \le 4\beta_{k,l}Md, \quad l \le n.$$

Hence, it follows from (8) that

$$\sum_{k\in KS}\left|\sum_{l=1}^{n}\Phi(n,l)P_{k,l}\gamma_{l-1}(v_{l-1})\right| \le Md,$$

which, combined with (16), gives us the estimates

$$\sum_{k\in KS}|\zeta_{k,n}| \le 2Md, \quad n \ge 0. \tag{17}$$

Similar reasoning based on inequalities (9) and (10) shows that analogs of estimates (17) are valid for $k \in KU$ as well. Thus, T maps $\mathcal{V}$ into itself.

Now we note that the same reasoning as in the proof of estimate (15) shows that if $V, V' \in \mathcal{V}$, then

$$|v_{l-1} - v'_{l-1}| \le \|v_{l-1} - v'_{l-1}\| \le \|V - V'\|_\infty,$$

and it follows from condition (13) that

$$|P_{k,l}\left(\gamma_{l-1}(v_{l-1}) - \gamma_{l-1}(v'_{l-1})\right)| \le$$

$$\le \beta_{k,l}|v_{l-1} - v'_{l-1}| \le \beta_{k,l}\|V - V'\|_\infty.$$

Thus,

$$|\zeta_{k,n} - \zeta'_{k,n}| = \left|\sum_{l=1}^{n}\Phi(n,l)P_{k,l}(\gamma_{l-1}(v_{l-1}) - \gamma_{l-1}(v'_{l-1}))\right| \le$$

$$\leq \sum_{l=1}^{n} \|\Phi(n,l)P_{k,l}\|\beta_{k,l}\|V-V'\|_\infty$$

for $k \in KS$ and a similar estimate is valid for $k \in KU$.

Hence, conditions (8) and (10) imply the estimate

$$\|T(V)-T(V')\|_\infty \leq \frac{1}{2}\|V-V'\|_\infty$$

for $V, V' \in \mathcal{V}$.

Thus, T is a contraction on $\mathcal{V}$; hence, T has a unique fixed point V in $\mathcal{V}$. Projections of coordinates of this point satisfy the equalities $w_{k,0}=0$ and

$$w_{k,n} = \sum_{l=1}^{n} \Phi(n,l)P_{k,l}(\delta_l + \gamma_{l-1}(v_{l-1})), \quad n>0,$$

for $k \in KS$ and

$$w_{k,n} = -\sum_{l=n+1}^{\infty} \Psi(n,l)P_{k,l}(\delta_l + \gamma_{l-1}(v_{l-1})), \quad n \geq 0,$$

for $k \in KU$.

Standard calculations from the theory of Perron series show that the sequence (14) with

$$v_n = \sum_{k \in K} w_{k,n}$$

is a solution of Eq. (11). □

Example 2. To avoid unnecessary complications, we give an example of application of Theorem 3 with a finite index set K; the reasoning used in this example can be easily extended to the general case.

Let $X = \mathbb{R}^4$ and $K = \{1,\dots,4\}$; fix the projections $P_{k,1}x = (x_1,0,0,0)$, $\dots$, $P_{k,4}x = (0,0,0,x_4)$.

Assume that the matrices A_n in mappings (4) are constant diagonal matrices

$$A_n = \operatorname{diag}(0, 1/2, 1, 2).$$

Take $KS = \{1,\dots,3\}$ and $KU = \{4\}$. Fix arbitrary positive numbers $m_1,\dots,m_4$.

Let $k=1$; then $\|\Phi(n,n)P_{1,n}\| = 1$ and $\|\Phi(n,l)P_{1,l}\| = 0$ for $n \neq l$. Thus, if we take $\alpha_{1,n} = m_1$ for all n, then

$$\sum_{l=1}^{n} \|\Phi(n,l)P_{1,l}\|\alpha_{1,l} \le m_1, \quad n \ge 0.$$

Let $k=2$; then $\|\Phi(n,l)P_{2,l}\| = 2^{l-n}$. Thus, if we take $\alpha_{2,n} = m_2/2$ for all n, then

$$\sum_{l=1}^{n} \|\Phi(n,l)P_{2,l}\|\alpha_{2,l} \le m_2, \quad n \ge 0.$$

If $k=3$, then $\|\Phi(n,l)P_{3,l}\| = 1$, and if we take a sequence $\alpha_{3,n}$ for which

$$\sum_{l=1}^{n} \alpha_{3,l} \le m_3, \quad n \ge 1,$$

then

$$\sum_{l=1}^{n} \|\Phi(n,l)P_{3,l}\|\alpha_{3,l} \le m_3.$$

Thus, we get estimate (7) in the form

$$\sum_{k\in KS} \sum_{l=1}^{n} \|\Phi(n,l)P_{k,l}\|\alpha_{k,l} \le M = m_1 + m_2 + m_3, \quad n \ge 0,$$

and one can repeat a similar procedure in the general case of infinite set KS with an arbitrary series $m_k,\ k \in KS$.

Finally, if $k=4$, then $\|\Psi(n,l)P_{4,l}\| = 2^{n-l}$ for $n < l$, and, as for $k=2$, we may take the constant sequence $\alpha_{4,n} = M$ to get estimate (9).

After that, we can similarly select the sequences $\beta_{k,n}$ (we leave details to the reader).

5 Perturbations of dynamical systems on simple time scales

Let us consider a simple variant of a time scale $\mathcal{T}$ that is a subset of $[0,\infty)$ and consists of isolated segments T_n, where $n = 1,2,\dots,\ T_n = [l_n, r_n]$, and $0 \le l_1 < r_1 < l_2 < \dots$.

The phase space is a Banach space X with norm $|\cdot|$. We denote by $\|\cdot\|$ the operator norm of a linear operator.

The system on T_n is generated by a differential equation

$$\dot{x} = A_n(t)x + a_n(t,x), \quad t \in T_n. \tag{18}$$

We assume that the operators $A_n(t)$ are continuous and bounded on T_n. The functions $a_n(t,x)$ are assumed to be continuous and Lipschitz continuous in x on $T_n \times X$ (with small Lipschitz constants).

Denote by $\phi_n(t,t_0,x_0)$ the solution of the Cauchy problem (t_0,x_0) for system (18).

Thus, for any $t_0 \in T_n$ and for any $x_0 \in X$ there exists a solution $\phi_n(t,t_0,x_0)$ defined on the whole segment T_n (in what follows, we work with such solutions).

Let $\Phi_n(t)$ and $\Psi_n(t)$ be the fundamental matrices of the system

$$\dot{x} = A_n(t)x$$

such that $\Phi_n(l_n) = E$ and $\Psi_n(r_n) = E$, respectively, where E is the identity map of X.

For any index $n = 1, 2, \dots$ we fix a map of X taking a point x to $B_n x + b_n(x)$, where B_n is a linear operator and $b_n(x)$ is a continuous function (it is not assumed, in general, that any operator B_n is an isomorphism of the space X).

The trajectory $x(t)$, $t \in \mathcal{T}$, of the appearing system starting at a point $x_0 \in X$ at the time moment l_1 (the left-hand end of the segment T_1) is defined as follows:

- $x(t) = \phi_1(t, l_1, x_0), \quad t \in T_1,$
- $x(l_2) = B_1 x(r_1) + b_1(x(r_1)),$
- $x(t) = \phi_2(t, l_2, x(l_2)), \quad t \in T_2,$
- $x(l_3) = B_2 x(r_2) + b_2(x(r_2))$, and so on.

We fix countable sets of indices KS and KU and assume that there exist families of continuous projections $P_k(t)$, $t \in \mathcal{T}$, of the space X indexed by $k \in K = KS \bigcup KU$ and having the following properties (19)–(23).

Let

$$P^+(t) = \sum_{k \in KS} P_k(t) \quad \text{and} \quad P^-(t) = \sum_{k \in KU} P_k(t).$$

We assume that

$$P^-(t) + P^+(t) = E, \qquad t \in \mathcal{T}; \tag{19}$$

$$P^-(t)P^+(t) = 0, \qquad t \in \mathcal{T}; \tag{20}$$

$$\Phi_n(t)P_k(l_n) = P_k(t)\Phi_n(t) \text{ and } \Phi_n^{-1}(t)P_k(t) = P_k(l_n)\Phi_n^{-1}(t),$$
$$t \in T_n; \quad k \in KS; \tag{21}$$

$$\Psi_n(t)P_k(r_n) = P_k(t)\Psi_n(t) \text{ and } \Psi_n^{-1}(t)P_k(t) = P_k(r_n)\Psi_n^{-1}(t),$$
$$t \in T_n; \quad k \in KU; \tag{22}$$

$$B_nP_k(r_n) = P_k(l_{n+1})B_n, \quad n \geq 1,\ k \in K. \tag{23}$$

Concerning the projections P_k, $k \in KU$, we assume, in addition, that the following property holds: the restriction of any map B_n, $n \geq 1$, to the subspace $P_k(r_n)X$, $k \in KU$, is an isomorphism of the subspace $P_k(r_n)X$ to the space $P_k(l_{n+1})X$.

For a trajectory $x(t)$, we denote $y_k(t) = P_k(t)x(t)$, $k \in KS$, and $z_k(t) = P_k(t)x(t)$, $k \in KU$.

Let us write down the analog of the Perron operator for the functions $y_k(t)$ and $z_k(t)$ on an interval T_n.

First we write the term of the "direct" operator which is the finite sum of summands including $a_1, \ldots, a_n$ and $b_1, \ldots, b_{n-1}$:

- including a_1:

$$\Phi_n(t)\left(B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2)B_1\int_{l_1}^{r_1}\Phi_1(r_1)\Phi_1^{-1}(s)P_k(s)a_1(s,x(s))\,ds\right);$$

- including a_2:

$$\Phi_n(t)\left(B_{n-1}\Phi_{n-1}(r_{n-1})\cdots\Phi_3(r_3)B_2\int_{l_2}^{r_2}\Phi_2(r_2)\Phi_2^{-1}(s)P_k(s)a_2(s,x(s))\,ds\right);$$

- $\ldots$;
- including a_n:

$$\Phi_n(t)\left(\int_{l_n}^{t}\Phi_n^{-1}(s)P_k(s)a_n(s,x(s))\,ds\right);$$

- including b_1:

$$\Phi_n(t)B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2))P_k(l_2)b_1(x(r_1));$$

- . . . ;
- including b_{n-1}:

$$\Phi_n(t)P_k(l_n)b_{n-1}(x(r_{n-1})).$$

The corresponding term of the "inverse " operator representing $z_k(t)$ for $t \in T_n$ is the infinite sum of summands including $a_n, a_{n+1}, \dots$ and $b_n, b_{n+1}, \dots$:

- including a_n:

$$\Psi_n(t)\int_{r_n}^{t}\Psi_n^{-1}(s)P_k(s)a_n(s,x(s))\,ds;$$

- including a_{n+1}:

$$\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})\int_{r_{n+1}}^{l_{n+1}}\Psi_{n+1}^{-1}(s)P_k(s)a_{n+1}(s,x(s))\,ds;$$

- including a_{n+2}:

$$\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}\Psi_{n+2}(l_{n+2})\int_{r_{n+2}}^{l_{n+2}}\Psi_{n+2}^{-1}(s)P_k(s)a_{n+2}(s,x(s))\,ds;$$

- . . . ;
- including b_n:

$$-\Psi_n(t)B_n^{-1}P_k(l_{n+1})b_n(x(r_n));$$

- including b_{n+1}:

$$-\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}P_k(l_{n+2})b_{n+1}(x(r_{n+1}));$$

- . . . .

Perturbations. The natural statement of the perturbation problem is as follows.

We replace systems (18) on the segments T_n by systems

$$\dot{x} = C_n(t)x + c_n(t,x)$$

(assuming that the operators $C_n(t)$ and the functions $c_n(t,x)$ have properties similar to those of $A_n(t)$ and $a_n(t,x)$) and the maps $B_nx + b_n(x)$ by similar maps $D_nx + d_n(x)$, take a trajectory $\xi(t)$ of the new system and look for a close trajectory $x(t)$ of the original system.

As usual, we are looking for functions $v(t)$ on T_n with values in X such that

$$x(t) = \xi(t) + v(t), \quad t \in T_n.$$

From the relations

$$\dot{x} = A_n(t)x + a_n(t, x) = A_n(t)(\xi + v) + a_n(t, \xi + v) = \dot{\xi} + \dot{v} = C_n(t)\xi + c_n(t, \xi) + \dot{v}$$

we deduce the equations for v:

$$\dot{v} = A_n(t)v + a_n^*(t, \xi, v), \quad t \in T_n, \tag{24}$$

where

$$a_n^*(t, \xi, v) = A_n(t)\xi + a_n(t, \xi + v) - C_n(t)\xi - c_n(t, \xi),$$

which we represent in the form

$$a_n^*(t, \xi, v) = A_n(t)\xi + a_n(t, \xi + v) - a_n(t, \xi) - C_n(t)\xi + a_n(t, \xi) - c_n(t, \xi),$$

and the summands of the right-hand side of the above formula have the following properties: $a_n(t, \xi + v) - a_n(t, \xi)$ vanishes for $v = 0$ and has small Lipschitz constant in v for small $|v|$ (of course, if we impose a similar condition on $a_k(t, x)$) and $a_n(t, \xi) - c_n(t, \xi)$ is small (if the perturbed system is close to the nonperturbed one).

Now we look at the "transition rule." From the equalities

$$x(l_{n+1}) = B_n x(r_n) + b_n(x(r_n)) = B_n\xi(r_n) + B_n v(r_n) + b_n(\xi(r_n) + v(r_n))$$
$$= \xi(l_{n+1}) + v(l_{n+1}) = D_n\xi(r_n) + d_n(\xi(r_n)) + v(l_{n+1})$$

we deduce the relations

$$v(l_{n+1}) = B_n v(r_n) + b_n^*(\xi(r_n), v(r_n)), \quad n \geq 1, \tag{25}$$

where

$$b_n^*(\xi(r_n), v) = (B_n - D_n)\xi(r_n) + b_n(\xi(r_n) + v(r_n)) - b_n(\xi(r_n)) + b_n(\xi(r_n)) - d_n(\xi(r_n)).$$

Thus, for $v(t)$ we get system (24)–(25) similar to the original one (with the same $A_n(t)$ and B_n but, of course, with different "small" nonlinear terms).

We solve this system in a standard way.

Let $\mathcal{V}$ be the space of continuous functions on $\mathcal{T}$ with values in X and with the norm

$$\|v\| = \sup_{n \geq 1} \max_{t \in T_n} |v(t)|.$$

Clearly, $\mathcal{V}$ is a complete metric space with the metric $\rho(v,w) = \|v-w\|$.

Our goal is to indicate conditions under which the "Perron operator" corresponding to system (24) and (25) has a fixed point in $\mathcal{V}$ whose norm we can control.

Main assumption. We make the following main assumption.

There exist sequences of positive numbers $\alpha_{n,k}, \beta_{n,k}$ and a number $M > 0$ such that

$$\|\Phi_n(t)\| \Bigg(\alpha_{1,k}\|B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2)B_1\int_{l_1}^{r_1}\Phi_1(r_1)\Phi_1^{-1}(s)P_k(s)\,ds\|+$$

$$+\alpha_{2,k}\|B_{n-1}\Phi_{n-1}(r_{n-1})\cdots\Phi_3(r_3)B_2\int_{l_2}^{r_2}\Phi_2(r_2)\Phi_2^{-1}(s)P_k(s)\,ds\|+$$

$$+\cdots+\alpha_{n,k}\|\int_{l_n}^{t}\Phi_n^{-1}(s)P_k(s)\,ds\| + \beta_{1,k}\|B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2))P_k(l_2)\|+$$

$$+\cdots+\beta_{n,k}\|P_k(l_{n+1})\|\Bigg) \le M, \quad t\in T_n,\ n\ge 1,\ k\in KS, \tag{26}$$

and

$$\|\Psi_n(t)\| \Bigg(\alpha_{n,k}\|\int_{r_n}^{t}\Psi_n^{-1}(s)P_k(s)\,ds\|+$$

$$+\alpha_{n+1,k}\|B_n^{-1}\Psi_{n+1}(l_{n+1})\int_{r_{n+1}}^{l_{n+1}}\Psi_{n+1}^{-1}(s)P_k(s)\,ds\|+$$

$$+\alpha_{n+2,k}\|B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}\Psi_{n+2}(l_{n+2})\int_{r_{n+2}}^{l_{n+2}}\Psi_{n+2}^{-1}(s)P_k(s)\,ds\|+\cdots+$$

$$+\beta_{n,k}\|B_n^{-1}P_k(l_{n+1}))\| + \beta_{n+1,k}\|B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}P_k(l_{n+2}))\|+$$

$$+\ldots\Bigg) \le M, \quad t\in T_n,\ n\ge 1,\ k\in KU. \tag{27}$$

Theorem 4 [10]. *Let conditions* (26) *and* (27) *be satisfied. Fix a positive d and assume that if $|v| \le 2Md$, then the following estimates hold for $t \in T_n$, $n \ge 1$, and $k \in K$, where* Lip_v *is a Lipschitz constant in variable v:*

$$|P_k(t)a_n^*(t,\xi(t),v)| \le \alpha_{n,k}d,$$

$$|P_k(l_{n+1})b_n^*(\xi(r_n),v)| \le \beta_{n,k}d,$$

$$\mathrm{Lip}_v\left(P_k(t)a_n^*(t,\xi(t),v)\right)\leq\frac{\alpha_{n,k}}{4M},$$

$$\mathrm{Lip}_v\left(P_k(l_{n+1})b_n^*(\xi(r_n),v)\right)\leq\frac{\beta_{n,k}}{4M}.$$

Then for the trajectory $\xi(t)$ of the perturbed system there exists a trajectory $x(t)$ of the unperturbed system such that

$$|x(t)-\xi(t)|\leq 2Md,\quad t\in\mathcal{T}.$$

Acknowledgements

The research of the author was supported by the Russian Science Foundation, grant no. 23-21-00025, https://rscf.ru/project/23-21-00025/

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