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Application of the Method of Approximation of Iterated Stochastic Itô Integrals Based on Generalized Multiple Fourier Series to the High-Order Strong Numerical Methods for Non-Commutative Semilinear Stochastic Partial Differential Equations

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Abstract

We consider a method for the approximation of iterated stochastic Itô integrals of arbitrary multiplicity with respect to the infinite-dimensional Wiener process using the mean-square approximation method of iterated stochastic Itô integrals with respect to the finite-dimensional Wiener process based on generalized multiple Fourier series. The case of Fourier-Legendre series is considered in details. The results of the article can be applied to construction of high-order strong numerical methods (with respect to the temporal discretization) for a mild solution of non-commutative semilinear stochastic partial differential equations.

Key words: non-commutative semilinear stochastic partial differential equation, infinite-dimensional Wiener process, iterated stochastic Itô integral, generalized multiple Fourier series, multiple Fourier-Legendre series, Legendre polynomials, mean-square approximation, expansion.

1 Introduction

There exists a lot of publications on the subject of numerical integration of stochastic partial differential equations (SPDEs) (see, for example [1]-[25]). One of the perspective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for SPDEs is based on the Taylor formula for operators and exponential formula for the mild solution of SPDEs [12] (2015), [13] (2016). As shown in [12] and [18] (2007) the exponential Milstein type approximation method has a strong order of convergence $1-\varepsilon$ (where ε is an arbitrary small posilive real number) [12] or 1 [18]. In [13] the exponential Wagner-Platen type numerical approximation method for SPDEs with strong order $3/2-\varepsilon$ (where ε is an arbitrary small posilive real number) has been considered. An important feature of these numerical methods is the presence in them the so-called iterated stochastic Itô integrals with respect to the infinite-dimensional Wiener process [19]. Approximation of these stochastic integrals is a complex problem. This problem can be significantly simplified if special commutativity conditions be fulfilled [12], [13]. In [25] (2019) two methods of the mean-square approximation of simplest double stochastic Itô integrals with respect to the infinite-dimensional Wiener process are considered and theorems on the convergence of these methods are given (the basic idea about Karhunen-Loeve expansion of the Brownian bridge process was taken from monograph [26] (1988, In Russian)). It is important to note that the approximation of iterated stochastic Itô integrals with respect to the infinite-dimensional Wiener process can be reduced to the approximation of iterated stochastic Itô integrals with respect to the finite-dimensional Wiener process. In a lot of author's publications [27]-[39] an effective method of the mean-square approximation of iterated stochastic Itô (and Stratonovich) integrals with respect to the finite-dimensional Wiener process was proposed and developed. This method is based on the generalized multiple Fourier series, in particular, on the multiple Fourier-Legendre series. The purpose of this article is an adaptation of the method [27]-[39] for the mean-square approximation of iterated stochastic Itô integrals of multiplicity k ($k \in \mathbb{N}$) with respect to the infinite-dimensional Wiener process.

Let U, H be separable $\mathbb{R}$-Hilbert spaces and $L_{HS}(U,H)$ be a space of Hilbert-Schmidt operators. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0,\bar{T}]\}$ [19], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0,\bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here $L(U)$ denotes all bounded linear operators on U.

Consider the semilinear parabolic SPDE

$$dX_t = (AX_t + F(X_t))\,dt + B(X_t)d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}], \tag{1}$$

where nonlinear operators F, B ($F : H \to H$, $B : H \to L_{HS}(U_0, H)$), linear operator $A : D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the mild solution of the SPDE (1) [22] (see also [12], [13]). Here U_0 is an $\mathbb{R}$-Hilbert space defined as $U_0 = Q^{1/2}(U)$.

As it is known, numerical methods of high orders of accuracy (with respect to the temporal discretization) for approximating the mild solution of the SPDE (1), which are based on the Taylor formula for operators and an exponential formula for the mild solution of SPDEs, contain iterated stochastic integrals with respect to the Q-Wiener process [8], [10]-[13], [18].

Note that an exponential Milstein type numerical scheme [12], [18], [24] and exponential Wagner-Platen type numerical scheme [13] contain, for example, the following iterated stochastic integrals

$$\int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \quad \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{2}$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}, \quad \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \tag{3}$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \tag{4}$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{5}$$

where $0 \le t < T \le \bar{T}$, $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and F', B', B'' denote Frêchet derivatives. At that, an exponential Milstein type scheme [12] contains integrals (2) while exponential Wagner-Platen type scheme [13] contains integrals (2) – (5). It is easy to notice that the numerical schemes for SPDEs with higher orders of convergence (with respect to the temporal discretization) in contrast with numerical schemes from [12], [13] will include iterated stochastic Itô integrals (with respect to the Q-Wiener process) with multiplicities $k > 3$ [21] (2012). So, this work is partially devoted to the approximation of iterated stochastic integrals of the form

$$I[\Phi^{(k)}(Z)]_{T,t} = \int\limits_t^T \Phi_k(Z) \left(\ldots \left(\int\limits_t^{t_3} \Phi_2(Z) \left(\int\limits_t^{t_2} \Phi_1(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) \ldots \right) d\mathbf{W}_{t_k}, \tag{6}$$

where $0 \le t < T \le \bar{T}$, $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and an operator $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v))\ldots\ ))$ is a k-linear Hilbert–Schmidt operator for all $v \in H$. In Sect. 5 we consider the approximation of more general iterated stochastic integrals than (6). In Sect. 6, 7 some other types of iterated stochastic integrals of multiplicities 2–4 with respect to the Q-Wiener process will be considered.

Note that the stochastic integral (5) is not a special case of the stochastic integral (6) for $k = 3$. Nevertheless, the expanded representation of the approximation of stochastic integral (5) has a close structure to (10) for $k = 3$. Moreover, the mentioned representation of stochastic integral (5) contains the same iterated stochastic Itô integrals of the third multiplicity as in (10) for $k = 3$ (see Sect. 6). These conclusions mean that the main result (Theorem 4, Sect. 5) for $k = 3$ can be reformulated naturally for the stochastic integral (5) (see Sect. 6).

It should be noted that by developing an approach from the work [13], which uses Taylor formula for operators and a formula for the mild solution of the SPDE (1), we obviously obtain a number of other iterated stochastic integrals. For example, the following stochastic integrals

$$\int\limits_t^T B'''(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2},$$

$$\int\limits_t^T B'(Z) \left(\int\limits_t^{t_3} B''(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) d\mathbf{W}_{t_3},$$

$$\int\limits_t^T B''(Z) \left(\int\limits_t^{t_3} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_3} B'(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) d\mathbf{W}_{t_3},$$

$$\int\limits_t^T F'(Z) \left(\int\limits_t^{t_3} B'(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) dt_3,$$

$$\int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2,$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}$$

will be considered in Sect. 7. Here $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and B', B'', B''', F', F'' are Frêchet derivatives.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i = (i_1, \ldots, i_d) \in J$, $x = (x_1, \ldots, x_d) \in U$, and $J = \{i : \ i \in \mathbb{N}^d, \ \text{and } \lambda_i > 0\}$.

The series representation of the Q-Wiener process has the form [19]

$$\mathbf{W}(t, x) = \sum_{i\in J} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)}, \ t \in [0, \bar{T}],$$

or in the shorter notations

$$\mathbf{W}_t = \sum_{i\in J} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)}, \ t \in [0, \bar{T}],$$

where $\mathbf{w}_t^{(i)}$, $i \in J$ are independent standard Wiener processes.

Note that eigenfunctions e_i, $i \in J$ form an orthonormal basis of U [19].

Consider the finite approximation of $\mathbf{W}_t$ [19]

$$\mathbf{W}_t^M = \sum_{i\in J_M} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)}, \ t \in [0, \bar{T}], \tag{7}$$

where $J_M = \{i : \ 1 \le i_1, \ldots, i_d \le M, \ \text{and } \lambda_i > 0\}$.

Using (7) and the relation [19]

$$\mathbf{w}_t^{(i)} = \frac{1}{\sqrt{\lambda_i}}\langle e_i, \mathbf{W}_t\rangle_U, \ i \in J \tag{8}$$

we obtain

$$\mathbf{W}_t^M = \sum_{i\in J_M} e_i \ \langle e_i, \mathbf{W}_t\rangle_U, \ t \in [0, \bar{T}], \tag{9}$$

where $\langle \cdot, \cdot\rangle_U$ is a scalar product in U.

Taking into account (8), (9) we note that the approximation $I[\Phi^{(k)}(Z)]_{T,t}^{M}$ of iterated stochastic integral $I[\Phi^{(k)}(Z)]_{T,t}$ (see (6)) can be rewritten with probability 1 (further w. p. 1) in the following form

$$I[\Phi^{(k)}(Z)]_{T,t}^{M} =$$

$$= \int\limits_t^T \Phi_k(Z) \left(\ldots \left(\int\limits_t^{t_3} \Phi_2(Z) \left(\int\limits_t^{t_2} \Phi_1(Z) d\mathbf{W}_{t_1}^{M} \right) d\mathbf{W}_{t_2}^{M} \right) \ldots \right) d\mathbf{W}_{t_k}^{M} =$$

$$= \sum_{r_1,\ldots,r_k \in J_M} \Phi_k(Z) \left(\ldots \left(\Phi_2(Z) \left(\Phi_1(Z) e_{r_1} \right) e_{r_2} \right) \ldots \right) e_{r_k} \times$$

$$\times \int\limits_t^T \ldots \int\limits_t^{t_3} \int\limits_t^{t_2} d\langle e_{r_1}, \mathbf{W}_{t_1} \rangle_U \ d\langle e_{r_2}, \mathbf{W}_{t_2} \rangle_U \ \ldots \ d\langle e_{r_k}, \mathbf{W}_{t_k} \rangle_U =$$

$$= \sum_{r_1,\ldots,r_k \in J_M} \Phi_k(Z) \left(\ldots \left(\Phi_2(Z) \left(\Phi_1(Z) e_{r_1} \right) e_{r_2} \right) \ldots \right) e_{r_k} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times$$

$$\times \int\limits_t^T \ldots \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)} d\mathbf{w}_{t_2}^{(r_2)} \ldots d\mathbf{w}_{t_k}^{(r_k)}, \qquad (10)$$

where $0 \leq t < T \leq \bar{T}$.

Remark 1. *Obviously, without the loss of generality we can write down* $J_M = \{1, 2, \ldots, M\}$.

When special conditions of commutativity for the SPDE (1) be fulfilled it is proposed to simulate numerically the stochastic integrals (2) – (5) using the simple formulas [12], [13]. In this case, the numerical simulation of mentioned stochastic integrals requires the use of increments of the Q-Wiener process only. However, if these commutativity conditions are not met (which often corresponds to SPDEs in numerous applications), the numerical simulation of stochastic integrals (2) – (5) becomes much more difficult. In [25] two methods for the mean-square approximation of simplest double stochastic Itô integrals with respect to the Q-Wiener process are proposed. In this article, we consider a substantially more general and effective method for the mean-square approximation of iterated stochastic Itô integrals of multiplicity k ($k \in \mathbb{N}$) with respect to the Q-Wiener process. The convergence analysis in the transition from J_M to J, i.e., from the finite-dimensional Wiener process to the infinite-dimensional one could be carried out similar to the proof of Theorem 1 [25].

Chapters 5 and 6 (pp. A.249 – A.628) of the monograph [35] (see also [28]-[34], [36]-[39]) are devoted to constructing of efficient methods of the mean-square approximation of iterated stochastic Itô integrals with respect to components of the finite-dimensional Wiener process. These results are also adapted to iterated stochastic Stratonovich integrals [28]-[39]. Below (Sect. 2 – 4) we consider a very short review of results from chapters 5 and 6 of the monograph [35] and some new results (Sect. 5 – 7).

2 Method of Approximation of Iterated Stochastic Itô Integrals Based on Generalized Multiple Fourier Series

Consider more general iterated stochastic Itô integrals than in (10)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{11}$$

where $0 \le t < T \le \bar{T}$, and every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous non-random function on $[t, T]$; $\mathbf{w}_\tau^{(i)}$ $(i = 1,\ldots,m)$ are independent standard Wiener processes (see Sect. 1) and $\mathbf{w}_\tau^{(0)} = \tau$; $i_1,\ldots,i_k = 0,\ 1,\ldots,m.$

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in $L_2([t, T])$ and define the following function on a hypercube $[t, T]^k$

$$K(t_1,\ldots,t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l<t_{l+1}\}}; \quad t_1,\ldots,t_k \in [t, T]; \quad k \ge 2, \tag{12}$$

and $K(t_1) \equiv \psi_1(t_1);\ t_1 \in [t, T]$, where $\mathbf{1}_A$ is the indicator of the set A.

The function $K(t_1,\ldots,t_k)$ is sectionally continuous on the hypercube $[t, T]^k$. At this situation it is well known that the generalized multiple Fourier series of

$K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ converges to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{13}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{14}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^N$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{if}\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{15}$$

Theorem 1 [28]-[39]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous on $[t,T]$ non-random function and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{16}$$

where

$$\mathcal{G}_k=\mathcal{H}_k\backslash\mathcal{L}_k;\ \mathcal{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\};$$

$$\mathcal{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\};$$

l.i.m. *is a limit in the mean-square sense*; $i_1,\ldots,i_k=0,1,\ldots,m$; *every*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{17}$$

is a standard Gaussian random variable for various i *or* j (*if* $i \neq 0$); $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (14); $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,\ 1, \ldots, m)$; $\{\tau_j\}_{j=0}^{N}$ *is the discretization of* $[t, T]$, *which satisfies the condition* (15).

It is not difficult to see that for the case of pairwise different numbers $i_1, \ldots, i_k = 1, \ldots, m$ from Theorem 1 we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.$$

In order to evaluate a significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 5$ [28]-[39] (cases $k = 6,\ 7$ and $k > 7$ can be found in [29], [32], [35])

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{18}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{19}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\prod_{l=1}^{3}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{20}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{21}$$

$$
\begin{aligned}
J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{aligned}
\tag{22}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider the generalization of the formulas (18) – (22) for the case of arbitrary multiplicity of $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$. In order to do this, let us consider the disordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r disordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}),
\tag{23}
$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an disordered set, and parentheses mean an ordered set.

We will say that (23) is a partition and consider the sum using all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{24}$$

Below there are several examples of sums in the form (24)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} = a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145}+$$

$$+a_{24,135} + a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4}+$$

$$+a_{15,23,4} + a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2}+$$

$$+a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can formulate Theorem 1 (formula (16)) using alternative form.

Theorem 2 [29]-[39]. *In conditions of Theorem* 1 *the following converging in the mean-square sense expansion is valid*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr). \tag{25}$$

In particular, from (25) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (22). Note that the rightness of formulas (18) – (22) can be verified by the fact that if $i_1=\ldots=i_5=i=1,\ldots,m$ and $\psi_1(s),\ldots,\psi_5(s)\equiv\psi(s)$, then we can derive from (18) – (22) the well known equalities, which be fulfilled w. p. 1 [29]-[32], [35]:

$$J[\psi^{(1)}]_{T,t}^{(i)}=\frac{1}{1!}\delta_{T,t}^{(i)},$$

$$J[\psi^{(2)}]_{T,t}^{(ii)}=\frac{1}{2!}\left(\left(\delta_{T,t}^{(i)}\right)^2-\Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}^{(iii)}=\frac{1}{3!}\left(\left(\delta_{T,t}^{(i)}\right)^3-3\delta_{T,t}^{(i)}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}^{(iiii)}=\frac{1}{4!}\left(\left(\delta_{T,t}^{(i)}\right)^4-6\left(\delta_{T,t}^{(i)}\right)^2\Delta_{T,t}+3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}^{(iiiii)}=\frac{1}{5!}\left(\left(\delta_{T,t}^{(i)}\right)^5-10\left(\delta_{T,t}^{(i)}\right)^3\Delta_{T,t}+15\delta_{T,t}^{(i)}\Delta_{T,t}^2\right),$$

where

$$\delta_{T,t}^{(i)}=\int\limits_t^T\psi(s)d\mathbf{w}_s^{(i)},$$

$$\Delta_{T,t}=\int\limits_t^T\psi^2(s)ds,$$

which can be independently obtained using the Itô formula and Hermite polynomials [16].

3 Calculation of the Mean-Square Error of Approximation of Iterated Stochastic Itô Integrals in Theorem 1

Assume that $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1\ldots p_k}$ is an approximation of (11), which is the prelimit expression in (16). Let us denote

$$E^{(i_1\ldots i_k)p_1,\ldots,p_k} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}\right)^2\right\},$$

$$E^{(i_1\ldots i_k)p} = E_k^{(i_1\ldots i_k)p_1,\ldots,p_k}\Bigg|_{p_1=\ldots=p_k=p},$$

$$I_k = \|K\|^2_{L_2([t,T]^k)}, \tag{26}$$

where $K(t_1,\ldots,t_k) \stackrel{\rm def}{=} K$.

In [32]-[35], [38], [39] it was shown that

$$E_k^{(i_1\ldots i_k)p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right), \tag{27}$$

for $i_1,\ldots,i_k = 1,\ldots,m$ $(T-t<\infty)$ or $i_1,\ldots,i_k = 0,1,\ldots,m$ $(T-t<1)$.

The exact calcutation of $E^{(i_1\ldots i_k)p}$ is presented in the following theorem.

Theorem 3 [33], [35], [38], [39]. *Suppose that the conditions of Theorem* 1 *be fulfilled for* $i_1,\ldots,i_k = 1,\ldots,m$. *Then*

$$E^{(i_1\ldots i_k)p} = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\}, \tag{28}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ *is the prelimit expression in* (16) (*see also* (46)) *for* $p_1 = \ldots = p_k = p$; $i_1,\ldots,i_k = 1,\ldots,m$; *expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum according to all possible permutations $(j_1, \ldots, j_k)$, *at the same time if* j_r *swapped with* j_q *in the permutation* $(j_1, \ldots, j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1, \ldots, i_k)$; *another notations see in Theorem* 1.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\} = C_{j_k\ldots j_1}$$

for $i_1 \ldots i_k = 1, \ldots, m.$

Then from Theorem 3 for $i_1, \ldots, i_k = 1, \ldots, m$ we obtain [33], [35]

$$E^{(i_1\ldots i_k)p} = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2 \quad (\text{pairwise different } i_1, \ldots, i_k), \tag{29}$$

$$E^{(i_1 i_2)p} = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}C_{j_1 j_2} \quad (i_1 = i_2),$$

$$E^{(i_1 i_2 i_3)p} = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3 j_2 j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3 j_1 j_2}C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3),$$

$$E^{(i_1 i_2 i_3 i_4)p} = I_4 - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2)} C_{j_4 j_3 j_2 j_1}\right)\right) \quad (i_1 = i_2 \neq i_3 = i_4),$$

$$E^{(i_1 i_2 i_3 i_4 i_5)p} = I_5 - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2,j_5)} C_{j_5 j_4 j_3 j_2 j_1}\right)\right)$$
$$(i_1 = i_2 = i_5 \neq i_3 = i_4).$$

4 Some Examples of the Mean-Square Approximations of Iterated Stochastic Itô Integrals Using Legendre Polynomials

Denote

$$I_{(1)T,t}^{(i_1)} = \int\limits_t^T d\mathbf{w}_{t_1}^{(i_1)}, \quad I_{(10)T,t}^{(i_1 0)} = \int\limits_t^T\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2, \quad I_{(01)T,t}^{(0 i_2)} = \int\limits_t^T\int\limits_t^{t_2} dt_1 d\mathbf{w}_{t_2}^{(i_2)},$$

$$I_{(11)T,t}^{(i_1 i_2)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \quad I_{(111)T,t}^{(i_1 i_2 i_3)} = \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)},$$

$$I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)} = \int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)},$$

$$I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)} = \int\limits_t^T \int\limits_t^{t_5} \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)},$$

where $i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m$.

The complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right); \ j = 0, 1, 2, \ldots, \tag{30}$$

where $P_j(x)$ is a Legendre polynomial.

Using the system of functions (30) and Theorem 1 we obtain the following approximations of iterated stochastic Itô integrals [27]-[39]

$$I_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(01)T,t}^{(0 i_1)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{31}$$

$$I_{(10)T,t}^{(i_1 0)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{32}$$

$$I_{(11)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q} \frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{33}$$

$$I_{(111)T,t}^{(i_1 i_1 i_1)} = \frac{(T-t)^{3/2}}{6}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right),$$

$$I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)q_2} = \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{34}$$

$$I_{(1111)T,t}^{(i_1 i_1 i_1 i_1)} = \frac{(T-t)^{2}}{24}\left(\left(\zeta_0^{(i_1)}\right)^4 - 6\left(\zeta_0^{(i_1)}\right)^2 + 3\right),$$

$$I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_3} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_3} C_{j_5 j_4 j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$

$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{35}
$$

$$
I_{(11111)T,t}^{(i_1i_1i_1i_1i_1)}=\frac{(T-t)^{5/2}}{120}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right),
$$

where

$$
C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3j_2j_1},
$$

$$
C_{j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}(T-t)^{2}}{16}\bar{C}_{j_4j_3j_2j_1},
$$

$$
C_{j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}(T-t)^{5/2}}{32}\bar{C}_{j_5j_4j_3j_2j_1},
$$

$$
\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,
$$

$$
\bar{C}_{j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,
$$

$$
\bar{C}_{j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv,
$$

the random variable $\zeta_j^{(i)}$ is defined by (17), and

$$
I_{(11)T,t}^{(i_1i_2)}=\underset{q\to\infty}{\text{l.i.m.}}\ I_{(11)T,t}^{(i_1i_2)q},
$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)} = \underset{q_1\to\infty}{\text{l.i.m.}}\ I_{(111)T,t}^{(i_1 i_2 i_3)q_1},$$

$$I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{q_2\to\infty}{\text{l.i.m.}}\ I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)q_2},$$

$$I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)} = \underset{q_3\to\infty}{\text{l.i.m.}}\ I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_3}.$$

Note that $T-t \ll 1$ ($T-t$ is an integration step with respect to the temporal variable). Thus $q_1 \ll q$ (see Table 1 [28]-[32], [35]). Moreover, the values $\bar{C}_{j_3 j_2 j_1}$, $\bar{C}_{j_4 j_3 j_2 j_1}$, $\bar{C}_{j_5 j_4 j_3 j_2 j_1}$ do not depend on $T-t$. This feature is important because we can use a variable integration step $T-t$. Coefficients $\bar{C}_{j_3 j_2 j_1}$, $\bar{C}_{j_4 j_3 j_2 j_1}$, $\bar{C}_{j_5 j_4 j_3 j_2 j_1}$ are calculated once and before the start of the numerical scheme. Some examples of exact calculation of coefficients $\bar{C}_{j_3 j_2 j_1}$, $\bar{C}_{j_4 j_3 j_2 j_1}$, $\bar{C}_{j_5 j_4 j_3 j_2 j_1}$ via DERIVE (computer algebra system) can be found in Tables 2–4 (another tables are presented in [28]-[32], [35]).

Denote

$$E^{(i_1 i_2)q} = \mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)} - I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3)q_1} = \mathsf{M}\left\{\left(I_{(111)T,t}^{(i_1 i_2 i_3)} - I_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3 i_4)q_2} = \mathsf{M}\left\{\left(I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3} = \mathsf{M}\left\{\left(I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)} - I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_3}\right)^2\right\}.$$

Then for pairwise different $i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m$ from Theorem 3 we obtain [27]-[39]

$$E^{(i_1 i_2)q} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{36}$$

$$E^{(i_1 i_2 i_3)q_1} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2, \tag{37}$$

$$E^{(i_1 i_2 i_3 i_4)q_2} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2, \tag{38}$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3} = \frac{(T-t)^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_3} C_{j_5 j_4 j_3 j_2 j_1}^2. \tag{39}$$

Table 1. Minimal numbers q, q_1 such that $E^{(i_1 i_2)q}$, $E^{(i_1 i_2 i_3)q_1} \le (T-t)^4$, $\quad q_1 \ll q$.

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 2. Coefficients $\bar{C}_{3jk}$.

${}_j{}^k$	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

Table 3. Coefficients $\bar{C}_{21kl}$.

${}_k{}^l$	0	1	2
0	$\frac{2}{21}$	$-\frac{2}{45}$	$\frac{2}{315}$
1	$\frac{2}{315}$	$\frac{2}{315}$	$-\frac{2}{225}$
2	$-\frac{2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 4. Coefficients $\bar{C}_{101lr}$.

${}_l{}^r$	0	1
0	$\frac{4}{315}$	0
1	$\frac{4}{315}$	$-\frac{8}{945}$

On the basis of the presented approximations of iterated stochastic Itô integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness according to $T-t$ $(T-t \ll 1)$ in the mean-square sense for iterated stochastic Itô integrals. This leads to sharp decrease of member quantities in the approximations of iterated stochastic Itô integrals, which are required for achieving the acceptable accuracy of the approximation $(q_1 \ll q)$.

From (37) – (39) we obtain [28]-[32], [35]

$$E^{(i_1 i_2 i_3)q_1}\Big|_{q_1=6} \approx 0.01956000(T-t)^3, \tag{40}$$

$$E^{(i_1 i_2 i_3 i_4)q_2}\Big|_{q_2=2} \approx 0.02360840(T-t)^4, \tag{41}$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3}\Big|_{q_3=1} \approx 0.00759105(T-t)^5. \tag{42}$$

It is not difficult to see that the accuracy in (41) and (42) is significantly better than in (40) $(T-t \ll 1)$ even for $q_2 = 2$ and $q_3 = 1$. This means that in such situation in formulas (34), (35) the number of terms can be chosen significantly less than 3^4 $(q_2 = 2)$ and 2^5 $(q_3 = 1)$. So, in practice, we can leave only few terms in these formulas.

5 Approximation of Iterated Stochastic Integrals of Multiplicity k with Respect to the Q-Wiener Process

Consider the iterated stochastic integral with respect to the Q-Wiener process in the form

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t} = \int\limits_t^T \Phi_k(Z)\Biggl(\ldots\Biggl(\int\limits_t^{t_3}\Phi_2(Z)\Biggl(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\Biggr)\times$$

$$\times\psi_2(t_2)d\mathbf{W}_{t_2}\Biggr)\ldots\Biggr)\psi_k(t_k)d\mathbf{W}_{t_k}, \tag{43}$$

where $Z:\Omega\rightarrow H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, for all $v\in H$ operator $\Phi_k(v)(\,\ldots(\Phi_2(v)(\Phi_1(v)))\ldots)$ is a k-linear Hilbert–Schmidt operator, and every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous on $[t,T]$ non-random function.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}$ be an approximation of the stochastic integral (43)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}=\int\limits_t^T\Phi_k(Z)\Biggl(\ldots\Biggl(\int\limits_t^{t_3}\Phi_2(Z)\Biggl(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}^{M}\Biggr)\times$$

$$\times\psi_2(t_2)d\mathbf{W}_{t_2}^{M}\Biggr)\ldots\Biggr)\psi_k(t_k)d\mathbf{W}_{t_k}^{M}=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)},\tag{44}$$

where $0\le t<T\le\bar{T}$, and

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated stochastic Itô integral (11).

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}$ be an approximation of the stochastic integral (44)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k},\tag{45}$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as a prelimit expression in (16)

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(r_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(r_k)}\Biggr)\tag{46}$$

or as a prelimit expression in (25)

$$J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k) p_1 \ldots p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(r_l)} + \sum_{m=1}^{[k/2]} (-1)^m \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\}) \\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{m} \mathbf{1}_{\{r_{g_{2s-1}} = \ r_{g_{2s}} \neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = \ j_{g_{2s}}\}} \prod_{l=1}^{k-2m} \zeta_{j_{q_l}}^{(r_{q_l})} \Biggr). \tag{47}$$

Let U, H be separable $\mathbb{R}$-Hilbert spaces, $U_0 = Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let $L(U,H)_0 = \{T|_{U_0} : \ T \in L(U,H)\}$. It is known [19] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 4. *Let the conditions of Theorem* 1 *be fulfilled, as well as the following conditions:*

1. *$Q \in L(U)$ is a nonnegative and symmetric trace class operator (λ_i and e_i ($i \in J$) are its eigenvalues and eigenfunctions (which form an orthonormal basis of U) correspondingly), and $\mathbf{W}_\tau$, $\tau \in [0,\bar{T}]$ is an U-valued Q-Wiener process.*

2. *$Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping.*

3. *$\Phi_1 \in L(U,H)_0$, $\Phi_2 \in L(H, L(U,H)_0)$, moreover for all $v \in H$ operator $\Phi_k(v)(\ \ldots (\Phi_2(v)(\Phi_1(v))) \ldots \ )$ is a k-linear Hilbert–Schmidt operator such that*

$$\left\| \Phi_k(Z) \left(\ldots \left(\Phi_2(Z) \left(\Phi_1(Z) e_{r_1} \right) e_{r_2} \right) \ldots \right) e_{r_k} \right\|_H^2 \leq L_k < \infty$$

w. p. 1 *for all $r_1, r_2, \ldots, r_k \in J_M$, $M \in \mathbb{N}$.*

Then

$$\mathsf{M} \left\{ \left\| I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1 \ldots p_k} \right\|_H^2 \right\} \leq$$

$$\leq L_k (k!)^2 \left(\mathrm{tr}\ Q\right)^k \left(I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2 \right), \tag{48}$$

where I_k *is defined by* (26), *and*

$$\mathrm{tr}\ Q = \sum_{i \in J} \lambda_i.$$

Remark 2. *It should be noted that the right-hand side of the inequality* (48) *is independent of* M *and tends to zero if* $p_1, \ldots, p_k \to \infty$ *due to the Parseval's equality.*

Proof. Using (27) we obtain

$$\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1\ldots p_k}\right\|_H^2\right\} =$$

$$= \mathsf{M}\left\{\left\|\sum_{r_1,r_2,\ldots,r_k \in J_M} \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2} \times \right.\right.$$

$$\left.\left. \times \left(J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)p_1,\ldots,p_k}\right)\right\|_H^2\right\} = \qquad (49)$$

$$= \left|\mathsf{M}\left\{\sum_{r_1,r_2,\ldots,r_k \in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{'}}\right)^{1/2} \times \right.\right.$$

$$\times \Big\langle \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\ ,$$

$$\Phi_k(Z)\Big(\ldots\Big(\Phi_2(Z)\Big(\Phi_1(Z)e_{r_1^{'}}\Big)e_{r_2^{'}}\Big)\ldots\Big)e_{r_k^{'}}\Big\rangle_H \times$$

$$\times\ \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)p_1,\ldots,p_k}\right) \times \right.$$

$$\left.\left.\left. \times \left(J[\psi^{(k)}]_{T,t}^{(r_1^{'} r_2^{'} \ldots r_k^{'})} - J[\psi^{(k)}]_{T,t}^{(r_1^{'} r_2^{'} \ldots r_k^{'})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right\}\right| \le \qquad (50)$$

$$
\le \sum_{r_1,r_2,\ldots,r_k\in J_M} \ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2} \left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2} \times
$$

$$
\times \mathsf{M}\left\{\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H \times\right.
$$

$$
\times\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{\prime}}\right)e_{r_2^{\prime}}\right)\ldots\right)e_{r_k^{\prime}}\right\|_H \times
$$

$$
\times\left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.\right.
$$

$$
\left.\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right|\right\} \le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} \ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2} \left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2} \times
$$

$$
\times \mathsf{M}\left\{\left|\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.\right.
$$

$$
\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)\right|\right\} \le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} \ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2} \left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2} \times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2} \le
$$

$$\leq L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} \ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times$$

$$\times\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\leq$$

$$\leq L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} k!\ \lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)=$$

$$=L_k\,(k!)^2\sum_{r_1,r_2,\ldots,r_k\in J_M}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\leq$$

$$\leq L_k\,(k!)^2\,(\mathrm{tr}\ Q)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),$$

where $\langle\cdot,\cdot\rangle_H$ is a scalar product in H, and

$$\sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}}$$

means the sum according to all possible permutations $(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime})$ such that $\{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}$.

The transition from (49) to (50) is based on the following theorem.

Theorem 5. *The following equality is true*

$$\mathsf{M}\Biggl\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\right)\times$$

$$\times\left(J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}-J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1\ldots p_k}\right)\Biggl|\mathbf{F}_t\Biggr\}=0 \qquad (51)$$

w. p. 1 *for all* $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ *such that* $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

Proof. Using the standard moment properties of stochastic Itô integral we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\Big|\mathbf{F}_t\right\}=0 \qquad (52)$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ ($M\in\mathbb{N}$) such that $(r_1,\ldots,r_k)\neq(m_1,\ldots,m_k)$.

Let us rewrite formulas (46), (47) (see also (18)–(22)) in the form

$$J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(m_l)}-S_{j_1,\ldots,j_k}^{(m_1\ldots m_k)}\right). \qquad (53)$$

From the proof of Theorem 5.1 in [35], p. A.261 or [36], p. 9 (see also [28]-[32]) it follows that

$$\prod_{l=1}^{k}\zeta_{j_l}^{(m_l)}-S_{j_1,\ldots,j_k}^{(m_1\ldots m_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\neq l_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(m_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(m_k)}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int_t^T\phi_{j_k}(t_k)\ldots\int_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\ \text{w. p. 1}, \qquad (54)$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum according to all possible permutations $(j_1,\ldots,j_k)$, at the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then m_r swapped with m_q in the permutation $(m_1,\ldots,m_k)$; another notations see in Theorem 1.

Then w. p. 1

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1\ldots p_k}\Big|\mathbf{F}_t\right\}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int_t^T\phi_{j_k}(t_k)\ldots\int_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\Big|\mathbf{F}_t\right\}.$$

From the standard moment properties of the stochastic Itô integral it follows that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\right|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ such that $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

Then

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1\ldots p_k}\right|\mathbf{F}_t\right\}=0 \tag{55}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ such that $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

From (53), (54) it follows that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\right|\mathbf{F}_t\right\}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{q_1=0}^{p_1}\ldots\sum_{q_k=0}^{p_k}C_{q_k\ldots q_1}\times$$

$$\times\mathsf{M}\left\{\left(\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\right)\times\right.$$

$$\left.\times\left(\sum_{(q_1,\ldots,q_k)}\int\limits_t^T\phi_{q_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{q_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\right)\right|\mathbf{F}_t\right\}=0 \tag{56}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ such that $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

From (52), (55), and (56) we obtain (51). Theorem 5 is proved.

Corollary 1. *The following equality is true*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\right)\times\right.$$

$$\left.\times\left(J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)}-J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)q_1\ldots q_l}\right)\right|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $l=1,2,\ldots,k-1,$ *and* $r_1,\ldots,r_k,$ $m_1,\ldots,m_l\in J_M,$ $p_1,\ldots,p_k,$ $q_1,\ldots,q_l=0,1,2,\ldots$

6 Approximation of Some Iterated Stochastic Integrals of Second and Third Miltiplicity with Respect to the Q-Wiener Process

This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 1)

$$I_0[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}^M, \tag{57}$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2, \tag{58}$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M. \tag{59}$$

Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v),B(v))$ be a trilinear Hilbert–Schmidt operator for all $v\in H$.

Then w. p. 1 we have (see (44))

$$I_0[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M} B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}I_{(01)T,t}^{(0r_1)}, \tag{60}$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M} F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}I_{(10)T,t}^{(r_10)}, \tag{61}$$

$$I_2[B(Z)]_{T,t}^M=\sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}. \tag{62}$$

Using the Itô formula we obtain

$$\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}=I_{(11)s,t}^{(r_1r_2)}+I_{(11)s,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}(s-t)\quad \text{w. p. 1.} \tag{63}$$

From (63) we have

$$\int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)} \int\limits_t^s d\mathbf{w}_\tau^{(r_2)} \right) d\mathbf{w}_s^{(r_3)} = I_{(111)T,t}^{(r_1 r_2 r_3)} + I_{(111)T,t}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(0 r_3)} \quad \text{w. p. 1.} \tag{64}$$

Note that in (60), (61), (63) and (64) we use the notations from Sect. 4.

After substituting (64) into (62) we have

$$I_2[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(I_{(111)T,t}^{(r_1 r_2 r_3)} + I_{(111)T,t}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(0 r_3)} \right) \quad \text{w. p. 1.} \tag{65}$$

Taking into account (31), (32) we put for $q = 1$

$$I_{(01)T,t}^{(0 r_3)q} = I_{(01)T,t}^{(0 r_3)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(r_3)} \right) \quad \text{w. p. 1,} \tag{66}$$

$$I_{(10)T,t}^{(r_1 0)q} = I_{(10)T,t}^{(r_1 0)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(r_1)} \right) \quad \text{w. p. 1,} \tag{67}$$

where $I_{(01)T,t}^{(0 r_3)q}$, $I_{(10)T,t}^{(r_1 0)q}$ denote the approximations of corresponding iterated stochastic Itô integrals.

Denote by $I_0[B(Z), F(Z)]_{T,t}^{M,q}$, $I_1[B(Z), F(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$ the approximations of iterated stochastic integrals (60), (61), (65)

$$I_0[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}} I_{(01)T,t}^{(0 r_1)q}, \tag{68}$$

$$I_1[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}} I_{(10)T,t}^{(r_1 0)q}, \tag{69}$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2, r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(I_{(111)T,t}^{(r_1 r_2 r_3)q} + I_{(111)T,t}^{(r_2 r_1 r_3)q} + \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(0 r_3)q} \right), \tag{70}$$

where $q \geq 1$, and the approximations $I_{(111)T,t}^{(r_1 r_2 r_3)q}$, $I_{(111)T,t}^{(r_2 r_1 r_3)q}$ are defined by (33).

From (60), (61), (65), (68) – (70) it follows that

$$I_0[B(Z),F(Z)]_{T,t}^{M}-I_0[B(Z),F(Z)]_{T,t}^{M,q}=0\quad \text{w. p. 1},$$

$$I_1[B(Z),F(Z)]_{T,t}^{M}-I_1[B(Z),F(Z)]_{T,t}^{M,q}=0\quad \text{w. p. 1},$$

$$I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$
$$\times\left(\left(I_{(111)T,t}^{(r_1r_2r_3)}-I_{(111)T,t}^{(r_1r_2r_3)q}\right)+\left(I_{(111)T,t}^{(r_2r_1r_3)}-I_{(111)T,t}^{(r_2r_1r_3)q}\right)\right)\quad \text{w. p. 1}.$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=3$ we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$
$$\le 4C(3!)^2\left(\text{tr }Q\right)^3\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^2\right),$$

where here and further constant C has the same meaning as constant L_k in Theorem 4 (k is the multiplicity of the iterated stochastic integral), and

$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_j(x)$ is a Legendre polynomial.

7 Approximation of Some Iterated Stochastic Integrals of Third and Fourth Miltiplicity with Respect to the Q-Wiener Process

In this section we consider an approximation of iterated stochastic integrals of the following form (see Sect. 1)

$$I_3[B(Z)]_{T,t}^{M}=\int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M,$$

$$I_4[B(Z)]^M_{T,t} =$$

$$= \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\mathbf{W}^M_{t_2}\right)d\mathbf{W}^M_{t_3},$$

$$I_5[B(Z)]^M_{T,t} =$$

$$= \int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}^M_{t_1}, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\mathbf{W}^M_{t_2}\right)d\mathbf{W}^M_{t_3},$$

$$I_6[B(Z),F(Z)]^M_{T,t} = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\mathbf{W}^M_{t_2}\right)dt_3,$$

$$I_7[B(Z),F(Z)]^M_{T,t} = \int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)dt_2,$$

$$I_8[B(Z),F(Z)]^M_{T,t} = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\mathbf{W}^M_{t_2}.$$

Consider the stochastic integral $I_3[B(Z)]^M_{T,t}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B'''(v)(B(v),B(v),B(v))$ be a 4-linear Hilbert–Schmidt operator for all $v \in H$.

We have (see (44))

$$I_3[B(Z)]^M_{T,t} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times \int\limits_t^T\left(\int\limits_t^s d\mathbf{w}^{(r_1)}_\tau \int\limits_t^s d\mathbf{w}^{(r_2)}_\tau \int\limits_t^s d\mathbf{w}^{(r_3)}_\tau\right)d\mathbf{w}^{(r_4)}_s \quad \text{w. p. 1.} \tag{71}$$

From [35] (pp. A.438 – A.439) or using the Itô formula we obtain

$$I^{(r_1)}_{(1)s,t}I^{(r_2)}_{(1)s,t}I^{(r_3)}_{(1)s,t} =$$

$$= I^{(r_1r_2r_3)}_{(111)s,t} + I^{(r_1r_3r_2)}_{(111)s,t} + I^{(r_2r_1r_3)}_{(111)s,t} + I^{(r_2r_3r_1)}_{(111)s,t} + I^{(r_3r_1r_2)}_{(111)s,t} + I^{(r_3r_2r_1)}_{(111)s,t} +$$

$$+\mathbf{1}_{\{r_1=r_2\}}\left(I^{(r_30)}_{(10)s,t} + I^{(0r_3)}_{(01)s,t}\right) + \mathbf{1}_{\{r_1=r_3\}}\left(I^{(r_20)}_{(10)s,t} + I^{(0r_2)}_{(01)s,t}\right) +$$

$$+\mathbf{1}_{\{r_2=r_3\}}\left(I_{(10)s,t}^{(r_1 0)}+I_{(01)s,t}^{(0 r_1)}\right)=$$

$$=\sum_{(r_1,r_2,r_3)} I_{(111)s,t}^{(r_1 r_2 r_3)}+(s-t)\left(\mathbf{1}_{\{r_2=r_3\}}I_{(1)s,t}^{(r_1)}+\mathbf{1}_{\{r_1=r_3\}}I_{(1)s,t}^{(r_2)}+\mathbf{1}_{\{r_1=r_2\}}I_{(1)s,t}^{(r_3)}\right)\quad \text{w. p. 1,} \tag{72}$$

where

$$\sum_{(r_1,r_2,r_3)}$$

means the sum according to all possible permutations (r_1, r_2, r_3) and we use the notations from Sect. 4.

After substituting (72) into (71) we obtain

$$I_3[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2},B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)} I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)}-\mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(r_3 r_4)}-\mathbf{1}_{\{r_1=r_3\}}J_{(01)T,t}^{(r_2 r_4)}-\mathbf{1}_{\{r_2=r_3\}}J_{(01)T,t}^{(r_1 r_4)}\right)\quad \text{w. p. 1,} \tag{73}$$

where

$$J_{(01)T,t}^{(r_1 r_2)}=\int_t^T (t-s)\int_t^s d\mathbf{w}_\tau^{(r_1)}d\mathbf{w}_s^{(r_2)}. \tag{74}$$

Denote by $I_3[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (73), which has the following form

$$I_3[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2},B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)} I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)q}-\mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(r_3 r_4)q}-\mathbf{1}_{\{r_1=r_3\}}J_{(01)T,t}^{(r_2 r_4)q}-\mathbf{1}_{\{r_2=r_3\}}J_{(01)T,t}^{(r_1 r_4)q}\right), \tag{75}$$

where the approximations $I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)q}$, $J_{(01)T,t}^{(r_1 r_2)q}$ are based on Theorem 1 and Legendre polynomials. The approximation $J_{(01)T,t}^{(r_1 r_2)q}$ of the stochastic integral $J_{(01)T,t}^{(r_1 r_2)}$ $(r_1, r_2 = 1, \ldots, M)$, which is based on Theorem 1 and Legendre polynomials, has the following form (see [35], formula (6.91) on the page A.544, and [32], formula (5.7) on the page A.249)

$$J_{(01)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}I_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_1)}\zeta_1^{(r_2)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(r_1)}\zeta_{i+2}^{(r_2)}-(i+1)\zeta_{i+2}^{(r_1)}\zeta_i^{(r_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \qquad (76)$$

$$I_{(11)T,t}^{(r_1r_2)q}=\frac{T-t}{2}\left(\zeta_0^{(r_1)}\zeta_0^{(r_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(r_1)}\zeta_i^{(r_2)}-\zeta_i^{(r_1)}\zeta_{i-1}^{(r_2)}\right)-\mathbf{1}_{\{r_1=r_2\}}\right), \quad (77)$$

where notations can be found in Theorem 1.

Moreover (see [35], formula (6.106) on the page A.551)

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\ (r_1\ne r_2). \qquad (78)$$

From (27), (29) we obtain

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}\le\frac{(T-t)^4}{8}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right),$$

where $r_1,r_2=1,\ldots,M$.

From (73), (75) it follows that

$$I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2,r_3,r_4\in J_M}B'''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2},B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)}\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)}-I_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right)-\mathbf{1}_{\{r_1=r_2\}}\left(J_{(01)T,t}^{(r_3r_4)}-J_{(01)T,t}^{(r_3r_4)q}\right)-\right.$$

$$\left.-\mathbf{1}_{\{r_1=r_3\}}\left(J_{(01)T,t}^{(r_2r_4)}-J_{(01)T,t}^{(r_2r_4)q}\right)-\mathbf{1}_{\{r_2=r_3\}}\left(J_{(01)T,t}^{(r_1r_4)}-J_{(01)T,t}^{(r_1r_4)q}\right)\right)\ \text{w. p. 1.} \qquad (79)$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k = 2, 4$ we obtain

$$\mathsf{M}\left\{\left\| I_3[B(Z)]^M_{T,t} - I_3[B(Z)]^{M,q}_{T,t}\right\|^2_H\right\} \le$$

$$\le C\,(\mathrm{tr}\ Q)^4\left(6^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C^2_{j_4j_3j_2j_1}\right) + 3^2(2!)^2 E_q\right),$$

where E_q is the right-hand side of (78), and

$$C_{j_4j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}(T-t)^2}{16}\bar{C}_{j_4j_3j_2j_1}, \qquad (80)$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu,$$

where $P_j(x)$ is a Legendre polynomial.

Consider the stochastic integral $I_4[B(Z)]^M_{T,t}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B'(v)(B''(v)(B(v), B(v)))$ be a 4-linear Hilbert–Schmidt operator for all $v \in H$.

We have (see (44))

$$I_4[B(Z)]^M_{T,t} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\int\limits_t^T\int\limits_t^s\left(\int\limits_t^\tau d\mathbf{w}_u^{(r_1)}\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}\right)d\mathbf{w}_\tau^{(r_3)}d\mathbf{w}_s^{(r_4)} \ \text{w. p. 1.} \qquad (81)$$

From (64) and (81) we obtain

$$I_4[B(Z)]^M_{T,t} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(I^{(r_1r_2r_3r_4)}_{(1111)T,t} + I^{(r_2r_1r_3r_4)}_{(1111)T,t} - \mathbf{1}_{\{r_1=r_2\}}J^{(r_3r_4)}_{(10)T,t}\right) \ \text{w. p. 1,} \qquad (82)$$

where

$$J^{(r_3r_4)}_{(10)T,t} = \int\limits_t^T\int\limits_t^s (t-\tau)d\mathbf{w}_\tau^{(r_3)}d\mathbf{w}_s^{(r_4)}. \qquad (83)$$

Denote by $I_4[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (82), which has the following form

$$I_4[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)q} + I_{(1111)T,t}^{(r_2r_1r_3r_4)q} - \mathbf{1}_{\{r_1=r_2\}}J_{(10)T,t}^{(r_3r_4)q}\right) \text{ w. p. 1}, \quad (84)$$

where the approximations $I_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $J_{(10)T,t}^{(r_1r_2)q}$ are based on Theorem 1 and Legendre polynomials.

The approximation $J_{(10)T,t}^{(r_1r_2)q}$ of the stochastic integral $J_{(10)T,t}^{(r_1r_2)}$ $(r_1, r_2 = 1, \ldots, M)$, which is based on Theorem 1 and Legendre polynomials, has the following form (see [35], formula (6.92) on the page A.544)

$$J_{(10)T,t}^{(r_1r_2)q} = -\frac{T-t}{2}I_{(11)T,t}^{(r_1r_2)q} - \frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_2)}\zeta_1^{(r_1)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(r_2)}\zeta_i^{(r_1)} - (i+2)\zeta_i^{(r_2)}\zeta_{i+2}^{(r_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \quad (85)$$

where the approximation $I_{(11)T,t}^{(r_1r_2)q}$ is defined by (77).

Moreover

$$\mathsf{M}\left\{\left(J_{(10)T,t}^{(r_1r_2)} - J_{(10)T,t}^{(r_1r_2)q}\right)^2\right\} = E_q \quad (r_1 \ne r_2), \quad (86)$$

where E_q is the right-hand side of (78) (see [35], formula (6.106) on the page A.551, [32], formula (5.19) on the pages A.252 – A.253).

From (82), (84) it follows that

$$I_4[B(Z)]_{T,t}^{M} - I_4[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)} - I_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right) + \left(I_{(1111)T,t}^{(r_2r_1r_3r_4)} - I_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right) -\right.$$

$$\left.-\mathbf{1}_{\{r_1=r_2\}}\left(J_{(10)T,t}^{(r_3r_4)} - J_{(10)T,t}^{(r_3r_4)q}\right)\right) \text{ w. p. 1}.$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k=2,4$ we obtain

$$\mathsf{M}\left\{\left\|I_4[B(Z)]_{T,t}^{M}-I_4[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le C\,(\mathrm{tr}\;Q)^4\left(2^2(4!)^2\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right)+(2!)^2E_q\right),$$

where E_q is the right-hand side of (78), and $C_{j_4j_3j_2j_1}$ is defined by (80).

Consider the stochastic integral $I_5[B(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v),B'(v)(B(v)))$ be a 4-linear Hilbert–Schmidt operator for all $v\in H$.

We have (see (44))

$$I_5[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M}B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)}\ \text{w. p. 1.}\tag{87}$$

Using the theorem on the integration order replacement in iterated stochastic Itô integrals (see [35], pp. A.146 – A.162 and example 3.1, p. A.163) or the Itô formula we obtain

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)}=$$

$$=I_{(1111)T,t}^{(r_2r_1r_3r_4)}+I_{(1111)T,t}^{(r_2r_3r_1r_4)}+I_{(1111)T,t}^{(r_3r_2r_1r_4)}+$$

$$+\mathbf{1}_{\{r_1=r_3\}}\left(J_{(10)T,t}^{(r_2r_4)}-J_{(01)T,t}^{(r_2r_4)}\right)-\mathbf{1}_{\{r_2=r_3\}}J_{(10)T,t}^{(r_1r_4)}\ \text{ w. p. 1,}\tag{88}$$

where we use the notations from Sect. 4, and $J_{(01)T,t}^{(r_1r_2)}$, $J_{(10)T,t}^{(r_1r_2)}$ are defined by (74), (83).

After substituting (88) into (87) we obtain

$$I_5[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M}B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\times$$

$$
\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\Biggl(I_{(1111)T,t}^{(r_2r_1r_3r_4)}+I_{(1111)T,t}^{(r_2r_3r_1r_4)}+I_{(1111)T,t}^{(r_3r_2r_1r_4)}+
$$

$$
+\mathbf{1}_{\{r_1=r_3\}}\left(J_{(10)T,t}^{(r_2r_4)}-J_{(01)T,t}^{(r_2r_4)}\right)-\mathbf{1}_{\{r_2=r_3\}}J_{(10)T,t}^{(r_1r_4)}\Biggr)\ \text{w. p. 1.} \tag{89}
$$

Denote by $I_5[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (89), which has the following form

$$
I_5[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M}B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\times
$$

$$
\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\Biggl(I_{(1111)T,t}^{(r_2r_1r_3r_4)q}+I_{(1111)T,t}^{(r_2r_3r_1r_4)q}+I_{(1111)T,t}^{(r_3r_2r_1r_4)q}+
$$

$$
+\mathbf{1}_{\{r_1=r_3\}}\left(J_{(10)T,t}^{(r_2r_4)q}-J_{(01)T,t}^{(r_2r_4)q}\right)-\mathbf{1}_{\{r_2=r_3\}}J_{(10)T,t}^{(r_1r_4)q}\Biggr)\ \text{w. p. 1.} \tag{90}
$$

where the approximations $I_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $J_{(01)T,t}^{(r_1r_2)q}$, and $J_{(10)T,t}^{(r_1r_2)q}$ are based on Theorem 1 and Legendre polynomials.

From (89), (90) it follows that

$$
I_5[B(Z)]_{T,t}^{M}-I_5[B(Z)]_{T,t}^{M,q}=
$$

$$
=\sum_{r_1,r_2,r_3,r_4\in J_M}B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times
$$

$$
\times\Biggl(\left(I_{(1111)T,t}^{(r_2r_1r_3r_4)}-I_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right)+\left(I_{(1111)T,t}^{(r_2r_3r_1r_4)}-I_{(1111)T,t}^{(r_2r_3r_1r_4)q}\right)+\left(I_{(1111)T,t}^{(r_3r_2r_1r_4)}-I_{(1111)T,t}^{(r_3r_2r_1r_4)q}\right)+
$$

$$
+\mathbf{1}_{\{r_1=r_3\}}\left(\left(J_{(10)T,t}^{(r_2r_4)}-J_{(10)T,t}^{(r_2r_4)q}\right)-\left(J_{(01)T,t}^{(r_2r_4)}-J_{(01)T,t}^{(r_2r_4)q}\right)\right)-
$$

$$
-\mathbf{1}_{\{r_2=r_3\}}\left(J_{(10)T,t}^{(r_1r_4)}-J_{(10)T,t}^{(r_1r_4)q}\right)\Biggr)\ \text{w. p. 1.}
$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k=2,4$ and taking into account (86) we obtain

$$
\mathsf{M}\left\{\left\|I_5[B(Z)]_{T,t}^{M}-I_5[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le
$$

$$
\leq C\,(\mathrm{tr}\ Q)^4\left(3^2(4!)^2\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right)+3^2(2!)^2E_q\right),
$$

where E_q is a right-hand side of (78), and $C_{j_4j_3j_2j_1}$ is defined by (80).

Consider the stochastic integral $I_6[B(Z),F(Z)]_{T,t}^M$. Let conditions 1 and 2 of Theorem 4 be fulfilled.

We have (see (44))

$$
I_6[B(Z),F(Z)]_{T,t}^M=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\int\limits_t^T\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_1)}d\mathbf{w}_\tau^{(r_2)}ds \ \text{w. p. 1.} \tag{91}
$$

Using the theorem on the integration order replacement in iterated stochastic Itô integrals (see [35], pp. A.146 – A.162 and example 3.1, p. A.163) or the Itô formula we obtain

$$
\int\limits_t^T\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_1)}d\mathbf{w}_\tau^{(r_2)}ds=(T-t)I_{(11)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_1r_2)} \ \text{w. p. 1.} \tag{92}
$$

After substituting (92) into (91) we have

$$
I_6[B(Z),F(Z)]_{T,t}^M=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left((T-t)I_{(11)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_1r_2)}\right) \ \text{w. p. 1.} \tag{93}
$$

Denote by $I_6[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (93), which has the following form

$$
I_6[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left((T-t)I_{(11)T,t}^{(r_1r_2)q}+J_{(01)T,t}^{(r_1r_2)q}\right), \tag{94}
$$

where the approximations $J_{(01)T,t}^{(r_1r_2)q}$, $I_{(11)T,t}^{(r_1r_2)q}$ are defined by (76), (77).

From (93), (94) it follows that

$$
I_6[B(Z),F(Z)]_{T,t}^M-I_6[B(Z),F(Z)]_{T,t}^{M,q}=
$$

$$
= \sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left((T-t)\left(I_{(11)T,t}^{(r_1r_2)} - I_{(11)T,t}^{(r_1r_2)q}\right) + \left(J_{(01)T,t}^{(r_1r_2)} - J_{(01)T,t}^{(r_1r_2)q}\right)\right) \text{ w. p. 1.}
$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k = 2$ we obtain

$$
\mathsf{M}\left\{\left\|I_6[B(Z),F(Z)]_{T,t}^{M} - I_6[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le
$$

$$
\le 2C(2!)^2\,(\text{tr }Q)^2\left((T-t)^2 G_q + E_q\right),
$$

where G_q and E_q are the right-hand sides of (36) and (78) correspondingly.

Consider the stochastic integral $I_7[B(Z),F(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled.

Then w. p. 1 we have (see (44))

$$
I_7[B(Z),F(Z)]_{T,t}^{M} = \sum_{r_1,r_2\in J_M} F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds. \tag{95}
$$

From (63) and (92) we have

$$
\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds = \int\limits_t^T I_{(11)s,t}^{(r_1r_2)}ds + \int\limits_t^T I_{(11)s,t}^{(r_2r_1)}ds + \mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2} =
$$

$$
= (T-t)\left(I_{(11)T,t}^{(r_1r_2)} + I_{(11)T,t}^{(r_2r_1)}\right) + J_{(01)T,t}^{(r_1r_2)} + J_{(01)T,t}^{(r_2r_1)} + \mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2} =
$$

$$
= (T-t)\Bigl(I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)} - \mathbf{1}_{\{r_1=r_2\}}(T-t)\Bigr) + J_{(01)T,t}^{(r_1r_2)} + J_{(01)T,t}^{(r_2r_1)} + \mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2} =
$$

$$
= (T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)} + J_{(01)T,t}^{(r_1r_2)} + J_{(01)T,t}^{(r_2r_1)} - \mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2} \text{ w. p. 1.} \tag{96}
$$

After substituting (96) into (95) we obtain

$$
I_7[B(Z),F(Z)]_{T,t}^{M} = \sum_{r_1,r_2\in J_M} F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$\times\left((T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}+J_{(01)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right)\ \text{w. p. 1.} \quad (97)$$

Denote by $I_7[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (97), which has the following form

$$I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}+J_{(01)T,t}^{(r_1r_2)q}+J_{(01)T,t}^{(r_2r_1)q}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right), \quad (98)$$

where the approximation $J_{(01)T,t}^{(r_1r_2)q}$ is defined by (76).

From (97), (98) it follows that

$$I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}}\left(\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)+\left(J_{(01)T,t}^{(r_2r_1)}-J_{(01)T,t}^{(r_2r_1)q}\right)\right)\ \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=2$ we obtain

$$\mathsf{M}\left\{\left\|I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 4C(2!)^2\left(\text{tr }Q\right)^2E_q,$$

where E_q is the right-hand side of (78).

Consider the stochastic integral $I_8[B(Z),F(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled.

Then w. p. 1 we have (see (44))

$$I_8[B(Z),F(Z)]_{T,t}^{M}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(01)T,t}^{(r_1r_2)}. \quad (99)$$

Denote by $I_8[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (99), which has the following form

$$I_8[B(Z),F(Z)]_{T,t}^{M,q}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(01)T,t}^{(r_1r_2)q}, \quad (100)$$

where the approximation $J_{(01)T,t}^{(r_1 r_2)q}$ is defined by (76).

From (99), (100) it follows that

$$I_8[B(Z), F(Z)]_{T,t}^{M} - I_8[B(Z), F(Z)]_{T,t}^{M,q} =$$

$$= - \sum_{r_1, r_2 \in J_M} B''(Z) \left(F(Z), B(Z)e_{r_1}\right) e_{r_2} \sqrt{\lambda_{r_1}\lambda_{r_2}} \left(J_{(01)T,t}^{(r_1 r_2)} - J_{(01)T,t}^{(r_1 r_2)q}\right) \text{ w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k = 2$ we obtain

$$\mathsf{M}\left\{\left\|I_8[B(Z), F(Z)]_{T,t}^{M} - I_8[B(Z), F(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le C(2!)^2 \left(\text{tr } Q\right)^2 E_q,$$

where E_q is the right-hand side of (78).

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Bibliography

[1] *Gyöngy I.* Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. *Potential Anal.* 9, 1 (1998), 1-25.

[2] *Gyöngy I.* Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal.* 11, 1 (1999), 1-37.

[3] *Gyöngy I.* and Krylov N. An accelerated splitting-up method for parabolic equations. *SIAM J. Math. Anal.* 37, 4 (2005), 1070-1097.

[4] *Hausenblas E.* Numerical analysis of semilinear stochastic evolution equations in Banach spaces. *J. Comp. Appl. Math.* 147, 2 (2002), 485-516.

[5] *Hausenblas E.* Approximation for semilinear stochastic evolution equations. *Potential Anal.* 18, 2 (2003), 141-186.

[6] *Hutzenthaler M. and Jentzen A.* Non-globally Lipschitz counterexamples for the stochastic Euler scheme. *arXiv:0905.0273* [math.NA] (2009), 22 pp.

[7] *Jentzen A.* Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. *Potential Anal.* 31, 4 (2009), 375-404.

[8] *Jentzen A.* Taylor expansions of solutions of stochastic partial differential equations. *arXiv:0904.2232* [math.NA] (2009), 32 pp.

[9] *Jentzen A. and Kloeden P.E.* Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. Lond. Ser. Math. Phys. Eng. Sci.* 465, 2102 (2009), 649-667.

[10] *Jentzen A. and Kloeden P.E.* Taylor expansions of solutions of stochastic partial differential equations with additive noise. *Ann. Prob.* 38, 2 (2010), 532-569.

[11] *Jentzen A. and Kloeden P.E.* Taylor approximations for stochastic partial differential equations. *SIAM*, Philadelphia, 2011, 224 pp.

[12] *Jentzen A. and Röckner M.* A Milstein scheme for SPDEs. *Foundations Comp. Math.* 15, 2 (2015), 313-362.

[13] *Becker S., Jentzen A. and Kloeden P.E.* An exponential Wagner-Platen type scheme for SPDEs. *SIAM J. Numer. Anal.* 54, 4 (2016), 2389-2426.

[14] *Zhang Z. and Karniadakis G.* Numerical methods for stochastic partial differential equations with white noise. *Springer*, 2017, 398 pp.

[15] *Jentzen A. and Röckner M.* Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. *J. Differ. Eq.* 252, 1 (2012), 114-136.

[16] *Kloeden P.E. and Platen E.* Numerical solution of stochastic differential equations. *Springer-Verlag*, Berlin, 1992, 632 pp.

[17] *Lord G.J. and Tambue A.* A modified semi-implict Euler-Maruyama scheme for finite element discretization of SPDEs. *arXiv:1004.1998* [math.NA] (2010), 23 pp.

[18] *Mishura Y.S. and Shevchenko G.M.* Approximation schemes for stochastic differential equations in a Hilbert space. *Theor. Prob. Appl.* 51, 3 (2007), 442-458.

[19] *Prévôt C. and Röckner M.* A concise course on stochastic partial differential equations, V. 1905 of Lecture Notes in Mathematics. *Springer*, Berlin, 2007, 148 pp.

[20] *Shardlow T.* Numerical methods for stochastic parabolic PDEs. *Numer. Funct. Anal. Optim.* 20, 1-2 (1999), 121-145.

[21] *Da Prato G., Jentzen A. and Röckner M.* A mild Itô formula for SPDEs. *arXiv:1009.3526* [math.PR] (2012), 39 pp.

[22] *Da Prato G. and Zabczyk J.* Stochastic equations in infinite dimensions. 2nd Ed. *Cambridge Univ. Press*, Cambridge, 2014, 493 pp.

[23] *Kruse R.* Optimal Error Estimates of Galerkin Finite Element Methods for Stochastic Partial Differential Equations with Multiplicative Noise. *IMA J. Numer. Anal.* 34, 1 (2014), 217-251.

[24] *Kruse R.* Consistency and stability of a Milstein-Galerkin finite element scheme for semilinear SPDE. *Stoch. PDE: Anal. Comp.* 2, 4 (2014), 471-516.

[25] *Leonhard C. and Rößler A.* Iterated stochastic integrals in infinite dimensions: approximation and error estimates. *Stoch. PDE: Anal. Comp.* 7, 2 (2019), 209-239.

[26] *Milstein G.N.* Numerical integration of stochastic differential equations. *Ural Univ. Press*, Sverdlovsk, 1988, 225 pp.

[27] *Kuznetsov D. F.* A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. *Differ. Eq. Control Processes*, 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[28] *Kuznetsov D.F.* Numerical integration of stochastic differential equations. 2. [In Russian]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227

[29] *Kuznetsov D.F.* Stochastic differential equations: theory and practice of numerical solution. With MatLab programs. 4th Ed. [In Russian]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2010, 816 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-231

[30] *Dmitriy F. Kuznetsov.* Strong approximation of multiple Itô and Stratonovich stochastic integrals: multiple Fourier series approach. 2nd Ed. [In English]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2011, 284 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-233

[31] *Dmitriy F. Kuznetsov.* Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234

[32] *Kuznetsov D.F.* Multiple Itô and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. *Differ. Eq. Control Processes*, 1 (2017), A.1-A.385. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[33] *Kuznetsov D.F.* Development and application of the Fourier method for the numerical solution of Itô stochastic differential equations. *Comp. Math. Math. Phys.* 58, 7 (2018), 1058-1070.
DOI: http://doi.org/10.1134/S0965542518070096

[34] *Kuznetsov D.F.* On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. *Autom. Remote Control.* 79, 7 (2018), 1240-1254.
DOI: http://doi.org/10.1134/S0005117918070056

[35] *Kuznetsov D.F.* Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Ed. [In Russian]. *Differ. Eq. Control Processes*, 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[36] *Kuznetsov D.F.* Expansion of multiple Itô stochastic integrals of arbitrary multiplicity, based on generalized multiple Fourier series, converging in the mean. [In English]. *arXiv:1712.09746* [math.PR] (2017), 22 pp.

[37] *Kuznetsov D.F.* Mean-square approximation of multiple Itô and Stratonovich stochastic integrals from the Taylor-Itô and Taylor-Stra-

tonovich expansions, using Legendre polynomials. [In English]. *arXiv: 1801.00231* [math.PR] (2017), 36 pp.

[38] *Kuznetsov D.F.* On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. *Autom. Remote Control.* 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[39] *Kuznetsov D.F.* Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. *Comp. Math. Math. Phys.* 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116