DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 4, 2020
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru

Filtration and identification

Numerical Investigation of the Optimal Measurement for a Semilinear Descriptor System with the Showalter–Sidorov Condition: Algorithm and Computational Experiment

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Abstract. The article deals with the problem of optimal measurement for a semilinear descriptor system with a distinguished linear part and a nonlinear term unsolved with respect to the derivative of the unknown vector function with the Showalter–Sidorov initial condition. Basing on the methods of the theory of optimal control we found sufficient conditions for the existence of solutions of the optimal measurement problem – the problem of recovering a dynamically distorted signal from a measuring device. An algorithm for finding a numerical solution uses the methods of decomposition, penalty and the Ritz method as well. The algorithm is based on the representation of the measurement components by polynomials of a given degree, which allows reducing the optimal control problem to a computer programming problem with respect to the unknown coefficients of the polynomials. As an example of a sensor we consider Fitz Hugh – Nagumo oscilloscope described by a nonlinear descriptor system. Computational experiments for the considered sensor model are presented.

Keywords: descriptor systems, optimal control problem, mathematical model of the optimal measurement.

1 Introduction

In the theory of dynamic measurements, an urgent problem is the problem of reconstructing measurements from observation. The traditional approach [1] to solving this problem is a method based on the theory of inverse problems [2]. Another approach [3] is a research method based on the automatic control theory [4]. In the meantime, the [5] approach has recently appeared, based on the theory of optimal control of solutions of linear Sobolev-type equations [6, 7, 8]. Article [5] initiated a several researches on the study of optimal measurement problems for linear deterministic and stochastic descriptor systems, also called Leontief-type systems [10, 11]. The problem

$$J(y) \to \min \tag{1}$$

will be called the optimal measurement problem if it is based on the search for the optimum of the target functional from the norm of the difference between real (i.e., fixed on a measuring device) and virtual (i.e., found by means of a computational algorithm) observations. Based on the ideas and approaches of the theory of optimal measurement for linear descriptor systems and extending the methods of the theory of optimal control for semilinear Sobolev-type equations, it will be possible to investigate optimal measurement problems for semilinear descriptor systems. This approach allows one to take into account nonlinear connections in sensors and expand the range of studied problems.

The main goal of our research is the numerical solution of the optimal measurement problem (1) for semilinear descriptor systems

$$L\dot{x}(t) + Mx(t) + N(x(t)) = u(t), \quad \det L = 0, \tag{2}$$

and

$$y(t) = Dx(t) \tag{3}$$

with the Showalter–Sidorov initial condition

$$L(x(0) - x_0) = 0. \tag{4}$$

A numerical study of linear problems of optimal dynamic measurement with the development of program algorithms was carried out in the works [12, 13, 14]. In these works, algorithms of numerical methods for solving optimal control problems for linear Sobolev-type equations were adapted to the finite-dimensional case. By modifying the algorithms of numerical methods developed for semilinear equations of Sobolev type [15] for the case of semilinear descriptor systems, this article will build a numerical method algorithm for the problem (1) – (3). The method obtained will be based on the methods of decomposition, penalty and Ritz [15].

The research of the optimal measurement problem for semilinear descriptor systems can be divided into three stages, each of which is one of the sections of this article. At the first stage, a mathematical model of the investigated sensor is created, the problem of finding the optimal measurement is set, and the conditions are given under which there is a unique solution to this problem. At the second stage, algorithms are constructed for approximate solutions to the problem of finding the optimal measurement. And finally, at the third stage of the optimal measurement problem research, on the basis of the algorithms of the second stage, programs are created and computational experiments are performed. In our case, we research the problem of optimal dynamic measurement for the mathematical model of the Fitz Hugh – Nagumo oscilloscope [16, 17].

2 Optimal Measurement Dynamic Problem

Consider a semilinear descriptor system of the form (2) and (3) with the Showalter–Sidorov initial condition (4). Here L, M are matrices of order n ($\langle Lx, x\rangle \geq 0$, $C_M\|x\|^2 \leq \langle Mx, x\rangle \leq C^M\|x\|^2$, $\|\cdot\|$ is the norm in $\mathbb{R}^n$) representing the mutual influence of the state and state velocities of the measuring sensor, respectively, and $\dim\ker L = \dim\operatorname{coker} L$; D is a square matrix of order n, N is a nonlinear operator defined by the formula $\langle N(x(t)), x(t)\rangle = a_1x_1^4 + a_2x_2^4 + ... + a_nx_n^4, \quad a_i \geq 0$, where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^n, x(t) = (x_1(t), x_2(t), ..., x_n(t))$ is vector functions of the state; $u(t) = (u_1(t), u_2(t), ..., u_n(t))$ and $y(t) = (y_1(t), y_2(t), ..., y_n(t))$ are measurement vector functions and observations of the measuring sensor, respectively.

Consider

$$\text{coim}L = \{x \in \mathcal{X} : [x, \varphi] = 0 \ \ \forall\varphi \in \ker L\backslash\{0\}\},$$

where $[\cdot,\cdot]$ is a scalar product in $L_2(\mathbb{R}^n)$. Let $\mathcal{X} = \{x \in L_4((0,T), \mathbb{R}^n) : \dot{x} \in \text{coim } L\}$ is the state space, $\mathcal{N} = D[\mathcal{X}]$ is the observation space for some fixed $T \in \mathbb{R}_+$, $\mathcal{U} = \{u \in L_{\frac{4}{3}}((0,T), \mathbb{R}^n\}$ is the measurement space. Let us single out in $\mathcal{U}$ is a closed, convex subset $\mathcal{U}_{ad}$ is the set of admissible measurements.

The main goal of this work is to minimize the values of the target functional

$$J(y)=\int\limits_0^T \|y(t)-y_0(t)\|^4_{L_4(\mathbb{R}^n)}dt, \tag{5}$$

where $y_0(t)=(y_{01}(t),y_{02}(t),...,y_{0n}(t))$ is an observation obtained during a full-scale experiment. Note that, by virtue of (2), (3), the vector function y depends on the functions x,u; therefore, we can assume that $J(y)=J(x,u)$.

Consider the set

$$\mathcal{M}=\{x\in\mathcal{X}:(I-Q)(Mx+N(x))=(I-Q)u\}. \tag{6}$$

By virtue of the properties of the matrix L, there exists a projector Q along coker L onto im L. Note that if $x=x(t)$ is a solution to equation (2), then it necessarily lies in the set $\mathcal{M}$. The set $\mathcal{M}$ will be called the phase manifold of equation (2).

Definition 1 *A weak solution to equation (2) is a vector function $x\in\mathcal{X}$ satisfying the condition*

$$\int\limits_0^T \varphi(t)\left[\frac{d}{dt}L\dot{x}+Mx+N(x),w\right]dt=\int\limits_0^T \varphi(t)[u,w]dt, \tag{7}$$

for any $w\in\mathcal{X}$ and any $\varphi\in L_2(0,T)$. A solution to equation (2) is called a solution to the Showalter–Sidorov problem if it satisfies (4).

It is required to find the optimal measurement $\tilde{u}\in\mathcal{U}_{ad}$ that satisfies system (2), (3), the Showalter–Sidorov initial condition (4), and

$$J(\tilde{u})=\min\ J(u),\quad u\in\mathcal{U}_{ad}. \tag{8}$$

Definition 2 *A pair $(\tilde{x},\tilde{u})\in\mathcal{X}\times\mathcal{U}_{ad}$ is called a solution to the optimal measurement problem (1) – (4), if*

$$J(\tilde{x},\tilde{u})=\min_{(x,u)}\ J(x,u),$$

where the pairs $(x,u)\in\mathcal{X}\times\mathcal{U}_{ad}$ satisfy (2) – (4) in the sense of Definition 1.

Theorem 1 *If $\mathcal{M}$ is a simple Banach C^∞-manifold, then for any $x_0\in\mathcal{X}$ and $y_0\in\mathcal{N}$ there is a unique solution $\tilde{x}\in\mathcal{U}_{ad}$ for which (8) holds.*

Proof. The main ideas of the proof are based on the reduction of the problem we are investigating to the general problem of optimal control for semilinear Sobolev-type equations. The conceptual proof of this Theorem is similar to the proof of Theorem 2 in [15] into account the transition to the finite-dimensional case.

□

The vector-function $\tilde{u} = \tilde{u}(t)$ existing by Theorem 1 will, in what follows, be called the exact optimal measurement. Note that the vector functions $\tilde{u} = \tilde{u}(t)$ and $y = y(t)$ obtained as a result of applying Theorem 1 are virtual exact optimal measurement and virtual exact optimal observation.

3 Algorithm for Numerical Method

Based on the theoretical results, an algorithm was developed for the approximate solution of the optimal control problem (1) – (4), where the objective functional has the form (5), based on modified decomposition and Ritz methods. Applying the decomposition method described in [15], we linearize system (2), by introducing the function $v = v(t)$, and obtain the equivalent Showalter–Sidorov problem (2) – (4) for the system of equations:

$$\begin{aligned} &L\dot{x}(t) + Mx(t) + N(v(t)) = u(t), \quad \det L = 0,\\ &x(t) = v(t),\\ &y(t) = Dx(t). \end{aligned} \tag{9}$$

Then the solution (x, u) of problem (1) – (4) reduces to finding the triple (x, v, u). Using the Ritz method, we represent the unknown functions $v(t)$, $u(t)$, by the expansion $v(t, H) = \sum\limits_{h=0}^{H} a_h t^h, u(t, H) = \sum\limits_{h=0}^{H} b_h t^h$. Next, we seek an approximate solution of the control problem (1), (4), (9) by the penalty method described in [15]. We consider the equivalent control problem, where the relation $x \to v$ is obtained for an approximate solution by introducing a new functional in the form

$$J_\theta^\varepsilon(x,u)=\theta\int\limits_0^T \|y(t)-y_0(t)\|^4_{L_4(\mathbb{R}^n)}\,dt+(1-\theta)\int\limits_0^T \|Dv(t)-y_0(t)\|^4_{L_4(\mathbb{R}^n)}\,dt+$$
$$+r_\varepsilon\int\limits_0^T \|x(t)-v(t)\|^2_{L_2(\mathbb{R}^n)}\,dt, \theta\in(0,1), \tag{10}$$

where the penalty parameter $r_\varepsilon=\frac{1}{\varepsilon}\to+\infty$ for $\varepsilon\to+0$.

The basic ideas of the algorithm were developed for the infinite-dimensional case, based on the methods of penalty, Ritz and decomposition. In this article, it is proposed to modify the already constructed algorithms for the finite-dimensional case, for this we represent the space $\mathcal{U}$ as $\mathcal{U}=\bigoplus\limits_{h=1}^{H}\mathcal{U}_h$, where $\mathcal{U}_h=\{u^h\in L_2((0,T);\mathbb{R})\}$. Denote by $\{\varphi_i\}$ an orthonormal sequence of basis vectors. It is clear that this sequence can be chosen the same in every $\mathcal{U}_h$. We construct a finite-dimensional lineal $\mathcal{U}_h^k=\text{span}\{\varphi_i : i=1,2,\ldots,k\}$ and the subspace $\mathcal{U}^k=\bigoplus\limits_{h=1}^{H}\mathcal{U}_h^k$. Being that $L_2(\mathbb{R})\subset L_{\frac{4}{3}}(\mathbb{R})$ densely condtruct the subset $\mathcal{U}_{ad}^k=\mathcal{U}^k\cap\mathcal{U}_{ad}$. The subset $\mathcal{U}_{ad}^k\subset\mathcal{U}_{ad}$ may turn out to be empty, but in any case it is closed and convex. It is clear that all members of the sequence $\{\mathcal{U}_{ad}^k\}$ cannot be empty sets because of the obvious monotonicity of this sequence and the fact that letting k go to infinity we obtain in the limiting case the set $\mathcal{U}_{ad}$. The main stages of the algorithm after the transition to the finite-dimensional case are presented in Fig. 1.

The program is designed to find a solution to the optimal measurement problem, written in the Maple language using the Maple 2017 compiler. The choice of the programming environment was due to the presence of a built-in analytical computing apparatus and built-in tools for solving systems of ordinary differential equations. The program is based on methods for solving optimal control problems for semilinear Sobolev type equations by the Showalter–Sidorov initial condition.

Data entry is carried out from the keyboard and is saved in a text file that sets the calculation parameters in the rest of the program. Also, functions and calculation coefficients are saved in separate files, from which the necessary data arrays are loaded. The following data is used as input:

- the type of quality functional that will be accepted for calculation;

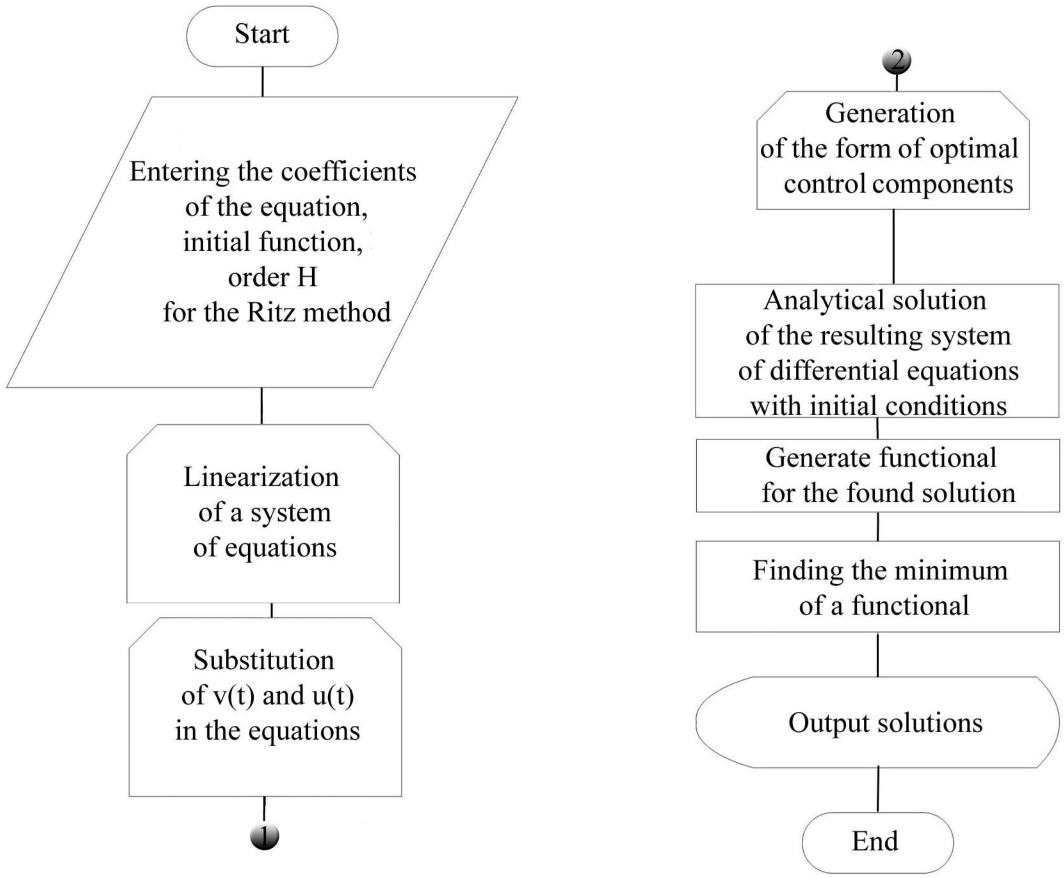

Fig 1. Generalized scheme of the algorithm for calculating the solution of the optimal measurement problem

- matrices L, M, D, size $n \times n$, matrix of coefficients of polynomials $u(t)$;
- array of observation data Y_0, array X_0 of initial values of the system state;
- the right boundary of the segment T on which the problem is considered.

When creating the program, methods of code optimization and calculations were used. To simplify the writing of program code, we used such an opportunity of the Maple programming language as built-in and specified structures for calculating results. Thus, the calculation formulas in the program text are always visible to the user, which affects the understanding of the program code and the ease of its modification. The program was also subjected to optimization of the computation speed. Due to the fact that when calculating many program procedures, the values of the function must be obtained not for an infinite number of values of variables, but mainly only for the integration nodes, then, taking into account the maximum of the double integral in the formula of the quality functional, it is sufficient to save the number of values for m^2, where $m+1$ is the number integration nodes.

4 Computational Experiment

For illustration, consider the Fitz Hugh–Nagumo oscillator, a simplified diagram of which is shown in Fig. 2. It represents an oscillatory RLC circuit with a nonlinear connected to it (in the classical version is a tunnel diode), the sum of the voltages acts on: a constant voltage supply of the circuit V_c and an alternating external signal $\xi(t)$. The current-voltage characteristic of the element (3) can be approximated by a cubic polynomial: $I = F(U) = aU^3 - gU$, where g is the negative differential conductivity, which allows "pumping energy" in the oscillatory circuit.

If we add Krichter's law, then we get the circuit equations:

$$L\frac{dI}{dt} = V_c + \varepsilon(t) - RI - U,$$
$$C\frac{dU}{dt} = I - F(U).$$

Let us introduce the dynamic variables $x_1 = -U$ and $x_2 = \frac{I}{g}$. For new variables, the equations will be written in the following form:

$$\begin{aligned} \varepsilon_1\dot{x}_1 &= x_1 - ax_1^3 - x_2, \\ \varepsilon_2\dot{x}_2 &= \gamma x_1 - \delta x_2 + \beta + \eta(t). \end{aligned} \tag{11}$$

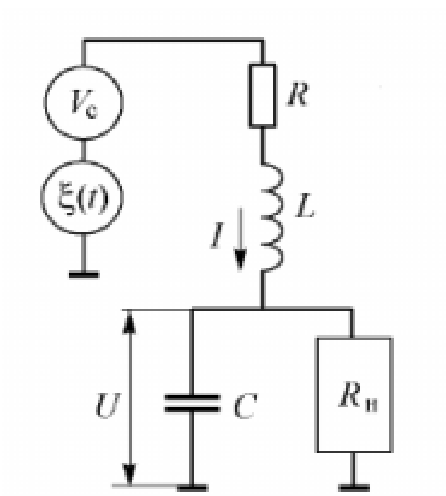

Fig 2. Oscillator circuit Fitz Hugh–Nagumo

The system of equations (11) was investigated in various aspects, and in many researches, along with the case $\varepsilon_1 > 0$ or $\varepsilon_2 > 0$, the case $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ is also discussed [18, 19]. The need to study cases when one of these parameters is equal to zero is associated with the fact that the rate of change of one of the components significantly exceeds the other.

It is required to find a solution to the problem (11), (3), (4), (8) under the following conditions: $\varepsilon_1 = 2,\ \varepsilon_2 = 0,\ \alpha = 2,\ \beta = 0,\ \delta = 1,\ H = 3,\ T = 1,\ \theta = \frac{1}{2},\ \varepsilon = \dfrac{1}{100},\ x_{01} = 0,\ x_{02} = 0,\ y_{01}(t) = t^3 + 3t^2 + 4t,\ y_{02}(t) = 0,\ D = \mathbb{I}$. Let us write the system of equations (11) under the formulated conditions:

$$\begin{cases} \frac{dx_1}{dt} = x_1 - x_2 - x_1^3 + u_1, \\ 0 = 3x_1 - x_2 + u_2. \end{cases} \tag{12}$$

We linearize system (11) by introducing the function $v = v(t)$ and we obtain the equivalent of problem (3), (4), (11), for the system of equations:

$$\begin{cases} \frac{dx_1}{dt} = x_1 - x_2 - v^3 + u_1, \\ 0 = 3x_1 - x_2 + u_2, \\ x_1 = v. \end{cases} \tag{13}$$

Then the solution of problem (3), (4), (13), (8) reduces to finding a triple (x, v, u), where $x = (x_1, x_2)$, $u = (u_1, u_2)$. Based on the Ritz method, we will search $v(t), u_1(t), u_2(t)$ in the form

$$\tilde{v}(t) = \sum_{h=0}^{H} a_h t^n, \quad \tilde{u}_1(t) = \sum_{h=0}^{H} b_h t^h, \quad \tilde{u}_2(t) = \sum_{h=0}^{H} c_h t^h, \tag{14}$$

considering that

$$\tilde{v}(0) = 0. \tag{15}$$

The condition (15) is introduced because $v(t)$ and $x_1(t)$ must coincide at the initial moment of time.

Let us solve problem (3), (4), (13), (8) with respect to unknowns a_h, b_h, c_h. Thus, the optimal control problem is reduced to finding the minimum of a function (10) of several variables with respect to a_h, b_h, c_h. As a result of calculations, an approximate solution of problem (3), (4), (13), (8) was found (see Figs. 3 – 5), while the value of the functional $J = 2.20209929769759327$. Note that $a_1 = 0.986982999441828, a_2 = 0.583924671603152, a_3 = 0.661978825001881, b_0 = 1.40814481467641, b_1 = 1.61882377202430, b_2 = 3.59308220022607, b_3 = 12.9361477673245, c_0 = -0.208877351370390, c_1 = 1.25085783597124, c_2 = 2.48199342550767, c_3 = -1.90233883350030$.

In the problem of optimal measurement, we are not interested in the found $\tilde{x}(t)$, because the primary task is to restore the signal $\tilde{u}(t)$ (Fig. 3), but since we use the decomposition method due to the nonlinearity of the investigated descriptor system, it is necessary in computational experiments to control the sufficient proximity of $\tilde{x}(t)$, $\tilde{v}(t)$, which is guaranteed by the smallness of the

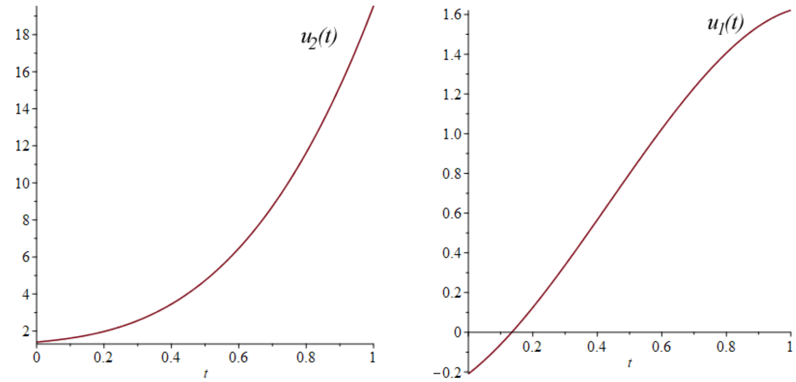

Fig 3. Proximity plot of virtual accurate optimal measurement and virtual accurate optimal observation at $t = 1$

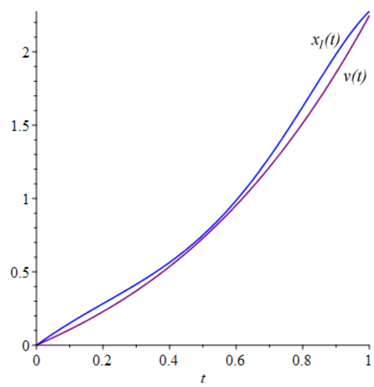

Fig 4. Graph of approaching the input signal of the model (virtual measurement) to the input signal of the sensor at $t = 1$

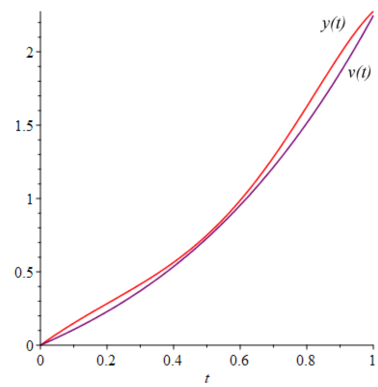

Fig 5. Graphs of the approximation of the virtual exact optimal observation to the incoming sensor signal at $t = 1$

parameter ε, in connection with the fact that in computational experiments this parameter can be adjusted. In our problem, for the given parameters, $\tilde{x}(t)$, $\tilde{v}(t)$ are close enough (Fig. 4), moreover, since $D = \mathbb{I}$, then $\tilde{y}(t)$ should be close to $\tilde{v}(t)$ (Fig. 5).

Acknowledgements. *This work was supported by a grant from the Ministry of Education and Science of the Russian Federation no. FENU-2020-0022 (2020072GZ).*

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