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# The tanh-function method and the $(G'/G)$-expansion method for the kinetic McKean system

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**Abstract.** In this paper, we consider the discrete kinetic McKean system. The McKean system is the Boltzmann kinetic equation, and for this model momentum and energy are not conserved. For the first time, new traveling wave solutions are found using the tanh-function method, the extended tanh-function method and the $(G'/G)$-expansion method. These methods are a powerful, reliable and effective tool for finding exact solutions to nonlinear partial differential equations in mathematical physics and engineering fields. With the help of computerized symbolic computation, we obtain kink waves, singular kink waves, periodic waves and rational solutions. Similarly, it is possible to find exact solutions for other kinetic models.

**Keywords:** McKean system, tanh-function method, Knudsen number, traveling wave solutions, symbolic computation

# 1 Introduction

We consider the one-dimensional McKean system [2, 13]:

$$\begin{aligned} &\partial_t u + \partial_x u = \frac{1}{\varepsilon}(w^2 - uw), \quad x \in \mathbb{R},\ t > 0, \\ &\partial_t w - \partial_x w = -\frac{1}{\varepsilon}(w^2 - uw). \end{aligned} \tag{1}$$

Here $u = u(x,t), w = w(x,t)$ are the densities of two groups of particles with velocities $c = 1, -1$, $\varepsilon$ is the Knudsen parameter from the kinetic theory of gases. This system describes a monatomic rarefied gas consisting of two groups of particles. The McKean system is a non-integrable system, i.e. the Painlevé test is not applicable. The interaction is as follows. The McKean system describes particles of two groups, namely, the first group of particles moves at a unit speed along the axis Ox, and the second group moves at a unit speed in the opposite direction. Particles of the first and second groups colliding cause a reaction that transfers into two particles of the second group. In turn, two particles of the second group transfers into particles of the first and second groups.

The main kinetic models are the Carleman, Godunov-Sultangazin, Broadwell, McKean systems (see [1, 2, 13, 21, 22, 23]). These models arise in the kinetic theory of gases, chemical kinetics, in various fields of science and technology [10, 11]. In particular, the Carleman system arises in autocatalysis [12]. There are many methods for finding exact solutions to nonlinear partial differential equations such as the homogeneous balance method [14], the Exp-function method [15], the Jacobi Elliptic function expansion method [16], the tanh-method and extended tanh-method [17, 18], the sine-cosine method [19] and many others. For Broadwell-type models in [20], solutions were obtained using the truncated Painlevé expansions. For the Carleman system in [3, 9], solutions were obtained using the generalized Bernoulli sub-ODE method and the $(G'/G)$-expansion method. The McKean system has been little studied. The McKean system (1) was studied recently in [5, 13]. Here a self-similar solution was found, as well as a solution by means of the truncated Painlevé expansion. In [3, 4, 5, 9, 20], traveling wave solutions can take both positive and negative values. This is some disadvantage from a physical point of view. Despite this, we will obtain new exact solutions for (1) by using the tanh-function method, the extended tanh-function method and the $(G'/G)$-expansion method. It should be noted that the sine-cosine method is not applicable to our system.

# 2 Our methods

Now we will describe the main steps of two well-known methods that allow us to find exact solutions.

## 2.1 Review of the tanh-method

Consider a given nonlinear equation

$$E(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2}$$

where $u = u(x,t)$ is an unknown function. To find the travelling wave solution of Eq. (2), we introduce the wave variable $\xi = x - ct, u = U(\xi)$. Then (2) is reduced to the ordinary differential equation:

$$E(U, -cU', U', c^2U'', -cU'', U'', ...) = 0. \tag{3}$$

We introduce a new independent variable

$$Y = \tanh(\mu\xi),$$

leads to the change of derivative,

$$\frac{d}{d\xi} = \mu(1 - Y^2)\frac{d}{dY},$$

where $\mu$ is a real parameter. We apply the following series expansion

$$U(\xi) = S(Y) = \sum_{n=0}^{N} a_n Y^n.$$

A balance procedure determines the degree $N$ of the power series. The coefficients follow from solving a nonlinear algebraic system. For more details, see [6, 17, 18]. The extended tanh-method is defined in a similar way.

## 2.2 Review of the $(G'/G)$-expansion method

We suppose that the solution of (3) can be expressed by a polynomial in the form:

$$U(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ satisfies

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where $a_n, a_{n-1}, ..., a_0, \lambda$ and $\mu$ are constants to be determined later; $a_n \neq 0$. Balancing between the highest order derivatives and nonlinear terms, we find the positive integer $n$. Then substituting (4) into (3), collecting the coefficients of $G'/G$ and equating each coefficient to zero, we can find $a_n, a_{n-1}, ..., a_0$. For more details, see [7, 8].

## 3 The tanh-method for the McKean system

We seek the solution in the following transformation

$$u = U(\xi), w = W(\xi), \xi = x - ct.$$

In this case we have

$$\begin{aligned} U'(1-c) &= \frac{1}{\varepsilon}(W^2 - UW), \\ -W'(1+c) &= -\frac{1}{\varepsilon}(W^2 - UW). \end{aligned} \tag{6}$$

The tanh method admits the use of finite series

$$U(\xi) = S(Y) = \sum_{m=0}^{M} a_m Y^m, W(\xi) = \bar{S}(Y) = \sum_{m=0}^{M_1} b_m Y^m, \tag{7}$$

where $Y = \tanh(\mu\xi)$, $M$ and $M_1$ are positive integers, $\mu$ is a parameter. Substituting (7) into (6) yields

$$\begin{aligned} \mu(1-c)(1-Y^2)\frac{dS}{dY} &= \frac{1}{\varepsilon}(\bar{S}^2 - S\bar{S}), \\ -\mu(1+c)(1-Y^2)\frac{d\bar{S}}{dY} &= -\frac{1}{\varepsilon}(\bar{S}^2 - S\bar{S}). \end{aligned} \tag{8}$$

After substitution of (7) into (8), we balance the highest powers of $Y$. Then we have

$$2 + M - 1 = 2M_1 = M + M_1,$$

$$2 + M_1 - 1 = 2M_1 = M + M_1,$$

so that $M = M_1 = 1$. We get the truncated expansion

$$\begin{aligned} S(Y) &= a_0 + a_1 Y, \\ \bar{S}(Y) &= b_0 + b_1 Y. \end{aligned} \tag{9}$$

Substituting (9) into (8) and collecting the coefficients of $Y$, we have

$$\begin{gathered} -\mu a_1 + \mu c a_1 + \frac{a_1 b_1}{\varepsilon} - \frac{b_1^2}{\varepsilon} = 0, \\ a_1 b_0 + a_0 b_1 - 2 b_0 b_1 = 0, \\ \mu a_1 - \mu c a_1 + \frac{a_0 b_0}{\varepsilon} - \frac{b_0^2}{\varepsilon} = 0 \end{gathered} \tag{10}$$

and

$$\begin{gathered} \mu b_1 + \mu c b_1 - \frac{a_1 b_1}{\varepsilon} + \frac{b_1^2}{\varepsilon} = 0, \\ -a_1 b_0 - a_0 b_1 + 2 b_0 b_1 = 0, \\ -\mu b_1 - \mu c b_1 - \frac{a_0 b_0}{\varepsilon} + \frac{b_0^2}{\varepsilon} = 0. \end{gathered} \tag{11}$$

Solving algebraic equations system with the aid of the Mathematica Package, we have the following solutions:

Case 1.

$$\begin{aligned} a_0 &= \frac{(1-3c)\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2(c-1)\sqrt{c+1}}, b_0 = -\frac{\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2\sqrt{c+1}}, \\ a_1 &= \frac{1}{2}(2+\frac{1}{c}+c)\mu\varepsilon, b_1 = -\frac{(c^2-1)\mu\varepsilon}{2c}, \mu \in \mathbb{R}. \end{aligned}$$

Case 2.

$$\begin{aligned} a_0 &= -\frac{(1-3c)\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2(c-1)\sqrt{c+1}}, b_0 = \frac{\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2\sqrt{c+1}}, \\ a_1 &= \frac{1}{2}(2+\frac{1}{c}+c)\mu\varepsilon, b_1 = -\frac{(c^2-1)\mu\varepsilon}{2c}, \mu \in \mathbb{R}. \end{aligned}$$

For case 1, the kink soliton solution has the form

$$\begin{aligned} u(x,t) &= \frac{(1-3c)\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2(c-1)\sqrt{c+1}} + \frac{1}{2}(2+\frac{1}{c}+c)\mu\varepsilon \tanh\Big(\mu(x-ct)\Big), \\ w(x,t) &= -\frac{\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2\sqrt{c+1}} - \frac{(c^2-1)\mu\varepsilon}{2c}\tanh\Big(\mu(x-ct)\Big). \end{aligned} \tag{12}$$

For case 2, we have the solution

$$u(x,t) = -\frac{(1-3c)\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2(c-1)\sqrt{c+1}} + \frac{1}{2}(2+\frac{1}{c}+c)\mu\varepsilon\tanh\Big(\mu(x-ct)\Big),$$

$$w(x,t) = \frac{\sqrt{\frac{(c-1)^2(c+1)^3}{c^2}}\mu\varepsilon}{2\sqrt{c+1}} - \frac{(c^2-1)\mu\varepsilon}{2c}\tanh\Big(\mu(x-ct)\Big).$$

# 4 The extended tanh-function method

The extended tanh method admits the use of finite series

$$\begin{aligned} U(\xi) = S(Y) &= \sum_{m=0}^{M} a_m Y^m + \sum_{l=1}^{L} b_l Y^{-l}, \\ W(\xi) = \bar{S}(Y) &= \sum_{p=0}^{P} c_p Y^p + \sum_{d=1}^{D} f_d Y^{-d}, \end{aligned} \tag{13}$$

where $M, L, N, D$ are nonnegative integers. Substituting (13) into (8) and balancing the highest, lowest powers of $Y$, we obtain

$$2 + M - 1 = 2P = M + P,$$

$$2 + P - 1 = 2P = M + P$$

and

$$-L - 1 = -2D = -L - D,$$

$$-D - 1 = -2D = -L - D,$$

so that $M = P = N = D = 1$. Then we seek the solution of (6) in the form

$$\begin{aligned} U(\xi) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}, \\ W(\xi) = \bar{S}(Y) = c_0 + c_1 Y + f_1 Y^{-1}. \end{aligned} \tag{14}$$

Substituting (14) into (8) and collecting the coefficients of $Y$, we have

$$\begin{aligned}
&-\mu a_1 + \mu c a_1 + \frac{a_1 c_1}{\varepsilon} - \frac{c_1^2}{\varepsilon} = 0, \\
&a_1 c_0 + a_0 c_1 - 2c_0 c_1 = 0, \\
&b_1 c_0 + a_0 f_1 - 2c_0 f_1 = 0, \\
&-\mu b_1 + \mu c b_1 + \frac{b_1 f_1}{\varepsilon} - \frac{f_1^2}{\varepsilon} = 0, \\
&\frac{a_0 c_0}{\varepsilon} - \frac{c_0^2}{\varepsilon} + \frac{b_1 c_1}{\varepsilon} + \frac{a_1 f_1}{\varepsilon} - \frac{2c_1 f_1}{\varepsilon} + \mu a_1 + \mu b_1 - \mu c a_1 - \mu c b_1 = 0
\end{aligned}$$

and

$$\begin{aligned}
&\mu c_1 + \mu c c_1 - \frac{a_1 c_1}{\varepsilon} + \frac{c_1^2}{\varepsilon} = 0,\\
&- a_1 c_0 - a_0 c_1 + 2c_0 c_1 = 0,\\
&- b_1 c_0 - a_0 f_1 + 2c_0 f_1 = 0,\\
&\mu f_1 + \mu c f_1 - \frac{b_1 f_1}{\varepsilon} + \frac{f_1^2}{\varepsilon} = 0,\\
&- \frac{a_0 c_0}{\varepsilon} + \frac{c_0^2}{\varepsilon} - \frac{b_1 c_1}{\varepsilon} - \frac{a_1 f_1}{\varepsilon} + \frac{2c_1 f_1}{\varepsilon} - \mu c_1 - \mu f_1 - \mu c c_1 - \mu c f_1 = 0.
\end{aligned}$$

We obtain solutions:

Case 1.

$$a_0 = -\frac{(1-3c)f_1}{c-1}, a_1 = 0, b_1 = -\frac{(c+1)}{c-1}f_1,$$
$$c_0 = f_1, c_1 = 0, \mu = \frac{2cf_1}{(1-c^2)\varepsilon}.$$

Case 2.

$$a_0 = \frac{(1-3c)f_1}{c-1}, a_1 = 0, b_1 = -\frac{(c+1)}{c-1}f_1,$$
$$c_0 = -f_1, c_1 = 0, \mu = \frac{2cf_1}{(1-c^2)\varepsilon}.$$

Case 3.

$$a_0 = \frac{2(1-3c)\sqrt{(c+1)f_1^2}}{(c-1)\sqrt{c+1}}, a_1 = -\frac{(c+1)f_1}{c-1}, b_1 = -\frac{(c+1)f_1}{c-1},$$
$$c_0 = -\frac{2\sqrt{(c+1)f_1^2}}{\sqrt{c+1}}, c_1 = f_1, \mu = \frac{2cf_1}{(1-c^2)\varepsilon}.$$

Case 4.

$$a_0 = -\frac{2(1-3c)\sqrt{(c+1)f_1^2}}{(c-1)\sqrt{c+1}}, a_1 = -\frac{(c+1)f_1}{c-1}, b_1 = -\frac{(c+1)f_1}{c-1},$$
$$c_0 = \frac{2\sqrt{(c+1)f_1^2}}{\sqrt{c+1}}, c_1 = f_1, \mu = \frac{2cf_1}{(1-c^2)\varepsilon}.$$

Here $f_1$ is any real number for cases 1-4. For case 1, we have

$$\begin{aligned}
&u(x,t) = -\frac{(1-3c)f_1}{c-1} - \frac{(c+1)}{c-1}f_1 \coth\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big),\\
&w(x,t) = f_1 + f_1 \coth\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big).
\end{aligned} \tag{15}$$

For case 3, we have

$$\begin{aligned} u(x,t) = & \frac{2(1-3c)\sqrt{(c+1)f_1^2}}{(c-1)\sqrt{c+1}} - \frac{(c+1)f_1}{c-1}\tanh\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big) \\ & -\frac{(c+1)f_1}{c-1}\coth\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big), \\ w(x,t) = & -\frac{2\sqrt{(c+1)f_1^2}}{\sqrt{c+1}} + f_1\tanh\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big) \\ & +f_1\coth\Big(\frac{2cf_1}{(1-c^2)\varepsilon}(x-ct)\Big). \end{aligned} \tag{16}$$

## 5 The $(G'/G)$-expansion method

We suppose that

$$\begin{aligned} U(\xi) &= \sum_{i=0}^{n} a_i\Big(\frac{G'}{G}\Big)^i, \\ W(\xi) &= \sum_{i=0}^{m} b_i\Big(\frac{G'}{G}\Big)^i, \end{aligned} \tag{17}$$

where $G = G(\xi)$ satisfies (5). Balancing between $U'$ and $UW$ yields

$$n+1 = n+m, m = 1.$$

Similarly

$$m+1 = n+m, n = 1.$$

Then

$$\begin{aligned} U(\xi) &= a_0 + a_1\Big(\frac{G'}{G}\Big), \\ W(\xi) &= b_0 + b_1\Big(\frac{G'}{G}\Big). \end{aligned} \tag{18}$$

Note that

$$U' = a_1\Big(-\lambda\Big(\frac{G'}{G}\Big) - \mu - \Big(\frac{G'}{G}\Big)^2\Big),$$

$$W^2 = b_0^2 + 2b_0b_1\Big(\frac{G'}{G}\Big) + b_1^2\Big(\frac{G'}{G}\Big)^2,$$

$$UW = a_0b_0 + a_0b_1\Big(\frac{G'}{G}\Big) + b_0a_1\Big(\frac{G'}{G}\Big) + a_1b_1\Big(\frac{G'}{G}\Big)^2.$$

Substituting (18) into (6) and collecting the coefficients of $G'/G$, we have

$$\begin{gathered}\frac{a_0b_0}{\varepsilon}-\frac{b_0^2}{\varepsilon}-\mu a_1+\mu c a_1=0,\\ \frac{a_1b_0}{\varepsilon}+\frac{a_0b_1}{\varepsilon}-\frac{2b_0b_1}{\varepsilon}-\lambda a_1+\lambda c a_1=0,\\ \frac{a_1b_1}{\varepsilon}-\frac{b_1^2}{\varepsilon}-a_1+ca_1=0\end{gathered}\tag{19}$$

and

$$\begin{gathered}-\frac{a_0b_0}{\varepsilon}+\frac{b_0^2}{\varepsilon}+\mu b_1+\mu c b_1=0,\\ -\frac{a_1b_0}{\varepsilon}-\frac{a_0b_1}{\varepsilon}+\frac{2b_0b_1}{\varepsilon}+\lambda b_1+\lambda c b_1=0,\\ -\frac{a_1b_1}{\varepsilon}+\frac{b_1^2}{\varepsilon}+b_1+cb_1=0.\end{gathered}\tag{20}$$

Solving (19) and (20) by the Wolfram Mathematica gives

Case 1.

$$a_0=\frac{-\varepsilon\lambda(c+1)^2(c-1)c+\sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(1-c)},$$

$$b_0=\frac{\varepsilon\lambda c(1-c^2)+\sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2},$$

$$a_1=\frac{(c+1)^2\varepsilon}{2c}, b_1=\frac{\varepsilon(1-c^2)}{2c}.$$

Case 2.

$$a_0=\frac{\varepsilon\lambda(c+1)^2(c-1)c+\sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(c-1)},$$

$$b_0=-\frac{\varepsilon\lambda c(c^2-1)+\sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2},$$

$$a_1=\frac{(c+1)^2\varepsilon}{2c}, b_1=\frac{\varepsilon(1-c^2)}{2c}.$$

Solving (5), we have for $\lambda^2-4\mu>0$

$$\frac{G'}{G}=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)-\frac{\lambda}{2},\tag{21}$$

where $C_1, C_2$ are arbitrary constants. When $\lambda^2 - 4\mu < 0$, we have

$$\frac{G'}{G} = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right) - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right) + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)}\right) - \frac{\lambda}{2}, \quad (22)$$

When $\lambda^2 - 4\mu = 0$, we have

$$\frac{G'}{G} = \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}. \quad (23)$$

Case 1. When $\lambda^2 - 4\mu > 0$, we have

$$u(\xi) = \frac{-\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(1-c)} + \frac{(c+1)^2\varepsilon}{2c}A, \quad (24)$$

$$w(\xi) = \frac{\varepsilon\lambda c(1-c^2) + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2} + \frac{(1-c^2)\varepsilon}{2c}A, \quad (25)$$

where $A = \frac{G'}{G}$ is determined by (21), $\xi = x - ct$.

When $\lambda^2 - 4\mu < 0$, we have

$$u(\xi) = \frac{-\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(1-c)} + \frac{(c+1)^2\varepsilon}{2c}B,$$

$$w(\xi) = \frac{\varepsilon\lambda c(1-c^2) + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2} + \frac{(1-c^2)\varepsilon}{2c}B,$$

where $B = \frac{G'}{G}$ is determined by (22), $\xi = x - ct$.

When $\lambda^2 - 4\mu = 0$, we have

$$u(\xi) = \frac{-\varepsilon\lambda(c+1)^2(c-1)c}{4c^2(1-c)} + \frac{(c+1)^2\varepsilon}{2c}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right),$$

$$w(\xi) = \frac{\varepsilon\lambda c(1-c^2)}{4c^2} + \frac{(1-c^2)\varepsilon}{2c}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right),$$

where $\xi = x - ct, C_1, C_2$ are arbitrary constants.

Case 2. When $\lambda^2 - 4\mu > 0$, we have

$$u(\xi) = \frac{\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(c-1)} + \frac{(c+1)^2\varepsilon}{2c}A,$$

$$w(\xi) = -\frac{\varepsilon\lambda c(c^2-1) + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2} + \frac{\varepsilon(1-c^2)}{2c}A,$$

where $A = \frac{G'}{G}$ is determined by (21), $\xi = x - ct$.

When $\lambda^2 - 4\mu < 0$, we have

$$u(\xi) = \frac{\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(c-1)} + \frac{(c+1)^2\varepsilon}{2c}B,$$

$$w(\xi) = -\frac{\varepsilon\lambda c(c^2-1) + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}}{4c^2} + \frac{\varepsilon(1-c^2)}{2c}B,$$

where $B = \frac{G'}{G}$ is determined by (22), $\xi = x - ct$.

When $\lambda^2 - 4\mu = 0$, we have

$$u(\xi) = \frac{\varepsilon\lambda(c+1)^2c}{4c^2} + \frac{(c+1)^2\varepsilon}{2c}\Big(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\Big),$$

$$w(\xi) = -\frac{\varepsilon\lambda c(c^2-1)}{4c^2} + \frac{\varepsilon(1-c^2)}{2c}\Big(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\Big),$$

where $\xi = x - ct, C_1, C_2$ are arbitrary constants. Thus, we obtain three types of solutions by the $(G'/G)$-expansion method.

**Remark.** Note that the solutions are related. Consider $c > 1$. The solutions (12), (15) can be written in the form

$$\begin{gathered} u(x,t) = \frac{(1-3c)(c+1)\mu\varepsilon}{2c} + \frac{(c+1)^2}{2c}\mu\varepsilon\tanh\Big(\mu\xi - \xi_0\Big), \xi = x - ct, \\ w(x,t) = -\frac{(c-1)(c+1)\mu\varepsilon}{2c} - \frac{(c^2-1)\mu\varepsilon}{2c}\tanh\Big(\mu\xi - \xi_0\Big). \end{gathered} \tag{26}$$

Assuming $\xi_0 = 0$ in (26), we obtain the solution (12). If $\xi_0 = \frac{i\pi}{2}, \mu = \frac{2cf_1}{(1-c^2)\varepsilon}$, we have the solution (15). Similarly, one can show the relationship of the formulas (15) and (16) using $\tanh(k\xi) + \coth(k\xi) = 2\coth(2k\xi)$. Also note that if $C_1 = 0, C_2 = 1$, we can reduce (24) to (12)

$$\begin{gathered} u(\xi) = \frac{-\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(1-c)} + \\ + \frac{(c+1)^2\varepsilon}{2c}\Big(\frac{1}{2}\sqrt{\lambda^2-4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi) - \frac{\lambda}{2}\Big) = \\ = b_0 + b_1\tanh(\mu^*\xi), \end{gathered}$$

where

$$b_0 = \frac{-\varepsilon\lambda(c+1)^2(c-1)c + \sqrt{(c^3-c)^2\varepsilon^2(\lambda^2-4\mu)}(1-3c)}{4c^2(1-c)} - \frac{(c+1)^2\varepsilon}{2c}\frac{\lambda}{2},$$

$$b_1 = \frac{(c+1)^2\varepsilon}{2c}\frac{\sqrt{\lambda^2-4\mu}}{2}, \mu^* = \frac{1}{2}\sqrt{\lambda^2-4\mu}.$$

Similarly, we can get for $w(x,t)$.

**Example.** Consider the McKean system

$$\begin{aligned} &\partial_t u + \partial_x u = w^2 - uw, \quad x \in \mathbb{R},\ t > 0, \\ &\partial_t w - \partial_x w = uw - w^2, \end{aligned} \tag{27}$$

with the boundary conditions

$$u(+\infty,t) = -2, w(+\infty,t) = 0.$$

The system (27) has the following analytical solution

$$\begin{aligned} u(x,t) &= -\frac{7}{4} - \frac{1}{4}\tanh(2t+x), \\ w(x,t) &= -\frac{3}{4} + \frac{3}{4}\tanh(2t+x). \end{aligned}$$

## Conclusion

In this work, we have found the exact travelling wave solutions of the kinetic McKean system by using the tanh–function method, the extended tanh–function method and the $(G'/G)$ method. All of the above solutions have been verified using the Mathematica package. In the future the solutions of the remaining kinetic models will be found.

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