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Ordinary differential equations

Inverse problem for incomplete Sobolev type equation of higher order

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Abstract. The article is devoted to the study of the inverse problem for a high-order Sobolev type equation. Mathematical models based on such equations describe various problems of hydrodynamics and elasticity theory. The main result of the article is to find sufficient conditions for the existence and uniqueness of a solution to the original problem. At first using the theory of relatively bounded operators, the original problem is reduced to the equivalent system of two problems, which are usually called regular and singular. Thus, the solution to the original problem is represented as the sum of the solutions of these two problems. Further, the regular problem is reduced to a first-order equation. Then, by the method of successive approximations, the required smoothness for the function q is achieved by obtaining sufficient conditions for the existence and uniqueness of a solution to the regular problem. To find a solution of the singular problem, the phase space method and the results of the study of the regular problem are used. The results obtained can be applied to the study of various mathematical models, such as the model of sound waves in smectics or the model of oscillation of a rotating viscous fluid taking into account the viscosity coefficient.

Keywords: Sobolev type equation, inverse problem, method of successive approximations, relative boundedness of operator.

1 Introduction

Let $\mathcal{U}, \mathcal{F}, \mathcal{Y}$ be Banach spaces, operators $L,\ M \in \mathcal{L}(\mathcal{U};\mathcal{F})$, $C \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, $\ker\ L \neq \{0\}$, given functions $\chi : [0,T] \to \mathcal{L}(\mathcal{Y};\mathcal{F})$, $f : [0,T] \to \mathcal{F}$, $\Psi : [0,T] \to \mathcal{Y}$. Consider the following problem

$$Lv^{(n)}(t) = Mv(t) + \chi(t)q(t) + f(t), \quad t \in [0,T], \tag{1}$$

$$v(0) = v_0,\ \dots,\ v^{(n-1)}(0) = v_{n-1}, \tag{2}$$

$$Cv(t) = \Psi(t), \tag{3}$$

where (1) is an incomplete, inhomogeneous high-order Sobolev type equation, (2) is the Cauchy condition, and (3) is the overdetermination condition. The problem of finding a pair of functions $v \in C^n([0,T];\mathcal{U})$ and $q \in C^1([0,T];\mathcal{Y})$ from relations (1) – (3) is called the inverse problem.

A large number of works is an devoted to the study of Sobolev type equations [1, 3, 4, 5, 6, 7, 11, 12]. The article [4] gives sufficient conditions for the existence of positive solutions to both the Showalter – Sidorov problem and the Cauchy problem for an abstract linear equation of the first order ($n = 1$). The Cauchy problem for a nonlinear Sobolev type differential equation in the space of continuous functions was studied in [11]. The paper [12] is devoted to initial-boundary value problems for a Sobolev type equation with a Gerasimov-Caputo fractional derivative with a memory effect.

Sobolev type equations find their application in mathematical modeling [3, 5, 6]. In [6], the sufficient conditions for the solvability of a high-order semilinear Sobolev type mathematical model were obtained. Consideration of optimal equation problems for linear Sobolev type models with the initial Cauchy condition is presented in [5].

Recently, a large number of works [3, 7, 8, 9, 10] are devoted to the study of inverse problems in various aspects. The question of well-posedness of the inverse problem of determining the source function for a second order quasilinear parabolic system in Sobolev spaces was considered in [9]. The article [8] is devoted to establishing the uniqueness of the solution to the coefficient inverse problem of wave tomography in the non-overdetermined setting. The solvability of the inverse problem for first-order partial differential equations was studied in [10]. In [7] the solvability of several inverse problems belonging to the same class of Sobolev type equations is studied. Work [3] establishes the unique solvability of the inverse problem for a complete second-order Sobolev type equation.

2 Preliminary Information

To find a pair of functions $v(t)$ and $q(t)$ we use the results obtained in the research of higher order Sobolev type equations [1].

Definition 1 *The sets*

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\}$$

and $\sigma^L(M) = \bar{\mathbb{C}} \backslash \rho^L(M)$ *will be called the* L*-resolvent set and the* L*-spectrum of the operator* M*, respectively.*

Definition 2 *The operator-function* $(\mu L - M)^{-1}$ *with domain* $\rho^L(M)$ *will be called the* L*-resolvent of the operator* M*.*

Definition 3 *The operator* M *is called* (L,σ)*-bounded if*

$$\exists a \in \mathbb{R}_+ \ \ \forall \mu \in \mathbb{C} \ \ (|\mu| > a) \Rightarrow \big((\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\big)\,.$$

Lemma 1 *Let the operator* M *be* (L,σ)*-bounded. Then the operators*

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} (\mu L - M)^{-1} L d\mu \in \mathcal{L}(\mathcal{U}),$$

$$Q = \frac{1}{2\pi i}\int\limits_{\gamma} L(\mu L - M)^{-1} d\mu \in \mathcal{L}(\mathcal{F})$$

are projectors. Here $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Put $\mathcal{U}^0 = \ker P$, $\mathcal{F}^0 = \ker Q$, $\mathcal{U}^1 = \operatorname{im} P$, $\mathcal{F}^1 = \operatorname{im} Q$. From the previous Lemma it follows that $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$. Let $L_k(M_k)$ denote the restriction of the operator $L(M)$ onto $\mathcal{U}^k$, $k = 0, 1$.

Theorem 1 *[1] Let the operator* M *be* (L,σ)*-bounded. Then the actions of the operators split:*

(i) $L_k \in \mathcal{L}(\mathcal{U}^k;\mathcal{F}^k)$, $k = 0, 1$;

(ii) $M_k \in \mathcal{L}(\mathcal{U}^k;\mathcal{F}^k)$, $k = 0, 1$;

(iii) there exists an operator $(L_1)^{-1} \in \mathcal{L}(\mathcal{F}^1;\mathcal{U}^1)$;

(iv) there exists an operator $(M_0)^{-1} \in \mathcal{L}(\mathcal{F}^0;\mathcal{U}^0)$.

Denote $H = M_0^{-1} L_0 \in \mathcal{L}(\mathcal{U}^0)$, $S = L_1^{-1} M_1 \in \mathcal{L}(\mathcal{U}^1)$.

Definition 4 *The point ∞ is called*

(i) a removable singular point of the L-resolvent of operator M, if $H \equiv \mathbb{O}$;

(ii) a pole of order $p \in \mathbb{N}$ of the L-resolvent of operator M, if $H^p \neq \mathbb{O}$, but $H^{p+1} \equiv \mathbb{O}$;

(iii) an essentially singular point of the L-resolvent of operator M, if $H^k \neq \mathbb{O}$ for any $k \in \mathbb{N}$.

Remark 1 *If the operator M is (L, σ)-bounded, and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of the L-resolvent of the operator M, then the operator M is called (L, p)-bounded.*

3 Reduction of the Initial Inverse Problem

Let the operator M be (L, p)-bounded, then $v(t)$ can be represented as $v(t) = Pv(t) + (I - P)v(t)$. Denote $Pv(t) = u(t), (I - P)v(t) = \omega(t)$. Suppose that $\mathcal{U}^0 \subset \ker C$. Then, by virtue of Theorem 1 and Lemma 1, problem (1) – (3) is equivalent to the problem of finding the functions $u \in C^n([0, T]; \mathcal{U}^1)$, $\omega \in C^n([0, T]; \mathcal{U}^0)$, $q \in C^1([0, T]; \mathcal{Y})$ from the relations

$$u^{(n)}(t) = Su(t) + (L_1)^{-1}Q\chi(t)q(t) + (L_1)^{-1}Qf(t), \tag{4}$$

$$u(0) = u_0, \ ..., \ u^{(n-1)}(0) = u_{n-1}, \tag{5}$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \tag{6}$$

$$H\omega^{(n)}(t) = \omega(t) + (M_0)^{-1}(I - Q)\chi(t)q(t) + (M_0)^{-1}(I - Q)f(t), \tag{7}$$

$$\omega(0) = \omega_0, \ ..., \ \omega^{(n-1)}(0) = \omega_{n-1}, \tag{8}$$

where $u_0 = Pv_0, \ ..., \ u_{n-1} = Pv_{n-1}$, $\omega_0 = (I - P)v_0, \ ..., \omega_{n-1} = (I - P)v_{n-1}$, $t \in [0, T]$. The inverse problem (4) – (6) is called regular, and problem (7), (8) is called singular.

4 Solution of the Regular Inverse Problem

Rewrite problem (4) – (6) in the notation [2]. Let $\mathcal{X} = \mathcal{U}^1$, operators $S \in \mathcal{L}(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, operator-function $\Phi : [0, T] \to \mathcal{L}(\mathcal{Y}; \mathcal{X})$, functions $h : [0, T] \to \mathcal{X}$, $\Psi : [0, T] \to \mathcal{Y}$

$$u^{(n)}(t) = Su(t) + \Phi(t)q(t) + h(t), \ t \in [0, T], \tag{9}$$

$$u(0) = u_0, \ ..., \ u^{(n-1)}(0) = u_{n-1}, \tag{10}$$

$$Cu(t) = \Psi(t). \tag{11}$$

Theorem 2 *Let the operator M be (L,p)-bounded, moreover, $C \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\Phi \in C^1([0,T];\mathcal{L}(\mathcal{Y};\mathcal{X}))$, $h \in C^1([0,T];\mathcal{X})$, $\Psi \in C^{n+1}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ the operator $C\Phi(t)$ be invertible and $(C\Phi)^{-1} \in C^1([0,T];\mathcal{L}(\mathcal{Y}))$. If the compatibility condition $Cu_{n-1} = \Psi^{(n-1)}(0)$ is satisfied, then there exists a unique solution $q \in C^1([0,T];\mathcal{Y})$, $u \in C^n([0,T];\mathcal{X})$ to the inverse problem (9) – (11).*

Proof. Reduce problem (9) – (11) to the problem for the first order equation:

$$z'(t) = Az(t) + Q(t)q(t) + H(t), \quad t \in [0,T], \tag{12}$$

$$z(0) = z_0, \tag{13}$$

$$Bz(t) = \bar{\Psi}(t), \tag{14}$$

where $z(t) = \begin{pmatrix} u(t) \\ \vdots \\ u^{(n-2)}(t) \\ u^{(n-1)}(t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \\ S & 0 & \dots & 0 \end{pmatrix}$, $Q(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi(t) \end{pmatrix}$,

$H(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h(t) \end{pmatrix}$, $z(0) = \begin{pmatrix} u(0) \\ \vdots \\ u^{(n-2)}(0) \\ u^{(n-1)}(0) \end{pmatrix}$, $z_0 = \begin{pmatrix} u_0 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}$,

$B = \begin{pmatrix} 0 & \dots & 0 & C \end{pmatrix}$, $\bar{\Psi}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Psi^{(n-1)}(t) \end{pmatrix}$.

Put $R(t) = -(C\Phi(t))^{-1}$. Therefore, all the conditions of Theorem 6.2.3 from [2], are fulfilled, and the function $q(t)$ satisfies the integral equation

$$q(t) = q_0(t) + R(t)CS\int_0^t V_{1\ n}(t-s)\Phi(s)q(s)ds, \tag{15}$$

where

$$q_0(t) = -R(t)\Big(\Psi^{(n)}(t) - CSV_{1\ 1}(t)u_0 - CSV_{1\ 2}(t)u_1 - ... -$$

$$-CSV_{1\ n}(t)u_{n-1} - CS\int\limits_0^t V_{1\ n}(t-s)h(s)ds - Ch(t)\Big).$$

Thus, there exists a unique solution $q \in C^1([0,T];\mathcal{Y})$, $z \in C^1([0,T];\mathcal{X}^n)$ to the inverse problem (12) – (14). And we obtain that the solution to the regular inverse problem (9) – (11) exists and is unique, with $q \in C^1([0,T];\mathcal{Y})$, $u \in C^n([0,T];\mathcal{X})$. $\square$

In order to obtain a solution to the singular problem, we need a greater smoothness of the function q from the solution of the regular problem, than class $C^1([0,T];\mathcal{Y})$. Next, we need the following Lemma from [3].

Lemma 2 *Let $l \in \mathbb{N}$, $V \in C^{l-1}([0,T];\mathcal{L}(\mathcal{X}))$, $g \in C^l([0,T];\mathcal{X})$. Then*

$$\left(\int_0^t V(t-s)g(s)ds\right)^{(l)} = \sum_{k=0}^{l-1} V^{(l-k-1)}(t)g^{(k)}(0) + \int_0^t V(t-s)g^{(l)}(s)ds.$$

The following theorem provides sufficient conditions for the existence of solution $q \in C^{n(p+1)}([0,T],\mathcal{Y})$ of the regular problem.

Theorem 3 *Let the operator M be (L,p)-bounded, $p \in \mathbb{N}_0$, moreover, $C \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\Phi \in C^{n(p+1)}([0,T];\mathcal{L}(\mathcal{Y};\mathcal{X}))$, $h \in C^{n(p+1)}([0,T];\mathcal{X})$, $\Psi \in C^{n(p+2)}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ operator $C\Phi(t)$ be invertible, with $(C\Phi)^{-1} \in C^{n(p+1)}([0,T];\mathcal{L}(\mathcal{Y}))$ and the compatibility condition $Cu_{n-1} = \Psi^{(n-1)}(0)$ be satisfied for some $u_{n-1} \in \mathcal{U}^1$. Then there exists a unique solution of (9) – (11). Moreover $q \in C^{n(p+1)}([0,T];\mathcal{Y})$.*

Proof. Write the propagators of the homogeneous equation (9) in a matrix, defining the resolving group of homogeneous equation (12)

$$V(t) = \begin{pmatrix} V_{1\ 1}(t) & V_{1\ 2}(t) & \dots & V_{1\ n-1}(t) & V_{1\ n}(t) \\ V_{2\ 1}(t) & V_{2\ 2}(t) & \dots & V_{2\ n-1}(t) & V_{2\ n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{n-1\ 1}(t) & V_{n-1\ 2}(t) & \dots & V_{n-1\ n-1}(t) & V_{n-1\ n}(t) \\ V_{n\ 1}(t) & V_{n\ 2}(t) & \dots & V_{n\ n-1}(t) & V_{n\ n}(t) \end{pmatrix} =$$

$$= \frac{1}{2\pi i} \int\limits_{\gamma} (\mu^n L - M)^{-1} \begin{pmatrix} \mu^{n-1}L & \mu^{n-2}L & \dots & \mu L & \mathbb{I} \\ M & \mu^{n-1}L & \dots & \mu^2 L & \mu\mathbb{I} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu^{n-3}M & \mu^{n-4}M & \dots & \mu^{n-1}L & \mu^{n-2}\mathbb{I} \\ \mu^{n-2}M & \mu^{n-3}M & \dots & M & \mu^{n-1}\mathbb{I} \end{pmatrix} e^{\mu t} d\mu,$$

where $\mathbb{I}$ is the identity operator.

Earlier, in the proof of Theorem 2 it was established that the function $q(t)$ satisfies integral equation (15). Take the natural number $l \leq n(p+1)$. Assuming that $q \in C^l([0,T];\mathcal{Y})$ by Lemma 2 we obtain the equality

$$q^{(l)}(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) \left(\Phi q\right)^{(m)}(0) +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int\limits_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) q^{(l-k-m)}(s) ds,$$

where $C_l^k = \frac{l!}{k!(l-k)!}$, $C_l^{k,m} = \frac{l!}{k!m!(l-k-m)!}$ and

$$q_0^{(l)}(t) = -\sum_{k=0}^{l} C_l^k R^{(k)}(t) \Bigg(\Psi^{(l-k+n)}(t) - CSV_{1\ 1}^{(l-k)}(t) u_0 -$$

$$-CSV_{1\ 2}^{(n-k)}(t) u_1 - \dots - CSV_{1\ n}^{(n-k)}(t) u_{n-1} - CS \int\limits_0^t V_{1\ n}(t-s) h^{(l-k)}(s) ds -$$

$$-Ch^{(l-k)}(t) \Bigg) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) h^{(m)}(0)$$

exists due to the conditions of the theorem for $l = 0, 1, \dots, n(p+1)$.

Show that $q \in C^{n(p+1)}([0,T],\mathcal{Y})$, for this purpose denote $r_0 = q_0(0)$, and for $l = 1, 2, \dots, n(p+1)$ determine the following values

$$r_l = q_0^{(l)}(0) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(0) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(0) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j}.$$

Consider the system of integral equations

$$\tilde{q}_0(t) = q_0(t) + R(t) CS \int\limits_0^t V_{1\ n}(t-s) \Phi(s) \tilde{q}_0(s) ds,$$

$$\tilde{q}_l(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m}(s) ds,$$

$$l = 1, 2, ..., n(p+1). \qquad (16)$$

Reduce (16) to the Volterra equation of the second kind

$$g(t) = g_0(t) + \int_0^t K(t,s) g(s) ds$$

on the space $(C([0,T];\mathcal{Y}))^{n(p+1)+1}$ with a matrix operator function $K(t,s)$ given on the triangle $\Delta = \{(t,s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s \leq t\}$. By virtue of the continuity of all data of system (16), it has a unique solution

$$(\tilde{q}_0, \tilde{q}_1, ..., \tilde{q}_{n(p+1)}) \in (C([0,T];\mathcal{Y}))^{n(p+1)+1}.$$

This solution will be the limit of the sequence of approximations

$$\tilde{q}_{0,i}(t) = q_0(t) + R(t) CS \int_0^t V_{1\ n}(t-s) \Phi(s) \tilde{q}_{0,i-1}(s) ds,$$

$$\tilde{q}_{l,i}(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m,i-1}(s) ds,$$

$$l = 1, 2, ..., n(p+1), \ i \in \mathbb{N}, \qquad (17)$$

which for $i \to \infty$ on the interval $[0,T]$ converge uniformly to the functions $\tilde{q}_l$, $l = 0, 1, ..., n(p+1)$. Set the initial approximation $\tilde{q}_{l,0} \equiv 0, \ l = 0, 1, ..., n(p+1)$, then $\tilde{q}_{l+1,0} = \tilde{q}'_{l,0}$, $l = 0, 1, ..., n(p+1) - 1$. In addition, from (17) it follows that

$$\tilde{q}_{l,i}(0) = r_l, \quad l = 0, 1, ..., n(p+1), \quad i \in \mathbb{N}. \qquad (18)$$

Assume that for all $\tau = 1, 2, ..., i$ the equalities $\tilde{q}_{l+1,\tau}(t) = \tilde{q}'_{l,\tau}(t)$, $l = 0, 1, ..., n(p+1) - 1$ are true. Then, using Lemma 2 and equalities (18) we obtain

$$\frac{d}{dt}\left(\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m,i}(s) ds \right) =$$

$$= \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k+1)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m,i}(s) ds +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS V_{1\ n}(t) \Phi^{(m)}(0) \tilde{q}_{l-k-m,i}(0) +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m+1)}(s) \tilde{q}_{l-k-m,i}(s) ds +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m+1,i}(s) ds =$$

$$= \sum_{k=1}^{l+1} \sum_{m=0}^{l-k+1} C_l^{k-1,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m+1,i}(s) ds +$$

$$+ \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS V_{1\ n}(t) \sum_{m=0}^{l-k} C_{l-k}^m \Phi^{(m)}(0) r_{l-k-m} +$$

$$+ \sum_{k=0}^{l} \sum_{m=1}^{l-k+1} C_l^{k,m-1} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m+1,i}(s) ds +$$

$$+ \sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m+1,i}(s) ds. \qquad (19)$$

Denote

$$a_{k,m} = R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s) \Phi^{(m)}(s) \tilde{q}_{l-k-m+1,i}(s) ds, \ \ l = 2, 3, ..., n(p+1).$$

Taking into account the equalities

$$C_l^k + C_l^{k-1} = C_{l+1}^k, \quad C_l^{k,m} + C_l^{k-1,m} + C_l^{k,m-1} = C_{l+1}^{k,m},$$

we get

$$\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} a_{k,m} + \sum_{k=1}^{l+1}\sum_{m=0}^{l-k+1} C_l^{k-1,m} a_{k,m} + \sum_{k=0}^{l}\sum_{m=1}^{l-k+1} C_l^{k,m-1} a_{k,m} =$$

$$= \left(\sum_{k=1}^{l}\sum_{m=1}^{l-k} C_l^{k,m} a_{k,m} + \sum_{k=1}^{l} C_l^{k,0} a_{k,0} + \sum_{m=0}^{l} C_l^{0,m} a_{0,m}\right) + \left(\sum_{k=1}^{l}\sum_{m=1}^{l-k} C_l^{k-1,m} a_{k,m} +\right.$$

$$\left. + \sum_{k=1}^{l} C_l^{k-1,0} a_{k,0} + \sum_{k=1}^{l} C_l^{k-1,l-k+1} a_{k,l-k+1} + C_l^{l,0} a_{l+1,0}\right) +$$

$$+ \left(\sum_{k=1}^{l}\sum_{m=1}^{l-k} C_l^{k,m-1} a_{k,m} + \sum_{m=1}^{l+1} C_l^{0,m-1} a_{0,m} + \sum_{k=1}^{l} C_l^{k,l-k} a_{k,l-k+1}\right) =$$

$$= \sum_{k=1}^{l}\sum_{m=1}^{l-k} C_{l+1}^{k,m} a_{k,m} + \sum_{k=1}^{l} C_{l+1}^{k,0} a_{k,0} + \sum_{m=1}^{l} C_{l+1}^{0,m} a_{0,m} + \sum_{k=1}^{l} C_{l+1}^{k,0} a_{k,l-k+1} +$$

$$+ C_l^{0,0} a_{0,0} + C_l^{0,l} a_{0,l+1} + C_l^{l,0} a_{l+1,0} = \sum_{k=0}^{l+1}\sum_{m=0}^{l-k+1} C_{l+1}^{k,m} a_{k,m}. \tag{20}$$

For $l = 0, 1$ fulfilment of (20) can be checked directly.

From (19) and (20) it follows that

$$\frac{d}{dt}\left(\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s)\Phi^{(m)}(s)\tilde{q}_{l-k-m,i}(s)ds\right) =$$

$$= \sum_{k=0}^{l+1}\sum_{m=0}^{l-k+1} C_{l+1}^{k,m} R^{(k)}(t) CS \int_0^t V_{1\ n}(t-s)\Phi^{(m)}(s)\tilde{q}_{n-k-m+1,i}(s)ds +$$

$$+ \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS V_{1\ n}(t) \sum_{m=0}^{l-k} C_{l-k}^m \Phi^{(m)}(0) r_{l-k-m}. \tag{21}$$

Changing the summation indices and re-grading the sums we get

$$\frac{d}{dt}\left(\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j}\right) =$$

$$= \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k+1)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} =$$

$$= \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} +$$

$$+ \sum_{k=1}^{l} C_l^{k-1} R^{(k)}(t) CS \sum_{m=0}^{l-k} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} =$$

$$= \left(\sum_{k=1}^{l-1} C_l^k R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} + \right.$$

$$\left. + C_l^0 R(t) CS \sum_{m=0}^{l-1} V_{1\ n}^{(l-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} \right) +$$

$$+ \left(\sum_{k=1}^{l-1} C_l^{k-1} R^{(k)}(t) CS \sum_{m=0}^{l-k-1} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} + \right.$$

$$+ \sum_{k=1}^{l-1} C_l^{k-1} R^{(k)}(t) CS V_{1\ n}(t) \sum_{j=0}^{l-k} C_{l-k}^j \Phi^{(j)}(0) r_{l-k-j} +$$

$$\left. + C_l^{l-1} R^{(l)}(t) CS V_{1\ n}(t) C_0^0 \Phi(0) r_0 \right) =$$

$$= \sum_{k=0}^{l} C_{l+1}^k R^{(k)}(t) CS \sum_{m=0}^{l-k} V_{1\ n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j \Phi^{(j)}(0) r_{m-j} -$$

$$- \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS V_{1\ n}(t) \sum_{m=0}^{l-k} C_{l-k}^m \Phi^{(m)}(0) r_{l-k-m}. \qquad (22)$$

Differentiating (17), and also using (21) and (22), we obtain the equalities $\tilde{q}'_{l,i+1} = \tilde{q}_{l+1,i+1}$, $l = 0, 1, ..., n(p+1) - 1$.

Thus, the sequence $\tilde{q}_{0,i}$ converges as $i \to \infty$ to the function $\tilde{q}_0$ uniformly on the interval $[0, T]$, and the sequence $\tilde{q}'_{0,i} = \tilde{q}_{1,i}$ converges as $i \to \infty$ to the function $\tilde{q}_1$ uniformly on the segment $[0, T]$. Therefore, the function

$\tilde{q}_0$ is continuously differentiable and $\tilde{q}_0' = \tilde{q}_1$. The equalities of $\tilde{q}_l' = \tilde{q}_{l+1}$, $l = 1, 2, ..., n(p+1) - 1$, are proved in the same way, which implies that $\tilde{q}_0 \equiv q \in C^{n(p+1)}([0, T]; \mathcal{Y})$ and, therefore, $q^{(l)} = \tilde{q}_l$, $l = 1, 2, ..., n(p+1)$. $\square$

5 Solvability of the Original Inverse Problem

Now we can formulate the solvability conditions for the problem (1) – (3).

Theorem 4 *Let the operator M be (L, p)-bounded, $p \in \mathbb{N}_0$, operator $C \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $\mathcal{U}^0 \subset \ker C$, $\chi \in C^{n(p+1)}([0, T]; \mathcal{L}(\mathcal{Y}; \mathcal{F}))$, $f \in C^{n(p+1)}([0, T]; \mathcal{F})$, $\Psi \in C^{n(p+2)}([0, T]; \mathcal{Y})$, for any $t \in [0, T]$ operator $C(L_1)^{-1}Q\chi$ be invertible, with $\left(C(L_1)^{-1}Q\chi\right)^{-1} \in C^{n(p+1)}([0, T]; \mathcal{L}(\mathcal{Y}))$, the condition $Cu_{n-1} = \Psi^{(n-1)}(0)$ be satisfied at some initial value $u_{n-1} \in \mathcal{U}^1$, and the initial values $w_k = (I - P)v_k \in \mathcal{U}^0$ satisfy*

$$w_k = -\sum_{j=0}^{p} H^j (M_0)^{-1} \frac{d^{nj+k}}{dt^{nj+k}} \Big[(I - Q)(\chi(0)q(0) + f(0))\Big], \quad k = 0, 1, ..., n-1.$$

Then there exists a unique solution (v, q) of inverse problem (1) – (3), where $q \in C^{n(p+1)}([0, T]; \mathcal{Y})$, $v = u + w$, whence $u \in C^n([0, T]; \mathcal{U}^1)$ is the solution of (4) – (6), and the function $w \in C^n([0, T]; \mathcal{U}^0)$ is a solution of (7), (8) given by

$$w(t) = -\sum_{j=0}^{p} H^j (M_0)^{-1} \frac{d^{nj}}{dt^{nj}} \Big[(I - Q)(\chi(t)q(t) + f(t))\Big]. \qquad (23)$$

Proof. The conditions of Theorems 2 and 3 are satisfied, and therefore there exists a unique solution (q, u) of problem (4) – (6), where $q \in C^{n(p+1)}([0, T]; \mathcal{Y})$, $u \in C^n([0, T]; \mathcal{U}^1)$.

Using the result of [1] and the required smoothness of the function q, we obtain that there exists a unique solution (7), (8) $w \in C^n([0, T]; \mathcal{U}^0)$, given by (23). $\square$

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