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Ordinary differential equations

Growth conditions for asymptotic behavior of solutions for certain time-varying differential equations

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Abstract. The question proposed in this paper is related to the study of the preservation of uniform h-stability and uniform boundedness of time-varying non-linear differential equations with a perturbation using Gronwall's inequalities and Lyapunov's theory. Moreover, we show the linearization technique for the uniform h-stability of a nonlinear system and give necessary and sufficient conditions for the global boundedness of perturbed systems. The last part is devoted to the study of the problem of h-stabilization for certain classes of nonlinear systems. Some examples and simulations are given to illustrate the main results.

Keywords: Global uniform boundedness, h-stabilization, Linearization system, Lyapunov function, uniform h-stability.

1 Introduction

The stability theory plays an important role in the area of the field of control systems and automation in engineering. There are different types of stability problems that arise in the study of dynamical systems, see [3, 9, 15] and has produced a vast body of important results. In this work, we will investigate the concept of h-stability of time-varying nonlinear systems. This notion has been presented by M. Pinto in [20, 21]. He introduced it for differential systems under some perturbations and extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. This notion is an important development of the exponential asymptotic stability within one common framework. The various notions of h-stability include several kinds of known stability as uniform stability, uniform Lipschitz stability (see [8]) and polynomial stability whose norm can increase not faster than exponentially. The most useful and general approach for studying the nonlinear control systems is the theory of Lyapunov. The relation between Lyapunov functions and various types of stability have been discussed by many authors, see [1, 4, 6, 7, 10, 11, 14, 19]. The general problem of motion stability includes two methods of stability analysis (the so-called linearization method and direct method) was first published in 1892. The linearization technique draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation, this result is proved in [14, 13] for the exponential stability. The direct method is not restricted to local motion, it determines the stability properties of a nonlinear system by constructing a scalar energy-like function for the system and examining the function's time variation, see [12]. Together, the linearization method and the direct method constitute the so-called Lyapunov stability theory.

Nevertheless, there are some systems that may be unstable and yet these systems may oscillate sufficiently near this state that its performance is acceptable. To deal with this situation, we need a notion of stability that is more suitable than Lyapunov stability such a concept is called uniform boundedness. For the boundedness as well as the stability, the Lyapunov theory is very useful and the relation between Lyapunov functions and various types of boundedness are very similar to those between Lyapunov functions and various types of stability (see [6, 14, 16, 17, 18, 24]). It is concerned with quantitative analysis as opposed to Lyapunov analysis which is qualitative in nature.

The contribution of this paper is to construct a Lyapunov equation and use it to show that an equilibrium point of a nonlinear system is uniformly h-stable if the linearization of the system about that point has a uniform h-stable equilibrium point at the origin. In addition, we use the Lyapunov theory to establish the global uniform boundedness of nonlinear perturbed systems of differential equations. The topic of Laypunov stability of control systems described by a system of differential equations was an interesting research area in the past decades (see [2, 5]). Under appropriate growth conditions on the nonlinear perturbation, a state control feedback is established based on the global uniform h-stabilizability of the nominal linear system to h-stabilize the perturbed control system using the Riccati differential equation. The remainder of this work is organized as follows. In Section 2, some preliminary results are summarized and the system description is given. The required assumptions and the statement of the main results are provided in Section 3. Section 4 is devoted to control applications. Finally, some numerical examples are given in Section 5 to demonstrate the effectiveness of the method put forward. Our conclusion is given in Section 6.

2 General definitions

We will use the following notations throughout this paper: $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with appropriate norm $\|.\|$. I and $A^T(t)$ denote the identity matrix and the transpose of the matrix $A(t)$, respectively.
Consider the nonlinear system:

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{1}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the state, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t, x) and locally Lipschitz in x.
Let $x(t, t_0, x_0)$, or simply by $x(t)$ the unique solution of (1) at time t_0 starting from the point x_0.
Firstly, let us introduce some basic definitions which we need in the sequel.

Definition 1 *Assume that $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ is positive, continuous and bounded function. The system (1) is said to be:*

1. *Uniformly h-stable if there exist constants $c \geq 1$ and $\delta > 0$, independent of t_0, such that for all $t_0 \in \mathbb{R}_+$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq \delta$, the solution $x(t)$ satisfies the estimation:*

$$\|x(t)\| \leq c\|x_0\| h(t) h(t_0)^{-1}, \quad \forall\ t \geq t_0. \tag{2}$$

2. *Globally uniformly h-stable if there exists constant $c \geq 1$, such that for all $t_0 \in \mathbb{R}_+$ and all $x_0 \in \mathbb{R}^n$, the solution $x(t)$ satisfies the estimation (2).*

Here, $h(t)^{-1} = \dfrac{1}{h(t)}\cdot$

Remark 1 *For some special cases of h, the h-stability coincides with known types of stability:*

i) *If $h(t) = a$ for $a > 0$, then the system (1) is stable.*

ii) *If $h(t) = e^{-\lambda t}$ for $\lambda > 0$, then the system (1) is uniformly exponentially stable.*

iii) *If $h(t) = \dfrac{1}{(1+t)^{\gamma}}$ for $\gamma \geq 1$, the system (1) is polynomially stable.*

v) *If $h(t)$ is a strictly decreasing function, such that $h(t)$ tends to 0 when $t \to +\infty$, then the origin is uniformly asymptotically stable. More precisely, the solutions of system (1) converge to the origin (i.e., $\limsup_{t\to+\infty} \|x(t)\| = 0$).*

Consider now the linear time-varying system:

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{3}$$

where $A(\cdot)$ is $n \times n$ matrix whose entries values are continuous functions of $t \in \mathbb{R}_+$. The general solution of system (3) is given by:

$$x(t) = \phi(t, t_0)x_0, \qquad x_0 \in \mathbb{R}^n, \qquad t \geq t \geq 0,$$

where $\phi(t, t_0)$ is the state transition matrix associated with $A(\cdot)$. We define the norm of matrices by: $\|A\| = \max_{\|x\|\leq 1} \|Ax\|$.

Lemma 1 *(See [20]) The system (3) is globally uniformly h-stable if and only if there exist $c \geq 1$ and a positive continuous bounded function h on $\mathbb{R}_+$, such that*

$$\|\phi(t, t_0)\| \leq ch(t)h(t_0)^{-1}, \quad \forall\ t \geq t_0 \geq 0.$$

Remark 2 *In linear systems there is the notion of an upper function which is related to upper Lyapunov exponent (see [3, 13, 16, 17, 18, 19, 22, 23]).*

Definition 2 *(See [23]) A bounded function $\mu(t)$ is an upper function for system (3) if there exists a constant λ, such that*

$$\|\phi(t, s)\| \leq \lambda \exp\Big(\int_s^t \mu(\tau)d\tau\Big), \qquad \forall\ t \geq s,$$

where $\phi(t, s)$ is the fundamental matrix of the system.

Remark 3 *In the case where μ is an upper function and*

$$h(t)h(t_0)^{-1} = \exp\Big(\int_{t_0}^{t} \mu(\tau)d\tau\Big)$$

the considered system is h-stable and vice versa.

Remark 4 *There is no relationship between the concept of polynomial stability and exponential stability as shown in the following example.*

Example 1 *Consider the scalar equation:*

$$\dot{x} = -\frac{1}{1+t}x, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{4}$$

The state transition matrix $\phi(t, t_0)$ is given by

$$\phi(t, t_0) = \frac{t_0+1}{t+1}, \quad \forall\ t \geq t_0.$$

Then, the system (4) is polynomially stable. On the contrary, if we suppose that (4) is exponentially stable, then there exists $\alpha > 0$, such that

$$(t_0+1) \leq e^{-\alpha(t-t_0)}(t+1), \quad \forall\ t \geq t_0.$$

For $t_0 = 0$ and $t \longrightarrow \infty$, we obtain a contradiction and hence the system (4) is not exponentially stable.

We prove now the following lemma which will be used later.

Lemma 2 *Consider the nonlinear system (1) with $f(t, 0) \equiv 0$, for all $t \in \mathbb{R}_+$. Let $\phi(\tau; t, x)$ be a solution of the system that starts at (t, x), and let $\phi_x(\tau; t, x) = \dfrac{\partial}{\partial x}\phi(\tau; t, x)$ and $\left\|\dfrac{\partial f}{\partial x}(t, x)\right\| \leq L$, where L is a positive constant. Then,*

$$\|\phi(\tau; t, x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}.$$

Proof. Let $\phi_x(\tau; t, x)$ be the solution of

$$\frac{\partial}{\partial \tau}\phi_x(\tau; t, x) = \frac{\partial f}{\partial x}(\tau, \phi(\tau; t, x))\phi_x, \qquad \phi_x(t; t, x) = I.$$

We have,

$$\begin{aligned} \left|\frac{\partial}{\partial \tau}\phi^T(\tau; t, x)\phi(\tau; t, x)\right| &= \Big|2\phi^T(\tau; t, x)f\big(\tau, \phi(\tau; t, x)\big)\Big| \\ &\leq 2\|\phi(\tau; t, x)\|\|f\big(\tau, \phi(\tau; t, x)\big)\| \\ &\leq 2L\|\phi(\tau; t, x)\|^2. \end{aligned}$$

Then,

$$\frac{\partial}{\partial\tau}\phi^T(\tau;t,x)\phi(\tau;t,x) \geq -2L\|\phi(\tau;t,x)\|^2. \tag{5}$$

Setting $\psi(\tau) = -\|\phi(\tau;t,x)\|$ and using (5), we conclude (as in [14], Example 3.9, pp. 103-104) that $D^+\psi(\tau) \leq {}'L\psi(\tau)$, with

$$D^+\psi(t) = \limsup_{h\to 0^+} \frac{1}{h}\Big(\psi\big(t+h\big) - \psi(t)\Big).$$

By the comparison lemma (see [14], pp. 102-103), we deduce that

$$\|\phi(\tau;t,x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}.$$

□

In this work, it is worth to notice the origin is not necessarily an equilibrium point for system (1). This brings us to the notion of global uniform boundedness.

Definition 3 *A solution of system* (1) *is said to be globally uniformly bounded if for every* $\eta > 0$ *there exists* $\theta = \theta(\eta) > 0$, *such that*

$$\|x_0\| \leq \eta \Longrightarrow \|x(t)\| \leq \theta, \qquad \forall\ t \geq t_0 \geq 0.$$

In order to solve the problem of such perturbed systems, we introduce the following technical lemma, that will be crucial in studying the global uniform boundedness of solutions.

Lemma 3 *Let* $\varpi,\ \rho : \mathbb{R}_+ \to \mathbb{R}$ *be continuous functions and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function, such that*

$$\dot{\varphi}(t) \leq \varpi(t)\varphi(t) + \rho(t), \quad \forall\ t \geq t_0. \tag{6}$$

Then, for any $t \geq t_0 \geq 0$, *we have the following inequality*

$$\varphi(t) \leq \varphi(t_0)\exp\Big(\int_{t_0}^t \varpi(v)dv\Big) + \int_{t_0}^t \exp\Big(\int_s^t \varpi(v)dv\Big)\rho(s)ds.$$

3 Basic results

3.1 Sufficient conditions for uniform boundedness

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate function of a system that satisfies certain conditions, then we can deduce the stability of this system. The following theorem discuss sufficient conditions on the global uniform boundedness of solutions of system (1) by using the Lyapunov's direct method.

Theorem 1 *Suppose that h is a positive, bounded, continuous, decreasing on $\mathbb{R}_+$ with h' exists and continuous on $\mathbb{R}_+$. Moreover, suppose that there exist constants $a_1,\ a_2,\ b > 0$, $k \geq 0$ and a function $V(t,x)$ satisfying the following properties:*

(i) $a_1\|x\|^b \leq V(t,x) \leq a_2\|x\|^b, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$

(ii) $\dot{V}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - kh'(t)h(t)^{-1}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Then, the system (1) is globally uniformly bounded.

Proof. Let $x(t) = x(t,t_0,x_0)$ be the solution of system (1) through $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, it follows from condition (ii) that

$$\dot{V}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - kh'(t)h(t)^{-1}.$$

By using Lemma 3 and the decreasing of h, we get for all $t \geq t_0$ and $x_0 \in \mathbb{R}^n$ the following estimation

$$\begin{aligned} V(t,x) &\leq V(t_0,x_0)h(t)h(t_0)^{-1} - k\int_{t_0}^{t} \exp\left(\int_s^t h'(\tau)h(\tau)d\tau\right) h'(s)h(s)^{-1}ds \\ &\leq V(t_0,x_0)h(t)h(t_0)^{-1} + k. \end{aligned}$$

We deduce for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$ that

$$\|x(t)\| \leq \left(\frac{a_2}{a_1}\|x_0\|^b h(t)h(t_0)^{-1} + \frac{k}{a_1}\right)^{\frac{1}{b}}. \tag{7}$$

1. If $b > 1$, by using the fact that $(\lambda_1 + \lambda_2)^\varepsilon \leq \lambda_1^\varepsilon + \lambda_2^\varepsilon$, for all $\lambda_1,\ \lambda_2 \geq 0$ and $\varepsilon \in]0,1[$, one obtains from the decreasing of h,

$$\|x(t)\| \leq \left(\frac{a_2}{a_1}\right)^{\frac{1}{b}} \|x_0\| + \left(\frac{k}{a_1}\right)^{\frac{1}{b}}. \tag{8}$$

2. If $b \leq 1$. Since $(\lambda_1 + \lambda_2)^p \leq 2^{p-1}(\lambda_1^p + \lambda_2^p)$, for all $\lambda_1,\ \lambda_2 \geq 0$ and $p \geq 1$, one can get from the decreasing of h,

$$\|x(t)\| \leq 2^{\frac{1-b}{b}} \left(\frac{a_2}{a_1}\right)^{\frac{1}{b}} \|x_0\| + 2^{\frac{1-b}{b}} \left(\frac{k}{a_1}\right)^{\frac{1}{b}}. \tag{9}$$

This yields that the solutions of system (1) are globally uniformly bounded.
Note that, if $b > 1$, the inequality (8) implies that

$$\|x(t)\| - \left(\frac{k}{a_1}\right)^{\frac{1}{b}} \leq \left(\frac{a_2}{a_1}\right)^{\frac{1}{b}} \|x_0\|.$$

Thus, for all $t \in \mathbb{R}_+$, if we take $\|x_0\| \geq \left(\frac{k}{a_1}\right)^{\frac{1}{b}}$, such that $\|x(t)\| \geq \left(\frac{k}{a_1}\right)^{\frac{1}{b}}$, we get that the solutions of system (1) approach to a compact set S, when $t \to \infty$, given by

$$S = \left\{ x \in \mathbb{R}^n, \|x\| \leq \left(\frac{k}{a_1}\right)^{\frac{1}{b}} \right\}.$$

If $b \leq 1$ and by using a similar reasoning as above on (9), we can deduce that the solutions of system (1) approach to a compact set S', when $t \to \infty$, given by

$$S' = \left\{ x \in \mathbb{R}^n, \|x\| \leq 2^{\frac{1-b}{b}} \left(\frac{k}{a_1}\right)^{\frac{1}{b}} \right\}.$$

This completed the proof. □

3.2 Converse Theorem

Although Lemma 1 may not be very helpful as a stability test, we will see that it guarantees the existence of a Lyapunov function for the linear system (3). That is, if we can find a continuously differentiable, positive, bounded and symmetric matrix $P(t)$, which is a solution of a differential equation for some continuous positive definite symmetric matrix $Q(t)$, then $V(t, x)$ is a Lyapunov function for the system. If the matrix $Q(t)$ is chosen to be bounded in addition to being symmetric, continuous, positive definite and if $A(t)$ is continuous and bounded, then it can be shown that when the origin is uniformly h-stable, there is a solution of system (3) that possesses the desired properties.
In this section, we state a converse theorem when the origin is a globally uniformly h-stable equilibrium point of the linear system (3), by defining a Lyapunov function that satisfies certain properties.

In what follows, we will denote by $\mathcal{H}$ the set of the functions $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that:

$$\mathcal{H}: \qquad \exists\, M > 0, \qquad \int_t^{\infty} h(\tau)^2 d\tau \leq M h(t), \qquad \forall\, t \geq 0.$$

Theorem 2 *Let the origin be globally uniformly h-stable equilibrium point of system (3). Assume that $h \in \mathcal{H}$ with h' exists and continuous on $\mathbb{R}_+$. Suppose that $A(t)$ is continuous and bounded on $\mathbb{R}^n$. Let $Q(t)$ be a continuous, bounded, positive definite and symmetric matrix. Then, there is a continuously differentiable, bounded, positive and symmetric matrix $P(t)$, which is the solution of the Riccati equation:*

$$\dot{P}(t) = h'(t)h(t)^{-1}P(t) - A^T(t)P(t) - P(t)A(t) - Q(t). \tag{10}$$

Proof. Assume that the system (3) is globally uniformly h-stable. Let $\phi(\tau; t, x)$ be the solution of system (3) that starts at (t, x). Due to linearity, $\phi(\tau; t, x) = \phi(\tau, t)x$. Let the matrix $P(t)$ defined by

$$P(t) = h(t)\int_t^\infty \phi^T(\tau, t)Q(\tau)\phi(\tau, t)d\tau. \tag{11}$$

Since $Q(t)$ is positive definite and bounded matrix, then there exist positive constants k_1 and k_2, such that

$$k_1 I \le Q(t) \le k_2 I, \qquad \forall\ t \ge 0. \tag{12}$$

On the one hand, we have

$$\begin{aligned} x^T P(t)x &\le k_2 h(t)\int_t^\infty \|\phi(\tau; t, x)\|^2 d\tau \\ &\le c^2 k_2 h(t)^{-1}\int_t^\infty h(\tau)^2 d\tau \|x\|^2. \end{aligned}$$

Thus,

$$x^T P(t)x \le c^2 k_2 M = c_1\|x\|^2.$$

On the other hand, since $A(t)$ is bounded, then there exists a positive constant L, such that

$$\|A(t)\| \le L, \qquad \forall\ t \in \mathbb{R}_+.$$

From Lemma 2, we have

$$\|\phi(\tau; t, x)\|^2 \ge \|x\|^2 e^{-2L(\tau - t)}.$$

$$\begin{aligned} x^T P(t)x &\ge k_1 h(t)\int_t^\infty e^{-2L(\tau - t)} d\tau \|x\|^2 \\ &= \frac{k_1 h(t)}{2L}\|x\|^2 \ge 0, \qquad \forall\ t \ge 0. \end{aligned}$$

Therefore, $P(t)$ is positive and bounded. In addition, the definition of $P(t)$ shows that it is symmetric and continuously differentiable. To calculate the differentiable of $P(t)$, we use the following property

$$\frac{\partial}{\partial t}\phi(\tau,t) = -\phi(\tau,t)A(t).$$

Hence,

$$\begin{aligned} \dot{P}(t) &= h'(t)h(t)^{-1}P(t) + h(t)\int_t^\infty \left[\frac{\partial}{\partial t}\phi^T(\tau,t)\right] Q(\tau)\phi(\tau,t)d\tau \\ &+ h(t)\int_t^\infty \phi^T(\tau,t)Q(\tau)\left[\frac{\partial}{\partial t}\phi(\tau,t)\right]d\tau - Q(t) \\ &= h'(t)h(t)^{-1}P(t) - A^T(t)P(t) - P(t)A(t) - Q(t). \end{aligned}$$

□

Theorem 3 *Let the solutions of system* (3) *be globally uniformly h-stable with $h \in \mathcal{H}$ and h' exists continuous on $\mathbb{R}_+$. Suppose that $A(t)$ is continuous and bounded on $\mathbb{R}^n$. Thus, there exists a function $V(t,x)$ satisfying the following properties:*

(*i*) $\|x\|^2 \le V(t,x) \le (c_1+1)\|x\|^2, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$

(*ii*) $\dot{V}(t,x) \le h'(t)h(t)^{-1}V(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$

where c_1 is a positive constant.

Proof. We choose a matrix $Q(t)$ continuous, bounded, positive definite, symmetric on $\mathbb{R}_+$, and there exists $c_2 > 0$, such that

$$x^T\Big(Q(t) + h'(t)h(t)^{-1}I - A(t) - A^T(t)\Big)x \ge c_2\|x\|^2. \qquad (13)$$

The linear system is globally uniformly h-stable, then by Theorem 2 we can find a matrix $P(t)$ which is a solution of the Riccati equation (10). We define the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(t,x) = x^T P(t)x + \|x\|^2.$$

It is easy to verify that,

$$V(t,x) \le (c_1+1)\|x\|^2.$$

Another,

$$\begin{aligned} V(t,x) &\geq k_1 h(t) \int_t^{\infty} e^{-2L(\tau - t)} d\tau \|x\|^2 + \|x\|^2 \\ &= \frac{k_1 h(t)}{2L} \|x\|^2 + \|x\|^2 \geq \|x\|^2. \end{aligned}$$

Hence, the first inequality of the theorem is hold.
Now, we shall show (ii). By taking the derivative of $V(t,x)$ along the trajectories of the linear system (3), we get

$$\begin{aligned} \dot{V}(t,x) &= \dot{x}^T P(t)x + x^T \dot{P}(t)x + x^T P(t)\dot{x} + x^T A^T(t)x + x^T A(t)x \\ &= h'(t)h(t)^{-1}V(t,x) - Q(t)\|x\|^2 + A(t)\|x\|^2 + x^T A^T(t)x \\ &+ x^T A(t)x - h'(t)h(t)^{-1}x^T x. \end{aligned}$$

It follows from (13) that

$$\dot{V}(t,x) \leq h'(t)h(t)^{-1}V(t,x),$$

which prove (ii). This ends the proof. □

Remark 5 *The above theorem is an extension of the global uniform exponential stability in [14], for* $h(t) = e^{-\beta t}$ *with* $\beta > 0$ *and the polynomial stability, for* $h(t) = \dfrac{1}{(1+t)^{\gamma}}$ *with* $\gamma \geq 1$.

Corollary 1 *Let the origin be globally uniformly h-stable equilibrium point of system* (3). *Assume that* $h \in \mathcal{H}$ *and* h' *exists continuous on* $\mathbb{R}_+$. *Suppose that* $A(t)$ *is continuous and bounded on* $\mathbb{R}^n$. *If* $Q(t) = C^T(t)C(t)$, *where* $C(t)$ *is a continuous matrix in* $t \in \mathbb{R}_+$, *then the Riccati equation is given by*

$$\dot{P}(t) = h'(t)h(t)^{-1}P(t) - A^T(t)P(t) - P(t)A(t) - C(t)^T C(t).$$

3.3 h-Linearized stability of nonlinear systems

The existence of Lyapunov functions for linear systems per Theorem 2 will now be used to prove a linearization result and to determine the uniform h-stability of the nonlinear system. In this section, the result of the preceding section is combined to obtain one of the most useful results in Lyapunov stability theory namely: **linearization method**. The advantage of this method lies in the fact that, under certain conditions, it enables one to draw conclusions about a nonlinear

system by studying the behavior of a linear system.
Consider the nonlinear non-autonomous system (1), where $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$ is continuously differentiable and D is a domain that contains the origin. Suppose that the origin is an equilibrium point of the system, with $f(t, 0) = 0$ and assume that the Jacobian matrix $\left[\dfrac{\partial f}{\partial x}\right]$ is uniformly bounded and Lipschitz on D, that is, there exist positive constants L_1 and L_2, such that

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\| \leq L_1, \qquad \forall\, x \in D, \qquad \forall\, t \geq 0,$$

$$\left\|\frac{\partial f}{\partial x}(t,x_1) - \frac{\partial f}{\partial x}(t,x_2)\right\| \leq L_2\|x_1 - x_2\|, \quad \forall\, x_1,\ x_2 \in D, \quad \forall\, t \geq 0.$$

We can write $f(t, x)$ in the form

$$f(t,x) = f(t,0) + \frac{\partial f}{\partial x}(t,z)x,$$

where $z \in]0, x[$. Since $f(t, 0) = 0$, then we have

$$\begin{aligned} f(t,x) &= \frac{\partial f}{\partial x}(t,z)x \\ &= \frac{\partial f}{\partial x}(t,0)x + \left[\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,0)\right] x \\ &= A(t)x + \chi(t,x), \end{aligned}$$

where $A(t) = \dfrac{\partial f}{\partial x}(t,0)$ and $\chi(t,x) = \left[\dfrac{\partial f}{\partial x}(t,z) - \dfrac{\partial f}{\partial x}(t,0)\right] x$.
Therefore, we may approximate the nonlinear system (1) by its linearization in a small neighborhood of the origin.
The next theorem states Lyapunov's indirect method for showing the uniform h-stability of the origin in the non-autonomous case. We will see that, the uniform h-stability of the linearized system reflects the uniform h-stability of the nonlinear system.

Theorem 4 *Let the origin be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{14}$$

where $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$ is continuously differentiable with $D = \{x \in \mathbb{R}^n / \|x\| < \varrho\}$. Suppose that the Jacobian matrix $\left[\dfrac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t,x)\bigg|_{x=0}.$$

Then, the origin is uniformly $h^{\frac{1}{2}}$-stable for system (14) *if it is uniformly h-stable for the linear system $\dot{x} = A(t)x$, with $h \in \mathcal{H}$.*

Proof. We can write the nonlinear system (14) as:

$$\dot{x} = A(t)x + \chi(t,x), \tag{15}$$

such that χ defined on $\mathbb{R}_+ \times D$ is continuous in (t,x), locally Lipschitz in x and verifies the following condition:

$$\|\chi(t,x)\| \leq \lambda \|x\|^2, \qquad \forall\, x \in D, \qquad \forall\, t \geq 0, \tag{16}$$

with λ is a positive constant. In addition, we assume that the linear system (3) has a uniform h-stable equilibrium point at the origin, $h \in \mathcal{H}$ with h' exists and continuous on $\mathbb{R}_+$, and $A(t)$ is continuous and bounded on $\mathbb{R}^n$. Then, Theorem 2 ensures the existence of a continuously differentiable, bounded, positive and symmetric matrix $P(t)$ that satisfies (10), where $Q(t)$ is continuous, bounded, positive definite and symmetric matrix that verifies (13). By Theorem 3, there exists a Lyapunov function $V(t,x)$ having the properties (i) and (ii). The derivative of $V(t,x)$ along the trajectories of system (15) is given by:

$$\begin{aligned} \dot{V}(t,x) &= \dot{x}^T P(t)x + x^T \dot{P}(t)x + x^T P(t)\dot{x} + \dot{x}^T x + x^T \dot{x} \\ &\leq h'(t)h(t)^{-1}V(t,x) - x^T Q(t)x + 2\big(\|P(t)\| + 1\big)\chi(t,x)\|x\| \\ &+ A^T(t)\|x\|^2 + A(t)\|x\|^2 - h'(t)h(t)^{-1}\|x\|^2. \end{aligned}$$

By using the inequality (13), we get

$$\dot{V}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - c_2\|x\|^2 + 2\big(\|P(t)\| + 1\big)\chi(t,x)\|x\|.$$

From condition (16) and the property on $P(t)$, we obtain

$$\dot{V}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \Big(c_2 - 2\rho\lambda\big(c_1+1\big)\Big)\|x\|^2, \qquad \forall\, \|x\| < \rho.$$

By choosing $\rho < \min\left\{\varrho, \dfrac{c_2}{2\lambda\big(c_1+1\big)}\right\}$, we obtain

$$V(t,x) \leq V(t_0,x_0)h(t)^{\frac{1}{2}}h(t)^{-\frac{1}{2}}.$$

Therefore, for all $t \geq t_0$ and all $x_0 \in D$ the solution $x(t)$ of system (15) is as follows:

$$\|x(t)\| \leq \sqrt{(c_1+1)}\|x_0\|h(t)^{\frac{1}{2}}h(t_0)^{-\frac{1}{2}},$$

which ensures that $\dot{V}(t,x)$ is negative definite in $\|x\| < \rho$. Hence, we conclude that the origin of the nonlinear system (14) is uniformly $h^{\frac{1}{2}}$-stable. □

Remark 6

- *The previous theorem is a generalization of uniform exponential stability, that is, the nonlinear system is uniformly exponentially stable if the linearized system is uniformly exponentially stable (see [14], Theorem 4.13).*
- *The linearization result is hold for polynomial stability:* $h(t) = \dfrac{1}{(1+t)^{\gamma}}$ *with* $\gamma \geq 1$.

Corollary 2 *If the nonlinear system is autonomous, that is,* $\dot{x} = f(x)$, *we can draw conditions about the stability of the origin as an equilibrium point for the system by investigating the stability for the linearization of the system, where* $A = \left.\dfrac{\partial f}{\partial x}\right|_{x=0}$, *such that* $h(t) = e^{-\beta t}$, *for all* $t \in \mathbb{R}_+$ *and* $\beta > 0$.

3.4 Boundedness of solutions of perturbed systems

We can use Lyapunov's indirect method to show the global uniform boundedness of the solutions. We consider a nonlinear perturbed system and we give sufficient conditions on the perturbed term. Our conditions are expressed as relations between the Lyapunov function and the interconnection term.

Theorem 5 *Consider the perturbed system:*

$$\dot{x} = A(t)x + \chi(t,x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{17}$$

where $A(t)$ *is continuous and bounded on* $\mathbb{R}^n$, $\chi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous in* (t,x), *locally Lipschitz in* x *and satisfies the following assumption:*

$$\|\chi(t,x)\| \leq \varphi(t)\|x\| + \mu(t), \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, t \geq 0,$$

with φ *and* μ *are non-negative continuous integrable functions on* $\mathbb{R}_+$. *Assume that the system* (3) *is globally uniformly* h*-stable with* $h \in \mathcal{H}$ *is decreasing and* h' *exists continuous on* $\mathbb{R}_+$, *then the solutions of system* (17) *are globally uniformly bounded.*

Proof. We have the system (3) is globally uniformly h-stable, $A(t)$ is continuous and bounded on $\mathbb{R}^n$. Let $Q(t)$ be a continuous, bounded, positive definite and symmetric matrix, such that (13) is hold. Then, Theorem 3 ensures that there exists a Lyapunov function candidate

$$V(t,x) = x^T P(t)x + \|x\|^2$$

that satisfies the certain properties. The derivative of $V(t,x)$ along the trajectories of system (17) is as follows:

$$\begin{aligned}\dot V(t,x) &= h'(t)h(t)^{-1}V(t,x) - x^TQ(t)x + h'(t)h(t)^{-1}x^Tx + 2x^T\Big(\|P(t)\|+1\Big)\chi(t,x)\\ &+ x^TA(t)x + x^TA^T(t)x.\end{aligned}$$

Using the inequality (13), we get

$$\begin{aligned}V(t,x) &\le h'(t)h(t)^{-1}V(t,x) + 2\big(c_1+1\big)\varphi(t)\|x\|^2 + 2\big(c_1+1\big)\mu(t)\|x\|\\ &\le \Big(h'(t)h(t)^{-1} + 2\big(c_1+1\big)\varphi(t)\Big)V(t,x) + 2\big(c_1+1\big)\mu(t)\sqrt{V(t,x)}.\end{aligned}$$

Put, $\varpi(t) = \sqrt{V(t,x)}$, then

$$\dot\varpi(t) = \frac{\dot V(t,x)}{2\sqrt{V(t,x)}}.$$

This yields,

$$\dot\varpi(t) \le \Big(\frac{1}{2}h'(t)h(t)^{-1} + 2\big(c_1+1\big)\varphi(t)\Big)\varpi(t) + 2\big(c_1+1\big)\mu(t).$$

By Lemma 3 and the decreasing of h, we have for all $t \ge t_0$

$$\varpi(t) \le \varpi(t_0)\exp\Big(2\big(c_1+1\big)M_2\Big)h(t)^{\frac{1}{2}}h(t_0)^{-\frac{1}{2}} + 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1,$$

where $M_1 = \displaystyle\int_0^\infty \mu(s)ds$ and $M_2 = \displaystyle\int_0^\infty \varphi(s)ds$. Therefore,

$$\begin{aligned}\|x(t)\| &\le \sqrt{(c_1+1)}\exp\Big(2\big(c_1+1\big)M_2\Big)\|x_0\|h(t)^{\frac{1}{2}}h(t_0)^{-\frac{1}{2}}\\ &+ 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1.\end{aligned}$$

Consequently, the solutions of system (17) are globally uniformly bounded.
From the decreasing of h, we obtain

$$\|x(t)\| - 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1 \le \sqrt{(c_1+1)}\exp\Big(2\big(c_1+1\big)M_2\Big)\|x_0\|.$$

Hence, for all $t \in \mathbb{R}_+$, if we take $\|x_0\| \ge 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1$, such that $\|x(t)\| \ge 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1$, then the solutions of system (17) approach, when t goes to infinity, to the compact set S defined by

$$S = \Big\{x \in \mathbb{R}^n,\ \|x\| \le 2\big(c_1+1\big)\exp\Big(2\big(c_1+1\big)M_2\Big)M_1\Big\}.$$

□

As a particular case of the forgoing theorem, when $\varphi(t) = 0$ we obtain the following corollary.

Corollary 3 *Consider the perturbed system* (17) *where* $A(t)$ *is continuous and bounded on* $\mathbb{R}^n$, χ *is defined on* $\mathbb{R}_+ \times \mathbb{R}^n$ *continuous in* (t, x) *and locally Lipschitz in* x. *Suppose that* χ *satisfies the following assumption:*

$$\|\chi(t,x)\| \leq \mu(t), \qquad \forall\ t \geq 0, \tag{18}$$

where μ *is a non-negative continuous integrable function on* $\mathbb{R}_+$. *If the system* (3) *is globally uniformly* h*-stable with* $h \in \mathcal{H}$ *is decreasing and* h' *exists continuous on* $\mathbb{R}_+$, *then the solutions of the perturbed system* (17) *are globally uniformly bounded.*

3.5 h-stabilization

Let us go to present the h-stabilizability problem of a nonlinear control system of the form:

$$\begin{cases} \dot{x} = A(t)x + B(t)u(t) + F(t,x,u), \\ x(t_0) = x_0, \end{cases} \tag{19}$$

where $x \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are matrices whose elements are continuous bounded functions on $\mathbb{R}_+$. The function $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous in (t, x, u) and satisfying the following inequality:

$$\|F(t,x,u)\| \leq \lambda(t)\|x\| + \gamma(t)\|u\|, \quad \forall\ x \in \mathbb{R}^n, \quad \forall\ u \in \mathbb{R}^m, \quad \forall\ t \in \mathbb{R}_+, \tag{20}$$

with λ and γ are non-negative continuous integrable functions on $\mathbb{R}_+$. The corresponding system without perturbation called the nominal system is described by

$$\begin{cases} \dot{x} = A(t)x + B(t)u(t), \\ x(t_0) = x_0, \end{cases} \tag{21}$$

Definition 4 *we say that the system* (19) *is stabilizable, if there exists at least a continuous function* $u(t)$, *such that the origin of the closed loop system* (19) *by* $u(t)$ *is asymptotically stable.* $u(t)$ *is called a feedback.*

Definition 5 *Let* $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ *be a continuous bounded function. The linear control system* (21) *is globally uniformly* h*-stabilizable if there exist a feedback* *control system $u(t) \in \mathbb{R}^m$ and a constant $c \geq 1$, such that for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$ the solution $x(t)$ of the closed-loop system satisfies the estimation:*

$$\|x(t)\| \leq c\|x_0\| h(t) h(t_0)^{-1}.$$

The goal of this section is to found a state feedback controller $u(t)$, such that the system (19) is globally uniformly h-stabilizable.

Theorem 6 *Assume that the linear system $\dot{x}(t) = A(t)x(t)$ is globally uniformly h-stable with $h \in \mathcal{H}$ and h' exists continuous on $\mathbb{R}_+$. We choose $Q(t)$ continuous, bounded, positive definite and symmetric matrix that verifies (13), then the Riccati differential equation (10) has a solution $P(t)$ bounded, positive, symmetric continuously differentiable matrix, and the nonlinear control system (19) is $h^{\frac{1}{2}}$-stabilizable by the feedback control*

$$u(t) = -B^T(t)P(t)x(t), \qquad t \geq t_0. \tag{22}$$

Proof. Suppose that the linear system $\dot{x}(t) = A(t)x(t)$ is globally uniformly h-stable where $A(t)$ is continuous and bounded on $\mathbb{R}^n$. Then, Theorem 2 ensures the existence of a continuously differentiable, bounded, positive and symmetric matrix $P(t)$ that satisfies the Riccati differential equation (10). Indeed, the derivative of $V(t,x)$ along the solutions $x(t)$ of system (19) using the chosen feedback control (22) is given by

$$\begin{aligned}
\dot{V}(t,x) &= \dot{x}^T P(t)x + x^T \dot{P}(t)x + x^T P(t)\dot{x} + \dot{x}^T x + x^T \dot{x} \\
&= h'(t)h(t)^{-1}V(t,x) + 2F(t,x,u)x^T P(t) + u^T(t)B^T(t)P(t)x \\
&+ x^T P(t)B(t)u(t) - x^T Q(t)x - h'(t)h(t)^{-1}x^T x + x^T A^T(t)x + x^T A(t)x \\
&+ u^T(t)B^T(t)x + x^T B(t)u(t) + 2F(t,x,u)x^T \\
&\leq \Big(h'(t)h(t)^{-1} + 2\big(c_1+1\big)\Big(\lambda(t) + c_1\gamma(t)\|B\|_\infty\Big)\Big)V(t,x),
\end{aligned}$$

where $\|B\|_\infty = \sup\limits_{t\geq 0} \|B(t)\|$. Hence,

$$\begin{aligned}
V(t,x) &\leq V(t_0,x_0)h(t)h(t_0)^{-1}\exp\left(2\big(c_1+1\big)\int_{t_0}^{t}\Big(\lambda(s) + c_1\gamma(s)\|B\|_\infty\Big)ds\right) \\
&\leq V(t_0,x_0)h(t)h(t_0)^{-1}\exp\Big(2\big(c_1+1\big)\Big(M_1 + c_1\|B\|_\infty\Big)\Big),
\end{aligned}$$

with $M_1 = \displaystyle\int_0^\infty \lambda(s)ds$ and $M_2 = \displaystyle\int_0^\infty \gamma(s)ds$. Therefore,

$$\|x(t)\| \leq c\|x_0\| h(t)^{\frac{1}{2}} h(t_0)^{-\frac{1}{2}},$$

where $c = \sqrt{(c_1+1)}\exp\Big(\big(c_1+1\big)\Big(M_1+c_1\|B\|_\infty\Big)\Big)$. This yields that the system (19) is $h^{\frac{1}{2}}$-stabilizable. The proof is completed. □

Next, we state another sufficient condition for the h-stabilizability of system (19) in the case when the linear system $\dot{x} = A(t)x$ is not globally uniformly h-stable, but the associated linear control system (21) is globally uniformly h-stable.

Theorem 7 *Assume that the linear control system* (21) *is h-stabilizable with $h \in \mathcal{H}$ and h' exists continuous on $\mathbb{R}_+$. Then the nonlinear system* (19) *is $h^{\frac{1}{2}}$-stabilizable.*

Proof. Assume that the system (21) is h-stabilizable, then there exists $K(t) \in \mathbb{R}^{m\times n}$, such that

$$\dot{x} = \Big(A(t) + B(t)K(t)\Big)x = \hat{A}(t)x, \qquad \forall t \geq t_0 \tag{23}$$

is globally uniformly h-stable. We choose $Q(t)$ a positive, continuous, bounded and symmetric matrix that satisfies (13), then we consider the Lyapunov function $V(t,x) = x^TP(t)x + \|x\|^2$ from Theorem 3. Hence, by taking the derivative of $V(t,x)$ along the solutions $x(t)$ of system (19) using the chosen feedback $u(t) = K(t)x(t)$, we obtain

$$\begin{aligned}
\dot{V}(t,x) &= \dot{x}^TP(t)x + x^T\dot{P}(t)x + x^TP(t)\dot{x} + \dot{x}^Tx + x^T\dot{x} \\
&\leq h'(t)h(t)^{-1}V(t,x) + 2F(t,x,u)\|P(t)\|\|x\| + 2\hat{A}(t)\|x\|^2 + 2F(t,x,u)\|x\| \\
&\leq \Big(h'(t)h(t)^{-1} + 2\big(c_1+1\big)\Big(\lambda(t) + \gamma(t)\|K\|_\infty\Big)\Big)V(t,x),
\end{aligned}$$

with $\|K\|_\infty = \sup\limits_{t\geq 0}\|K(t)\|$. Thus,

$$\begin{aligned}
V(t,x) &\leq V(t_0,x_0)h(t)h(t_0)^{-1}\exp\left(2\big(c_1+1\big)\int_{t_0}^{t}\Big(\lambda(s)+\gamma(s)\|K\|_\infty\Big)ds\right) \\
&\leq V(t_0,x_0)h(t)h(t_0)^{-1}\exp\Big(2\big(c_1+1\big)\Big(M_1 + M_2\|K\|_\infty\Big)\Big)
\end{aligned}$$

with $M_1 = \displaystyle\int_0^\infty \lambda(s)ds$ and $M_2 = \displaystyle\int_0^\infty \gamma(s)ds$. Therefore, for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$ the solution of system (19) is given by:

$$\|x(t)\| \leq c\|x_0\|h^{\frac{1}{2}}(t)h(t_0)^{-\frac{1}{2}},$$

where $c = \sqrt{(c_1+1)}\exp\Big(\big(c_1+1\big)\Big(M_1 + M_2\|K\|_\infty\Big)\Big)$. This ends the proof. □

Example 2 *Consider the second order problem:*

$$\begin{cases} \dot{x}_1 = \Big(\dfrac{-1}{1+t} - 1\Big)x_1 + \dfrac{e^{-t}}{1+t}u(t) + \dfrac{x_1 e^{-t}}{(1+t^2)\sqrt{1+x_1^2}}u(t) \\ \dot{x}_2 = \Big(\dfrac{-1}{1+t} - 1\Big)x_2 + \dfrac{1}{1+t^2}x_2 + \dfrac{x_1 e^{-t}}{\sqrt{1+x_1^2}}u(t), \qquad t \geq 0 \end{cases} \tag{24}$$

The above system is exactly the system (19), *where*

$$A(t) = \begin{pmatrix} -\dfrac{1}{1+t} - 1 & 0 \\ 0 & -\dfrac{1}{1+t} - 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \dfrac{e^{-t}}{1+t} \\ 0 \end{pmatrix}, \qquad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and

$$F(t,x,u) = \frac{1}{1+t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{x_1}{\sqrt{1+x_1^2}}\begin{pmatrix} \dfrac{e^{-t}}{1+t^2} \\ e^{-t} \end{pmatrix} u(t).$$

The nominal system $\dot{x}(t) = A(t)x(t)$ *is globally uniformly* h*-stable with* $h(t) = \dfrac{1}{1+t}$. *Then, for a matrix*

$$Q(t) = \begin{pmatrix} \dfrac{1}{2(1+t)} + 1 & 0 \\ 0 & \dfrac{1}{2(1+t)} + 1 \end{pmatrix},$$

which solves the hypothesis (13), *there exists a matrix* $P(t)$,

$$P(t) = \begin{pmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{pmatrix}$$

satisfies the Riccati equation (10). *Moreover, the function* F *satisfies the assumption* (20) *with* $\lambda(t) = \dfrac{1}{1+t^2}$ *and* $\gamma(t) = \sqrt{2}e^{-t}$.

We conclude that the conditions of Theorem 6 *are hold. Therefore, the system* (24) *is globally uniformly* $h^{\frac{1}{2}}$*-stabilizable under the closed-loop linear feedback*

$$u(t) = -B^T(t)P(t)x(t) = -\frac{e^{-t}}{2(1+t)}(x_1(t) + x_2(t)).$$

For simulation of system (24) *we select the initial state* $\Big(x_1(0), x_2(0)\Big) = (1,1)$. *The result of simulation is depicted in Figure* 1.

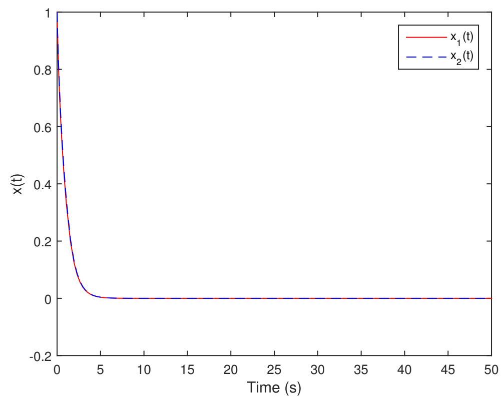

Figure 1: *The trajectory of the state* $x(t) = (x_1(t), x_2(t))$ *of system* (24).

3.6 Examples

In this section, we give some numerical examples and simulations to prove the applicability of the theoretical results.

Example 3 *Consider the scalar equation:*

$$\dot{x} = -\frac{x}{t + \sin x} + \frac{2}{1+t}, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_+. \tag{25}$$

Setting, $V(t,x) = x^2$ *and* $h(t) = \dfrac{1}{(1+t)^2}$*, which is positive, bounded, continuous and decreasing on* $\mathbb{R}_+$ *with* h' *exists and is continuous on* $\mathbb{R}_+$*. Then, Theorem 1 holds with* $a_1 = a_2 = 1$ *and* $b = k = 2$*. This yields the global uniform boundedness of system* (25)*, that is, the solutions of system* (25) *approach to a compact set* S'*, when* $t \to +\infty$*, given by:*

$$S' = \left\{ x \in \mathbb{R}, \ |x| \leq \sqrt{2} \right\}.$$

For simulation of system (25) *we select the initial state* $x(0) = 1$*. The result is depicted in Figure 2*

Example 4 *We consider the second order problem:*

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_1^2 x_2^2 - \dfrac{a}{1+t} x_1 \\ \dot{x}_2 = -x_2^3 + x_1 \sin x_2 - \dfrac{a}{1+t} x_2, \qquad\qquad a \geq 1, \end{cases} \tag{26}$$

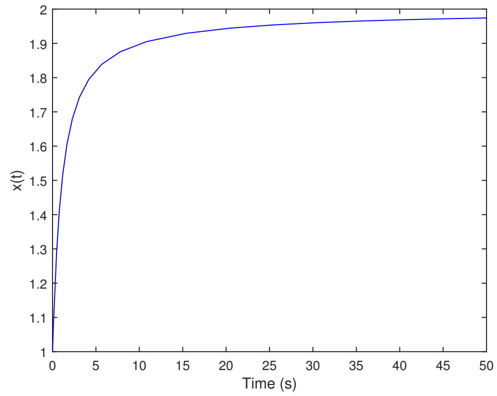

Figure 2: *The trajectory of the state $x(t)$ of system (25).*

where $x = (x_1, x_2)^T \in D \subset \mathbb{R}^2$ and $t \in \mathbb{R}_+$. The Jacobian matrix of the nonlinear system (26) is given by:

$$\frac{\partial f}{\partial x}(x_1, x_2)\Big|_{(0,0)} = \begin{pmatrix} -\dfrac{a}{1+t} & 0 \\ 0 & -\dfrac{a}{1+t} \end{pmatrix} = A(t).$$

The linear system $\dot{x}(t) = A(t)x(t)$ is uniformly h-stable with $c = 1$ and $h(t) = \dfrac{1}{(1+t)^a} \in \mathcal{H}$. By applying Theorem 4, the system (26) is uniformly $\frac{1}{(1+t)^{\frac{a}{2}}}$-stable. For simulation of system (26) we select the initial state $\Big(x_1(0), x_2(0)\Big) = (1, 2)$ and $a = 3$. The result of simulation is depicted in Figure 3

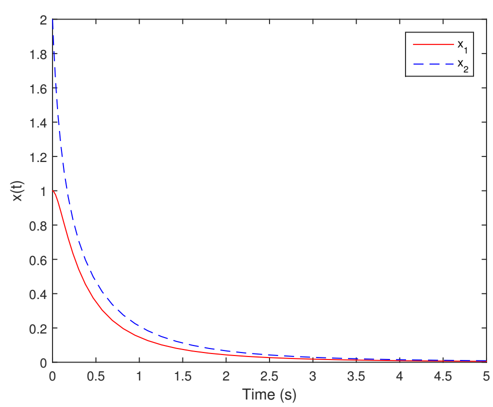

Figure 3: *The trajectory of the state $x(t) = (x_1, x_2)$ of system (26).*

Example 5 *We consider the first order problem:*

$$\dot{x} = -\left(\frac{2}{1+t}+1\right)x + \frac{\sqrt{t}}{1+t^3}x + \frac{\sin t}{1+t^2}, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}_+. \tag{27}$$

The previous system can be written as:

$$\dot{x}(t) = A(t)x(t) + \chi(t,x), \qquad t \geq 0.$$

The linear system $\dot{x}(t) = A(t)x(t)$ *is globally uniformly* h*-stable and* $h(t) = \dfrac{1}{(1+t)^2} \in \mathcal{H}$ *is positive, bounded and decreasing where* h' *exists and is continuous on* $\mathbb{R}_+$. *Then, for a function* $Q(t) = \dfrac{1}{1+t} + 1$ *verifies* (13), *there exists* $P(t) = \dfrac{1}{2}$ *satisfies the Riccati equation* (10). *Furthermore,*

$$|\chi(t,x)| \leq \frac{\sqrt{t}}{1+t^3}|x| + \frac{1}{1+t^2}, \qquad \forall\, x \in \mathbb{R}, \qquad \forall\, t \in \mathbb{R}_+.$$

Put, $\varphi(t) = \dfrac{\sqrt{t}}{1+t^3}$ *and* $\mu(t) = \dfrac{1}{1+t^2}$ *which are non-negative continuous and integrable functions on* $\mathbb{R}_+$. *From Theorem 5, we deduce the global uniform boundedness of system* (27).

Likewise, solutions of system (27) *approximate, when* t *goes to infinity, to the compact set* S *given by:*

$$S = \{x \in \mathbb{R}, |x| \leq \frac{3\pi}{2}e^{\pi}\}.$$

For simulation of system (27) *we select the initial state* $x(0) = 1$. *The result of simulation is depicted in Figure* 4

4 Conclusion

We have introduced some new conditions for global uniform boundedness of nonlinear systems of differential equations. A converse theorem has been established to guarantee the global uniform h-stability of a nonlinear system when its linearization has a global uniform h-stability equilibrium point. One of the main interests of this paper is that it serves to establish that property for nonlinear perturbed systems when global uniform h-stability of the nominal system has been showed with Lyapunov theory. We have illustrated this use in the global uniform h-stabilization for control systems. To guarantee that the closed-loop system is globally uniformly h-stable, a continuous linear controller has been provided and sufficient conditions has been given. The effectiveness of the conditions obtained in this paper has been verified in some numerical examples.

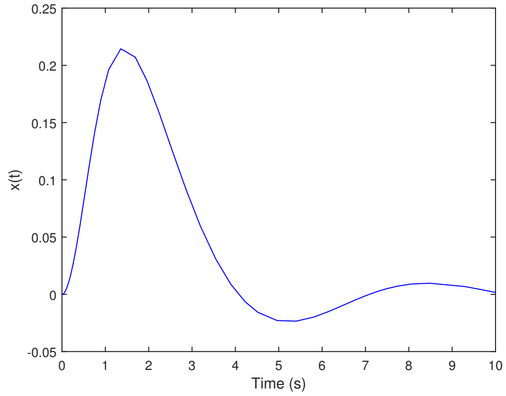

Figure 4: *The trajectory of the state $x(t)$ of system* (27).

Annex

Proof of Lemma 3. We write (6) as

$$\dot{\varphi}(t) - \varpi(t)\varphi(t) \leq \rho(t), \qquad \forall\, t \geq t_0.$$

On the other hand,

$$\frac{d}{ds}\left(\varphi(s)\exp\left(-\int_{t_0}^{s}\varpi(\tau)d\tau\right)\right) \leq \rho(s)\exp\left(-\int_{t_0}^{s}\varpi(\tau)d\tau\right), \quad s \geq t_0.$$

Thus, for all $t_0 \in \mathbb{R}_+$

$$\int_{t_0}^{t}\frac{d}{ds}\left(\varphi(s)\exp\left(-\int_{t_0}^{s}\varpi(\tau)d\tau\right)\right) \leq \int_{t_0}^{t}\rho(s)\exp\left(-\int_{t_0}^{s}\varpi(\tau)d\tau\right)ds, \qquad \forall\, t \geq t_0,$$

which implies

$$\varphi(t)\exp\left(-\int_{t_0}^{t}\varpi(\tau)d\tau\right) - \varphi(t_0) \leq \int_{t_0}^{t}\rho(s)\exp\left(-\int_{t_0}^{s}\varpi(\tau)d\tau\right)ds.$$

Then,

$$\varphi(t) \leq \varphi(t_0)\exp\left(\int_{t_0}^{t}\varpi(\tau)d\tau\right) + \int_{t_0}^{t}\rho(s)\exp\left(\int_{s}^{t}\varpi(\tau)d\tau\right)ds, \qquad \forall\, t \geq t_0.$$

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