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Applications to physics

Mathematical IQM Gauge Theory for Interaction Processes

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Abstract We provide a mathematical development of generalized gauge theory for interactions of an individual particle represented by a complex energy-density wave-packet Ψ in the Minkowski time-space. So, we obtained a general gauge 4-potential field which determinates the acceleration of this particle, by using the complex phase transformation of Ψ, generated during any interaction process, with local symmetry of the Lagrangian density: the Euler-Lagrange equation derived from this Lagrangian density represents the partial differential equation of motion of Ψ.
This developed mathematical theory is then applied for the famous example of the Aharonov-Bohm effect for the electrons.

Keywords: Interaction Processes, Mathematical gauge theory, Covariant derivative, Local symmetries, Aharonov-Bohm effect.

1 Basic Equations and Interaction Processes in the IQM Theory

Quantum mechanics, based on the Schrödinger equation is an epistemic statistical theory, here denominated as Statistical Quantum Mechanics (SQM), to differentiate it from the new part of the ontological quantum theory, provided in [5, 6] and [7], denominated Individual particles Quantum Mechanics (IQM).

Both of them are necessary components of the quantum theory, as are the Classical Mechanics for Individual objects (ICM),based on the Newton equations, Hamiltonian-Jacobi equations or the Euler-Lagrange equation of motion of individual objects) and the Statistical Classical Mechanics (SCM), based on the Liouville equations.

In the IQM theory there is a deeper specification of the state of the particle, and in this approach to completion provided in [5], these states are specified by the energy-density distributions of a given particle in the Minkowski time-space. Such an ontic state, also not fully accessible (non fully observable by the measurements, and/or with non accessible small compactified higher-dimensions for the electric charge (5th timelike dimension with the coordinate $q_4 = ct_4$) and spin (6th spacelike dimension with the coordinate q_5), for example), has to represent the complete description of an individual elementary particle, in order to be able to compute from it all properties of a particle as its rest-mass, position, speed, momentum, total energy, etc...

It was shown [4, 11, 12, 5, 17] that, generally, any massive particle can be defined in the Minkowski time-space (we will not use the real higherdimensional expressions but only its reduced forms to the 4-D representation) with the signature $(+, -, -, -)$, by the complex wave-packet

$$\Psi = \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T} \tag{1}$$

where $\overrightarrow{\mathbf{r}} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$ (for the 3-D Minkowski space orthonormal basis vectors $\mathbf{e}_j$, with $\mathbf{e}_j \cdot \mathbf{e}_j = -1$ for $1 \le j \le 3$ and $\mathbf{e}_0 \cdot \mathbf{e}_0 = 1$ for the time-coordinate $q_0 = ct$) composed by two sub components: by the shape $\Phi(t, \overrightarrow{\mathbf{r}})$ of particle's body that is a real function which defines the real *rest-mass energy-density* $\Phi_m \equiv \Psi\overline{\Psi} = \Phi^2(t, \overrightarrow{\mathbf{r}}) \ge 0$, and by the de Broglie 'phase (pilot) wave' with phase $\varphi_T(t, \overrightarrow{\mathbf{r}_T}) = -\frac{1}{\hbar}S_{t_0=0}$, where $S_{t_0=0} = \int_{0,\overrightarrow{\mathbf{r}_0}}^{t,\overrightarrow{\mathbf{r}_T}} L(t', \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{v}})dt'$ is the Hamiltonian principal function for the initial particle's position $(t_0, \overrightarrow{\mathbf{r}_0})$ and the current position at $t \ge 0$ (its barycenter) at $\overrightarrow{\mathbf{r}_T}(t) \equiv \frac{1}{\mathbf{1}_\Phi} \int \overrightarrow{\mathbf{r}}\Phi_m(t, \overrightarrow{\mathbf{r}})dV$, and particle's Lagrangian at time t', $L(t', \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{v}}) = -E - \overrightarrow{\mathbf{v}}\,\overrightarrow{\mathbf{p}}$ where E is particle's total energy and $\overrightarrow{\mathbf{p}}$ its canonical (conjugate) momentum, and $\mathbf{1}_\Phi \equiv \int \Phi_m(t, \overrightarrow{\mathbf{r}})dV$ is the particle's invariant energy (equal to rest-mass energy m_0c^2 for massive particles and energy E_0 of a boson, measured in the frame in which massive source of this boson is in rest).

Thus, for a *free* (non accelerated) particle which propagates with constant speed v and momentum p, so that $\overrightarrow{\mathbf{v}}\,\overrightarrow{\mathbf{p}} = -vp$, with barycenter position $\overrightarrow{\mathbf{r}_T}(t) =$

$\overrightarrow{\mathbf{r}_0} + \overrightarrow{\mathbf{v}}t$, we obtain that the phase change linearly in time $t \geq 0$,

$$\varphi_T(t) = \frac{E - pv}{\hbar}t \tag{2}$$

When a particle propagates in the vacuum with constant speed $\overrightarrow{\mathbf{v}}$ it has the time-invariant spherically-symmetric distribution [10], $\Phi_m = \frac{K}{\sqrt{r}}$, where $r = \|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}_T\|$ is the distance from its barycenter $\overrightarrow{\mathbf{r}_T}$, corresponding to particle's hydrostatic equilibrium where each infinitesimal amount of particle's material body $\Phi_m(t, \overrightarrow{\mathbf{r}})$ is in rest w.r.t. particle's barycenter. However, generally, during an acceleration each infinitesimal amount of energy-density $\Phi_m(t, \overrightarrow{\mathbf{r}})$ moves with a different speed $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})$ w.r.t. the group velocity $\overrightarrow{\mathbf{v}}(t) = \frac{d}{dt}\overrightarrow{\mathbf{r}_T}(t) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, with $v = \|\overrightarrow{\mathbf{v}}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$, of particle's energy-density wave-packet and it is shown [5] that is,satisfied the following relationship $\overrightarrow{\mathbf{v}}(t) = \frac{1}{\mathbf{1}_\Phi}\int \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})\Phi_m(t, \overrightarrow{\mathbf{r}})dV$, so we can introduce the variation-velocity of the particle's matter flux $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) - \overrightarrow{\mathbf{v}}(t)$ at each space-time point $(t, \overrightarrow{\mathbf{r}})$ inside particle's matter (where $\Phi_m(t, \overrightarrow{\mathbf{r}}) > 0$). As shown in [5], during an inertial propagation when the particle is in a hydrostatic equilibrium, we have that Φ_m is spherically symmetric around particle's barycenter with $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$ in every point inside particle's matter, so that every infinitesimal amount of Φ_m propagates with the constant wave-packet group velocity $\overrightarrow{\mathbf{v}}$. Only during the particle's accelerations we have that $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \neq 0$, so that particle's body changes dynamically its shape in time.

In the assumption [5] of the topology of the matter of an elementary massive particle, the wave-packet do not undergo a spreading, also when it changes its matter density distribution (i.e., its energy-density Φ_m), and tends to its stable stationary spherically symmetric distribution during inertial propagation in the vacuum. That is, the matter has some internal self-gravitational autocohesive force analogously to the peace of *perfect fluid* in the vacuum, so that at any instance of time, the 3-D space topology of particle's matter distribution, and consequently its compressible energy-density Φ_m is simply connected, closed, continuous and differentiable.

The Lagrangian density $\mathcal{L}$ of a particle [5], is given by

$$\mathcal{L} = \frac{\hbar}{\mathbf{1}_\Phi}(-\frac{\partial \varphi_T}{\partial t}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\partial_0\Psi - \Psi\partial_0\overline{\Psi} - \overline{\Psi}\overrightarrow{\mathbf{w}}\nabla\Psi + \Psi\overrightarrow{\mathbf{w}}\nabla\overline{\Psi})) \tag{3}$$

with previously introduced speed of particle's matter/enegy density $\Phi_m(t, \overrightarrow{\mathbf{r}})$

$$\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \tag{4}$$

Thus, each massive elementary particle satisfies the following conservation laws: Analogously to the Euler first equation of fluid dynamics (continuity equation), which represents the conservation of mass, here we have the analog equation for the conservation of matter (that is of the particle's rest-mass energy),

$$\frac{\partial \Phi_m(t, \overrightarrow{\mathbf{r}})}{\partial t} + \nabla \cdot (\Phi_m(t, \overrightarrow{\mathbf{r}})\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})) = 0 \tag{5}$$

and hence, from the fact that for the real component $\Phi(t, \overrightarrow{\mathbf{r}})$ of the wave-packet $\Psi(t, \overrightarrow{\mathbf{r}})$ in (1), we have that $\Phi_m = \Phi^2$, we obtain its first-order differential equation

$$\frac{\partial \Phi(t, \overrightarrow{\mathbf{r}})}{\partial t} = \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})\nabla\Phi(t, \overrightarrow{\mathbf{r}}) - \frac{1}{2}(\nabla \cdot \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}))\Phi(t, \overrightarrow{\mathbf{r}}) \tag{6}$$

In what follows, for the Cartesian coordinate system, $\nabla = \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z}$ is the gradient (for $x \equiv q_1, y \equiv q_2$ and $z \equiv q_3$) so that the Laplacian is defined by $\triangle = -\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (we are using positive-time metric signature (+,-,-,-))

It holds also for bosons when they become unstable after an initial 'space explosion' and, consequently, assume the massive particle behavior and a finite but non-zero energy-density volume in open 3-D space. We need that the body of the particle Φ_m provides also the physical internal pressure $P(t, \overrightarrow{\mathbf{r}})$ (which is a non-geometrical property) in order to guarantee the hydrostatic equilibrium of the massive particles. The hydrostatic equilibrium of an massive elementary particle demonstrated that the body of this particle Φ_m is a material substance [10], which is fluid and elastic, and which can not be reduced to the time-space geometry.

Hence, in this IQM theory [5] for individual elementary particles based on energy-density wave-packets, the point-like particles are only the stable-state bosons when they propagate with speed of light in the vacuum, and with their energy-density distributed in higher compactified dimensions [6]. In Section 2.7 in [5], dedicated to the 3-D radial expansion of the bosons w.r.t. the direction of particle's propagation, to the tunneling and reflections, has been considered the cylindrical expansion of the massive boson with energy density Φ_m (that is, during the unstable boson's states where the variation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})) \neq 0$.
The real physical hyperdimensional representation of the massless bosons energy-density, for a given instance of time t, for the Euclidean space point $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}_0} + \overrightarrow{\mathbf{c}}t$, is given by $\Phi_m = \Phi^2(\mathbf{r}_4, t_4, q_5) = \sigma(q_5)\ \delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{c}}t)$ where, $\sigma(q_5) = \frac{\mathbf{1}_\Phi}{L}$, with the length of the 6th dimension is $L = 4\pi R_5$, denotes the constant energy-density distributed in 6th dimension with radius R_5.

Thus, by integration of this hyperdimensional density over 6th dimension with coordinate q_5, from [6] we obtain the common point-like 4-D representation of the massless boson's energy-density in the 4-dimensional Minkowski time-space by the Dirac function (note that its pilot-wave phase is $\varphi_T = 0$),

$$\Phi_m(t, \overrightarrow{\mathbf{r}}) = \mathbf{1}_\Phi \delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{c}}t) \tag{7}$$

where $\mathbf{1}_\Phi$ is a constant (equal to a total energy $E = pc$ of a boson in the frame where the source of this boson is in the rest), which is consequently *only mathematically* correct point-like representation of the massless boson. In fact, now the total energy, for a given time-instance t, can be obtained by integration in the ordinary 3-D space, by $E = \int \Phi_m dV = \mathbf{1}_\Phi \int \delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{c}}t) dxdydz = \mathbf{1}_\Phi \cdot 1 = \mathbf{1}_\Phi$. However, it is not physically correct, because we would have an infinity density of energy Φ_m in the single point of the boson's barycenter $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}_0} + \overrightarrow{\mathbf{c}}t$. In such case, the Schwarzschild radius r_s would be greater (or equal) than the radius of the point (boson's barycenter) which is zero, so that the boson would become a black hole, which does not correspond to physical facts. Note that this fact can't happen in the case when we are using the complete 6-D expression for the wave-packet, where $\Phi^2(t, \overrightarrow{\mathbf{r}}, q_5) = \sigma(q_5)\delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{c}}t)$ is also physically composed expression where the energy density is only $\sigma(q_5)$ and there exists only in the 6th dimension and not in M^4, and hence the Dirac 'function' δ in the Minkowski time-space M^4 defines only the *position* of the boson and not its energy-density. In effect, by the integration in 6-D time-space of boson's energy density, its total energy is $E = \int \Phi^2(t, \overrightarrow{\mathbf{r}}, q_5) dq_1 dq_2 dq_3 dq_5 = \int \sigma(q_5)(\delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{c}}t)dV)dq_5 = \int \sigma(q_5)dq_5 = \mathbf{1}_\Phi$.

So, from (7) the volume of the massless boson in the ordinary 3-dimensional space is equal to zero. Only in such conditions a particle can travel with the maximal possible speed of light. But the matter/energy of the boson exists also in such conditions: it is uniformly distributed only in the spacelike sixth dimension (used for the spin) where it propagates with a constant speed v_5. Consequently, the hidden matter of a boson in the compactified higher dimensions results in zero rest-mass in the ordinary flat Minkowski time-space and explains why the boson can propagate with maximal possible speed. During this massless stable-state, the gravitational anti-black-hole barrier acting in the boson's barycenter (in 3-D space) does not permit the leaking of the matter from 6th into ordinary 3-dimensional space.

We consider the vacuum as the perfect 3-dimensional space symmetry where each possible direction of the propagation has the same physical conditions. Thus, the propagation of the particles in the vacuum is inertial and the particle propagates along GR geodesics with constant speed as a stable particle[1]. The asymmetry due to the presence of an infinitesimal inertial particle in flat Minkowski spacetime is purely circumstantial, because the spacetime is considered to be unaffected by the presence of this particle. However, according to general relativity, the presence of any inertial entity disturbs the symmetry of the manifold even more profoundly, because it implies an intrinsic curvature of the spacetime manifold, i.e., the manifold takes on an intrinsic shape that distinguishes the location and rest frame of the particle. Note that, from the fact that the stable bosons have no matter/energy in the ordinary 3-dimensional (open) space, the stable bosons do not generate any local time-space curvature, differently from the fermions. Thus, the local time-space neighborhood of a massless boson is always a locally flat Minkowski time-space, differently from the fermions (and also unstable massive bosons). The fact that the stable bosons have no any curved island-metrics in the ordinary 4-dimensional time-space, results in missing of any physical resistance of the neighborhood time-space to their propagation (differently from the massive particles with energy-density present in the 4-dimensional time-space and, generated from it, curved micro-island metrics). Consequently, they propagate with maximal possible speed in the ordinary 4-dimensional time-space. Thus, the bosons have the point-like 4-dimensional structure corresponding to their position (barycenter), but physically their total energy-density is $\Phi_m(q_5) = \sigma(q_5) = \frac{1_\Phi}{4\pi R_5} = const.$

But there are the situations when a stable, stationary, boson becomes excited for a short interval of time, as in the situations when the *space symmetry* during its propagation is sharply broken. Thus, the time-space boundary conditions for the particle's propagation are drastically changed, by considering that particle's wave-packet is a time-space perturbation and, if such a perturbation meets another perturbation, it changes its form. These events we analyzed in details for the phenomena of refraction and 'wave-behaviors' of an individual photon [5]. In all these situations a photon may change its momentum, direction of propagation and its velocity, without changing its total energy, because these 'interactions' are not based on collisions with another particles (as Compton effects, or annihilations), but on instantaneous 3-D space expansions of their geometric wave-packet scalar field Φ in the presence of a local sharply broken space symmetry. These are strong General Relativity effects correlated with the particle's 'micro-island' curvature metrics, caused by a dynamical changing of the boundary conditions in the local space around this particle.

[1]Such a 3-D space symmetry during an inertial propagation of a massive particle causes a spherical symmetry of its stable energy-density distribution $\Phi_m = \frac{K}{\sqrt{r}}$, for $r \leq r_0$ in a sphere with a radius r_0

Example 1 *The breaking of 3-D pace-symmetry for massless bosons and their transformation into massive bosons (short-range bosons) and derived theory of clouds of short-range photons generated by electrically charged particles that generate the electromagnetic field is provided in [17].*

In order to generate a static electric field around a single charged fermion (as electron, for example), the short-range massive photons must be constantly emitted in all *radial directions from this charged fermion, so that the number $N >> 1$ of emitted photons must be very high at each fixed instance of time. However, in order to maintain the constant energy-level of the charged fermion, it means that practically all of irradiated short-range photons will come back to be absorbed by the same fermion. So, the total emitted energy from a charged fermion corresponds to the total absorbed energy of the same fermion.*
Moreover, in [17] we explained that such a cloud of emmitted/absorbed short-range photons around any charged particle, generates the electromagnetic 4-potential (gauge field),

$$\boldsymbol{A}_4 = (A_0, A_1, A_2, A_3) = (\frac{\phi(t, \overrightarrow{\boldsymbol{r}})}{c}, \overrightarrow{A}(t, \overrightarrow{\boldsymbol{r}})) \tag{8}$$

where $A_0 = \frac{\phi}{c}$ and ϕ is the scalar Coulomb potential derived from the the density ρ of these short-range massive photons, and $\overrightarrow{A} = A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3$ is a 3-D vector potential, such that the electric and magnetic forces are defined by:

$$\overrightarrow{E} = -\nabla\phi - \frac{\partial\overrightarrow{A}}{\partial t}, \qquad \overrightarrow{B} = \nabla \times \overrightarrow{A} \tag{9}$$

The physical origin of the continuous emission of short-range photons from charged particles is explained in [7], Section 5.1.2, based on the Kaluza-Klein theory of the small compactified time-like 5th dimension (generating electric charge of particle by propagation of particle's energy-density in this closed dimension).

The interactions between any two wave-packets (particles) can be obtained only by their local collisions. In dependence on their energy and velocities, they can produce a kind of Compton's effects (elastic collisions) in which they survive the collisions by changing their momentum and energy (with the conservation of total momentum and energy), or they can produce a total fusion with a possible creation of the new stable particles (in Feynman's diagrams).

Thus, for any two massless bosons with the Dirac function energy-distributions, it is impossible to have the collisions during their stable states,

but only when they are excited and involved in their temporary 3-D "spatial explosions". Such 3-D spatial explosions can happen also when two stable particles are at a very small mutual distance, during which the ideal spatial symmetry for a free particle in the vacuum does not hold more for both of them. The bosons have a physical role as the intermediators between the massive particles (that have the rest-mass and the 3-D volume V_t greater than zero), that is, they are a quantum-source that generates the fields (the phenomena as electromagnetic fields are the statistical results of actions of a high number of photons). In the case when the bosons are massless (long range interactions as for massless photons) then we have no the significant interference between themselves. This situation can be obtained at the quantum level only if the collisions between photons, for example, are practically improbable. Consequently, a number of photons can coexist in the same small 3-D region of space without any significant direct interference between them, heaving contemporarily the collisions with fermions which have the rest-mass and volume V_t greater than zero. Also in such a situation, we can have the rare cases of the interference between the photons. In normal situations, these interferences statistically can be neglected, while in the cases of very strong field interactions (when the local density of photons is extremely high) these inter-boson's interactions are significant. Thus, we have the following assumption:
Interaction-dynamics Assumption: The interactions between the bosons and fermions are realized always between two non-point like particles. That is, between a massive fermion and a massive unstable boson with a small but finite energy-density volume V_t.

2 Introduction to Gauge Invariance: Local Symmetries

A key innovation of the 20th century was Herman Weyl's invention of gauge theory, in which a global physical symmetry is replaced by a local one; the arbitrary phase in Hamiltonian-based quantum wave function becomes a function of time-space, a change that requires the existence of the electromagnetic gauge field. Weyl's gauge method, where the global symmetry is transformed into a local one, applied to the Standard-Model symmetry group $SU(3)\times SU(2)\times U(1)$, is enough in the SQM theory to yield the strong, weak and electromagnetic interactions.

It is well known the relativistic invariance or Poincare symmetry and the internal symmetry based on the Lie group $U(1)$ symmetry of phase transformations (the conservation of matter and, dually, of electric charge for Dirac equation). Analogously, we can consider the isospin symmetry $SU(2)$ (used for the electroweak force interactions) and the flavour symmetry $SU(3)$ of the strong force interactions. All of them are continuous *global* symmetry transformations (they give rise to conserved *currents* and *charges* as described by Noether's theorem, that is, they presuppose that, at least in principle, we can measure all the components of a field Ψ at all points $\overrightarrow{\mathbf{r}}$ in space at the same time.

However, here we have to consider the theories which are invariant if the symmetry operations are performed *locally* where the transformation parameters are dependent on local space coordinates (for example, if the rotation angle θ of Lorentz transformation is not constant for all infinitesimal pieces of matter $\Phi_m = \overline{\Psi}\Psi$, but is dependent on its space position $\overrightarrow{\mathbf{r}}$, that is, the rotation angle is a function $\theta(\overrightarrow{\mathbf{r}})$). A gauge theory is a theory where the action is invariant under a continuous group symmetry that *depends on time-space* and such local symmetries introduce these *gauge fields* to the theory which *mediate* a force. Let us consider, for instance, the internal phase transformation, when θ is not a constant phase, but depends on space position $\overrightarrow{\mathbf{r}}$, and when we require that the Lagrangian density $\mathcal{L}$ in equation (3) be invariant under such local smooth changes of phase:

$$\Psi|_{(t,\overrightarrow{\mathbf{r}})} \quad \mapsto \quad \Psi'|_{(t,\overrightarrow{\mathbf{r}})} = \mathrm{e}^{i\theta(\overrightarrow{\mathbf{r}})}\Psi|_{(t,\overrightarrow{\mathbf{r}})} \tag{10}$$

However, since the Lagrangian density $\mathcal{L}$ is invariant under *global* internal symmetry when θ is constant, it is not invariant under local phase transformations given by (10). The problem is that the derivatives of the field Ψ does not transform like the field in (10). In fact we have for $j = 1, 2, 3$ that:

$$\partial_j\Psi|_{(t,\overrightarrow{\mathbf{r}})} \quad \mapsto \quad \partial_j\Psi'|_{(t,\overrightarrow{\mathbf{r}})} = \partial_j[\mathrm{e}^{i\theta(\overrightarrow{\mathbf{r}})}\Psi|_{(t,\overrightarrow{\mathbf{r}})}] = \mathrm{e}^{i\theta(\overrightarrow{\mathbf{r}})}[\partial_j\Psi + i\Psi\partial_j\theta(\overrightarrow{\mathbf{r}})] \tag{11}$$

If we want to consider the phase transformations $\theta(\overrightarrow{\mathbf{r}})$ that differ from a point to point, we have to define *a connection* that specifies the mode how we suppose to transport the phase of Ψ from $\mathbf{r}_4 = (t, \overrightarrow{\mathbf{r}})$ to $\mathbf{r}'_4$ as we travel along some path γ. Let us consider the infinitesimal transport $\mathbf{r}'_4 = \mathbf{r}_4 + \delta\mathbf{r}_4 = \mathbf{r}_4 + \sum_{i=0}^{3}\delta q_i\mathbf{e}_i$, so that with this infinitesimal transport we have the change of Ψ:

$$\delta\Psi|_{\mathbf{r}_4} = \Psi|_{(\mathbf{r}_4+\delta\mathbf{r}_4)} - \Psi|_{\mathbf{r}_4} \tag{12}$$

This problem is analog to the problem of the derivation of a vector field $\mathbf{W}$, with a vector at a point $p = \mathbf{r}_4$, $\mathbf{w}_p = \mathbf{W}(p) = \sum_{j=0}^{3} w_j\mathbf{e}_j$, along a particular curve lying in a given manifold in the differential geometry. In contrast to differential geometry, the vector field Ψ are not vectors in the tangent space (plane) of a manifold, but belong to an "internal" vector space V (like isospin, flavor, etc..) and hence the local transformation (10) can be considered as a time-space dependent change of the basis in V and hence it is a passive transformation.

We require that the physics do not depend on the local choice of the basis, so that the differentiation has to be defined based on the change of Ψ (or $\overline{\Psi}$) relative to the parallel transported Ψ^p, so we have $\delta\Psi|_{\mathbf{r}_4} = \Psi^p|_{\mathbf{r}'_4} - \Psi|_{\mathbf{r}_4}$. In this case we have no Christoffel symbols Γ^j_{ik}, but an imaginary term $-i\alpha\mathcal{A}_k(\mathbf{r}_4)$ where $\mathcal{A}_k(\mathbf{r}_4)$ is a suitably chosen vector field (an element of the Lie algebra belonging to the gauge group element, for example for the unitary group[2], $e^{i\theta(\mathbf{r}_4)} \in U(\mathbf{r}_4)$) and α is a *coupling constant*. Thus, analogously to the case used for definition of covariant derivative in General Relativity, we have,

$$\delta\Psi|_{\mathbf{r}_4} = (i\alpha\sum_k \mathcal{A}_k(\mathbf{r}_4)dq_k)\Psi|_{\mathbf{r}_4} \quad and \quad D\Psi|_{\mathbf{r}_4} = \sum_{k=0}^{3}(\partial_k - i\alpha\mathcal{A}_k(\mathbf{r}_4))\Psi|_{\mathbf{r}_4}dq_k \quad (13)$$

with the gauge covariant derivative at point $\mathbf{r}_4$,

$$D_k = \partial_k - i\alpha\mathcal{A}_k(\mathbf{r}_4), \qquad k = 0,1,2,3 \quad (14)$$

(we denote by D'_k the covariant derivative at a point $p = \mathbf{r}'_4$ and $\mathcal{A}'_k$ the k-th component of a 4-vector field

$$\mathbf{A} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4)$$

at a point $\mathbf{r}'_4$). From the fact that we want that the Lagrangian density becomes invariant under the covariant derivatives, i.e., $D'_k\Psi|_{\mathbf{r}'_4} = (\partial_k - i\alpha\mathcal{A}'_k)(e^{i\theta(\mathbf{r}_4)}\Psi|_{\mathbf{r}_4}) = e^{i\theta(\mathbf{r}_4)}(\partial_k - i\alpha\mathcal{A}_k)\Psi|_{\mathbf{r}_4} = e^{i\theta(\mathbf{r}_4)}D_k\Psi|_{\mathbf{r}_4}$, we obtain that $\mathcal{A}_k$ should transform like

$$\mathcal{A}_k(\mathbf{r}_4) \quad \mapsto \quad \mathcal{A}'_k(\mathbf{r}'_4) = \mathcal{A}_k(\mathbf{r}_4) + \frac{1}{\alpha}\partial_k\theta(\mathbf{r}_4) \qquad k = 0,1,2,3 \quad (15)$$

Consequently, we conclude that we can promote a global symmetry to local (i.e., gauge) symmetry (for example, from $U(1)$ into $U(\mathbf{r}_4)$) by replacing the standard derivative operators ∂_k by the covariant derivative operators D_k. The standard model of elementary particles, which is based on the concept of local

[2]Historically, while trying to explain the quantum effects of electrodynamics, it was found that Quantum Electrodynamic (QED) could be explained by a $U(1)$ Abelian gauge theory. Yang and Mills then generalized this Abelian $U(1)$ gauge theory to the non Abelian gauge theory case (with the self interactions).

gauge invariance, has shown to be very successful theory[3]. Let us consider a parallel transport at finite distances, for example in the case of the Abelian symmetry group $U(1)$, from the initial point $\mathbf{r}_4$ into the final point $\mathbf{r}'_4$, so that an infinitesimal parallel transport along a finite curve $\gamma(\mathbf{r}_4, \mathbf{r}'_4)$ is given by (13) which formally may be integrated

$$\Psi^p|_{\mathbf{r}'_4} = \mathrm{e}^{i\alpha \int \mathcal{A}_k(z)dz^k} \Psi|_{\mathbf{r}_4} \tag{16}$$

In fact, for infinitesimal transport, by taking the Taylor approximation, we obtain $\Psi^p|_{\mathbf{r}'_4} = \Psi^p|_{(\mathbf{r}_4+\delta\mathbf{r}_4)} \approx (1 + i\alpha \sum_k \mathcal{A}_k(\mathbf{r}_4)dq_k)\Psi|_{\mathbf{r}_4} = \Psi|_{\mathbf{r}_4} + \delta\Psi|_{\mathbf{r}_4}$.
Remark: There is a subset of configurations of material fields Ψ that changes only because of the presence of the gauge field. They are *geodesic* configurations which satisfy the equation $D_k\Psi = 0$ for $k = 0, 1, 2, 3$, which is equivalent to linear equation $\partial_k\Psi = i\alpha\mathcal{A}_k\Psi$.
□

Minimal Coupling. Thus, we can make a system invariant under local gauge transformation at the expense of introducing a gauge vector field, which defines in a given point $p = \mathbf{r}_4$ the 4-vector $\mathbf{A}(\mathbf{r}_4)$, that plays the role of connection. This procedure, that relates the matter field (of a particle) and gauge field (by which this particle interacts) through the covariant derivative is known as 'minimal coupling'.

3 Unified IQM Gauge Theory for Individual Particles

Let us consider how such a kind of localization of the internal phase transformation is supported by the particle's matter/energy-density wave-packet which propagates with an arbitrary acceleration in a given locally flat Minkowski reference frame; we want to derive a *general gauge field* which is a source of such an acceleration of this particle. That is, the gauge theory provided in this section represents the *unification of the three basic forces* of the nature: electromagnetic, weak and strong forces. This is done by the general *complex* local phase transformation $\theta(t, \overrightarrow{\mathbf{r}}) = \theta_R(t) + i\theta_I(t, \overrightarrow{\mathbf{r}})$ valid for each of these three basic forces.

[3]In hopes of duplicating the success of the standard model, general relativity (which is non-renormalizable) was also formulated as a local gauge theory [24]. A Poincare transformation thet consists of constants and effects all of space instantly is called a global transformation: in GR such transformations violates special relativity, since such transformation takes us from one inertial reference frame to another, and, in presence of gravity, reference frames will be accelerated so that would be possible to have the information transmitted faster than the speed of light!

Unified Local Symmetry. Let us consider the evolution of the particle's wave-packet $\Psi \equiv \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T}$ from initial time-instance t to another $\Psi' \equiv \Phi(t+\delta t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T|_{(t+\delta t)}}$ at a time-instance $t+\delta t > t$, in a given field of (combination of) basic forces. We would like to express this evolution as the result of the local phase transformation $\delta\theta(t, \overrightarrow{\mathbf{r}})$, that is, by $\Psi \mapsto \Psi' = \mathrm{e}^{i\delta\theta(t, \overrightarrow{\mathbf{r}})}\Psi$. We assume that δt is an infinitesimal amount of time during which a particle has the interaction with only one boson of the field so that, after this interaction, this particle changed its velocity momentum and energy and its phase for a small amount $\delta\theta$. Hence, this local phase transformation $\theta(t, \overrightarrow{\mathbf{r}})$ *is complex* (differently from the usual theory for the point-like particles). The fact that this complex local phase transformation is a local symmetry for this particle holds from the fact that both Ψ and Ψ' are the solutions of the *same* Lagrangian $\mathcal{L}$ for the field Ψ only at two different instances of time.

Thus, by substitution of $\Psi' \equiv \mathrm{e}^{i\theta(t, \overrightarrow{\mathbf{r}})}\Psi$ into $\mathcal{L}$ we obtain the same Lagrangian form, that is, we obtain that $\mathcal{L}$ is invariant under this local phase transformation. This complex local phase transformation supports any kind of basic forces or their combinations and hence mathematically represents the unification of these basic forces.

□

For a free particle (without any external field that influence its propagation) we obtain the case when the external gauge field is zero, that is, when $\mathbf{A}(\mathbf{r}_4) = 0$.

In the case when we have some external gauge field (i.e., the *total* field which can be represented by a sum of different types of the fields as well) $\mathbf{A}(\mathbf{r}_4) \neq 0$ which influences the propagation of a particle (without annihilation of this particle), with Lagrangian density denoted by $\mathcal{L}_{gauge}$. The total Lagrangian density $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{gauge}$ is given by equation in (3), where the gauge field is indirectly represented by the matter-density speed $\overrightarrow{\mathbf{w}}(\mathbf{r}_4) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(\mathbf{r}_4)$, where $\overrightarrow{\mathbf{v}}(t)$ is the speed of particle's barycenter which changes in time (the accelerations and matter-density speed are the direct results of the interaction of a particle with this external gauge field $\mathbf{A}(\mathbf{r}_4)$).

Consequently, any local phase transformation $\Psi \mapsto \Psi' = \mathrm{e}^{i\theta(\overrightarrow{\mathbf{r}})}\Psi$ transforms the matter-density speed $\overrightarrow{\mathbf{w}}(\mathbf{r}_4)$ and, consequently, the particle's trajectory, accelerations and momentum/energy. This mathematical consequence explains that the local phase changing of the phase indeed means that we have some gauge field which modifies the original (free particle) particle's trajectory, speed and momentum/energy. Thus, in what follows, we will use the general local phase transformations $\theta(\mathbf{r}_4) = \theta(t, \overrightarrow{\mathbf{r}})$.

Let us represent the free Lagrangian of the free particle (stationary situation when $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$) which propagates with a *constant* speed $\overrightarrow{\mathbf{v}}$, momentum p and energy E, and from (4) $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}$, and hence from (3),

$$\mathcal{L}_{free} = \frac{\hbar}{\mathbf{1}_\Phi}(-\frac{\partial \varphi_T}{\partial t}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\frac{\partial \Psi}{\partial t} - \Psi\frac{\partial \overline{\Psi}}{\partial t} - \overline{\Psi}\overrightarrow{\mathbf{v}}\nabla\Psi + \Psi\overrightarrow{\mathbf{v}}\nabla\overline{\Psi}))$$
$$= \frac{\hbar}{\mathbf{1}_\Phi}(-\frac{E-pv}{\hbar}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\partial_0\Psi - \Psi\partial_0\overline{\Psi} + \sum_{j=1}^{3} v_j(\overline{\Psi}\partial_j\Psi + \Psi\partial_j\overline{\Psi})))$$

It is not invariant under the local phase transformations $\Psi \mapsto \Psi' = \mathrm{e}^{i\theta(t, \overrightarrow{\mathbf{r}})}\Psi$. In fact, for such a transformations, we obtain

$$\mathcal{L}_{free} \quad \mapsto \quad \mathcal{L}_{free} + (-\partial_0\theta\mathbf{e}_0 + \nabla\theta)\mathbf{J} = \mathcal{L}_{free} - \sum_{i=0}^{3} J_i\partial_i\theta$$

where $\mathbf{J} = J_0\mathbf{e}_0 + \overrightarrow{J}$ (with $\overrightarrow{J} = \overrightarrow{\mathbf{v}}\overline{\Psi}\Psi = J_0\overrightarrow{\mathbf{v}}$, for the Noether charge density $J_0 = \overline{\Psi}\Psi = \Phi_m$) is the Neother current of the free particle's Lagrangian, and $-\partial_0\theta\mathbf{e}_0 + \nabla\theta(t, \overrightarrow{\mathbf{r}})$ is the time-space function, generally different from zero.

The physical meaning of this invariance is that the local phase transformation necessarily changes the particle's trajectory, so that instead of the constant speed $\overrightarrow{\mathbf{v}}$ we will have an acceleration and corresponding matter-density speed $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})$. That is, we obtain the general normalized Lagrangian density $\mathcal{L}$ in (3) used previously for the Ponicare symmetries, valid in all possible cases for the accelerated particles:

$$\mathcal{L} = \frac{\hbar}{\mathbf{1}_\Phi}(-\frac{\partial \varphi_T}{\partial t}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\partial_0\Psi - \Psi\partial_0\overline{\Psi} + \sum_{j=1}^{3} w_j(\overline{\Psi}\partial_j\Psi - \Psi\partial_j\overline{\Psi}))) \tag{17}$$

Thus, we can obtain the invariant free Lagrangian $\mathcal{L}^I_{free}$ by introducing a *general* gauge field as explained by (15),

$$\mathbf{A}^\mu = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4)$$

and by replacing the derivatives ∂_j by the covariant derivatives D_j, given by (14), so that

$$\mathcal{L}^I_{free}(\Psi, \overline{\Psi}, \mathcal{A}_j) = \frac{\hbar}{\mathbf{1}_\Phi}(-\frac{\partial \varphi_T}{\partial t}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}D_0\Psi - \Psi\overline{D_0\Psi} + \sum_{j=1}^{3} v_j(\overline{\Psi}D_j\Psi - \Psi\overline{D_j\Psi})))$$
$$= \frac{\hbar}{\mathbf{1}_\Phi}[-\frac{E-pv}{\hbar}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\partial_0\Psi - \Psi\partial_0\overline{\Psi} + \sum_{j=1}^{3} v_j(\overline{\Psi}\partial_j\Psi - \Psi\partial_j\overline{\Psi})) + \alpha(\mathcal{A}_0 + \sum_{j=1}^{3} v_j\mathcal{A}_j)\overline{\Psi}\Psi] \tag{18}$$

where $\overrightarrow{\mathbf{v}} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, E and p are the velocity, energy and momentum of the free particle, respectively. Consequently, the general Lagrangian density, valid for accelerated particles as well, must be equal to the sum of the free invariant Lagrangian $\mathcal{L}^I_{free}$ above and the Lagrangian of the introduced gauged field $\mathbf{A}^\mu$, which interacts with the previously free particle and determines its final trajectory, speed and momentum/energy.

Let us consider the properties of the general gauge filed $\mathbf{A}^\mu$ (the resulting combination of all forces acting on this moving particle) and its effect on the propagation of a given particle which at a given instance of time t has a given speed $\overrightarrow{\mathbf{v}}$ and a given momentum/energy. If we would eliminate the gauge field (external force), then this particle would continue to propagate with this constant speed as a free particle with the Lagrangian density $\mathcal{L}_{free}$. But if we do not eliminate this gauge field then, after an infinitesimal amount of time $t' = t + \delta t$, it will change its speed and hence the Lagrangian density will be equal to that of accelerated particles in equation (17) where $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{u}}(t', \overrightarrow{\mathbf{r}})$. Consequently, the gauge Lagrangian is obtained by

$$\mathcal{L}_{gauge} = \mathcal{L} - \mathcal{L}^I_{free}(\Psi, \overline{\Psi}, \mathcal{A}_j) \tag{19}$$

that is, $\mathcal{L}_{gauge} = \frac{\hbar}{\mathbf{1}_\Phi}[-(\frac{\partial \varphi_T}{\partial t} - \frac{E - pv}{\hbar})\overline{\Psi}\Psi + \frac{i}{2}\sum_{j=1}^3 (w_j - v_j)(\overline{\Psi}\partial_j\Psi - \Psi\partial_j\overline{\Psi}) - \alpha(\mathcal{A}_0 + \sum_{j=1}^3 v_j\mathcal{A}_j)\overline{\Psi}\Psi]$, and from the fact that $\overline{\Psi}\partial_j\Psi - \Psi\partial_j\overline{\Psi}) = 0$ (the Euler-Lagrange equation solutions of the Lagrangian $\mathcal{L}$ are the fields $\Psi = \Phi e^{-i\varphi_T}$), and by definition of the pilot-wave phase changing during this infinitesimal amount of time δt, from (2),

$$-\frac{\delta\theta}{\delta t} \equiv \frac{\partial \varphi_T}{\partial t}|_{t'} - \frac{\partial \varphi_T}{\partial t}|_t = \frac{E' - p'v'}{\hbar} - \frac{E - pv}{\hbar} \tag{20}$$

we obtain the following gauge Lagrangian

$$\mathcal{L}_{gauge} = \frac{\hbar}{\mathbf{1}_\Phi}(\frac{\delta\theta}{\delta t} - \alpha(\mathcal{A}_0 + \sum_{j=1}^3 v_j\mathcal{A}_j))\overline{\Psi}\Psi \tag{21}$$

which is invariant for the local phase transformations. Hence, from (19), also $\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}^I_{free}$ must is invariant for the local phase transformations, as expected.

Remark: Note that in this approximative derivation of the gauge Lagrangian, we considered only the pilot-wave changing of the particle and not also of its density Φ distribution, that is, the method of derivation of $\mathcal{L}_{gauge}$ was as for a point-like particle, and because of that we obtained from (20) that $\delta\theta$ is a real amount. If we consider also that during δt changes the energy-density Φ^2 as well, then we have to assume that $\delta\theta$ *is a complex* value in the gauge Lagrangian

$\mathcal{L}_{gauge}$. That is, for the non point-like massive particles the gauge Lagrangian $\mathcal{L}_{gauge}$ in (21) is valid only for the complex values $\delta\theta$, as it will be shown.
$\square$

Now we have to understand if the components $\mathcal{A}_j$ of this gauge Lagrangian are independent variables, as it is so if we *define* the gauge field independently of the particle's field Ψ, or not. The answer is *no*. In fact, in our approach the gauge Lagrangian is derived from the particle's Lagrangian densities for two possible particle's states: when it is free particle and when it is not. Thus the gauge field $\mathbf{A}^\mu$ is derived indirectly from the particle's behavior. Here we have the final behavior of a particle (its trajectory, velocity, accelerations, momentum-energy changes, etc...) and after that we analyze the properties of the gauge field which caused such a particular particle's behavior (which is a resulting effect of a number of different fundamental forces as electromagnetic, weak, strong). Hence, the total gauge field is derived from this concrete particle's behavior, that is, from its field Ψ. Consequently, in this framework, we have the gauge Lagrangian dependent only on two variables, so that, from (21), it can be expressed by $\mathcal{L}_{gauge}(\Psi, \overline{\Psi})$.

The second consideration is that the gauge Lagrangian (21) does not correspond exactly to the real situation because the fields $\Psi = \Phi e^{-i\varphi_T}$ (where Φ is a real density distribution function) used in $\mathcal{L}$ and $\mathcal{L}^I_{free}$ in equation (19) are different not only in the pilot-wave phases but also in the densities Φ, and hence (21) can be used only for the massive but *point-like* bosons. In effect, from the fact that the interactions of the observed massive particle are always with the unstable (massive) bosons, it means that the term in (21) in front of $\overline{\Psi}\Psi$ must be a real value.

However, if we want to use (21) for Ψ that *are not* point-like massive particles, then we need to use the complex fields $\mathcal{A}_j$ instead of real, in order to obtain that such complex fields compensate the fact that the observed particle's density Φ in $\mathcal{L}$ and in $\mathcal{L}^I_{free}$ are slightly different also for infinitesimal δt.

In fact, let us consider an infinitesimal phase transformation for a real amount $\delta\theta_R > 0$ during an infinitesimal amount of time $\delta t > 0$ of the observed massive particle which, at a time instance $t \geq 0$, is given by the complex field (particle's wave-packet) $\Psi|_{(t,\overrightarrow{\mathbf{r}})} = \Phi(t, \overrightarrow{\mathbf{r}})e^{-i\varphi_T}$ and its changing into the complex field at the time $t + \delta t$, $\Psi' = \Psi|_{(t+\delta t, \overrightarrow{\mathbf{r}})} = \Phi(t + \delta t, \overrightarrow{\mathbf{r}})e^{-i(\varphi_T|_t - \delta\theta_R(t))}$.

Then, for the changing of the density Φ, we may always define the *real* function:

$$\delta\theta_I(t,\overrightarrow{\mathbf{r}}) \equiv \begin{cases} -ln(\frac{\Phi(t+\delta t,\overrightarrow{\mathbf{r}})}{\Phi(t,\overrightarrow{\mathbf{r}})}) \approx -\frac{\delta t}{\Phi(t,\overrightarrow{\mathbf{r}})}\frac{\partial\Phi(t,\overrightarrow{\mathbf{r}})}{\partial t}, & \text{if } \Phi(t+\delta t,\overrightarrow{\mathbf{r}})\Phi(t,\overrightarrow{\mathbf{r}}) \neq 0 \\ 0 & \text{, otherwise} \end{cases} \tag{22}$$

Note that natural logarithm is always well defined for particle's barycenter, where $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}_T}(t)$, for an infinitesimal δt. So, from $\frac{\partial\Phi}{\partial t} \approx \frac{\Phi(t+\delta t,\overrightarrow{\mathbf{r}})-\Phi(t,\overrightarrow{\mathbf{r}})}{\delta t}$ and $\frac{\delta t}{\Phi(t,\overrightarrow{\mathbf{r}})}\frac{\partial\Phi(t,\overrightarrow{\mathbf{r}})}{\partial t} << 1$, we have that $-ln(\frac{\Phi(t+\delta t,\overrightarrow{\mathbf{r}})}{\Phi(t,\overrightarrow{\mathbf{r}})}) = -ln(1+\frac{\delta t}{\Phi(t,\overrightarrow{\mathbf{r}})}\frac{\partial\Phi(t,\overrightarrow{\mathbf{r}})}{\partial t}) \approx -\frac{\delta t}{\Phi(t,\overrightarrow{\mathbf{r}})}\frac{\partial\Phi(t,\overrightarrow{\mathbf{r}})}{\partial t}$.

Consequently, the changing of the particle's wave-packet satisfies local transformation $\Psi' = (\Phi(t,\overrightarrow{\mathbf{r}})\mathrm{e}^{-\delta\theta_I})\mathrm{e}^{-i(\varphi_T|_t-\delta\theta_R(t))} = \mathrm{e}^{i(\delta\theta_R+i\delta\theta_I)}\Psi = \mathrm{e}^{i\delta\theta}\Psi$. That is, the complex-phase transformation for this observable massive non point-like particle is given by:

$$\Psi \quad \mapsto \quad \Psi' = \mathrm{e}^{i\delta\theta}\Psi \qquad \textit{where} \qquad \delta\theta(t,\overrightarrow{\mathbf{r}}) = \delta\theta_R + i\delta\theta_I \tag{23}$$

So, for the non point-like massive particle expressed by the field Ψ, we need to considering an infinitesimal local *complex-phase* $\delta\theta = \delta\theta_R + i\delta\theta_I$ where only real component θ_R corresponds to the particle's pilot-wave phase transformation, i.e., $\delta\Psi = \mathrm{e}^{i\delta\theta}\Psi - \Psi$ and for $\overline{\delta\Psi}$ as well. From the fact that $\mathcal{L}_{gauge}$ in (21) is invariant w.r.t. local phase transformation, we must have that

$$0 = \delta\mathcal{L}_{gauge} = \frac{\partial\mathcal{L}_{gauge}}{\partial\Psi}\delta\Psi + \sum_{j=0}^3 \frac{\partial\mathcal{L}_{gauge}}{\partial(\partial_j\Psi)}\partial_j(\delta\Psi) + \frac{\partial\mathcal{L}_{gauge}}{\partial\overline{\Psi}}\overline{\delta\Psi} + \sum_{j=0}^3 \frac{\partial\mathcal{L}_{gauge}}{\partial(\partial_j\overline{\Psi})}\partial_j(\overline{\delta\Psi})$$
$$= \frac{\partial\mathcal{L}_{gauge}}{\partial\Psi}\delta\Psi + \frac{\partial\mathcal{L}_{gauge}}{\partial\overline{\Psi}}\overline{\delta\Psi} = \frac{\hbar}{\mathbf{1}_\Phi}(\frac{\delta\theta}{\delta t} - \alpha(\mathcal{A}_0 + \sum_{j=1}^3 v_j\mathcal{A}_j))(\overline{\Psi}\delta\Psi + \Psi\overline{\delta\Psi}), \text{ i.e.,}$$

$$\delta\theta = \delta\theta_R + i\delta\theta_I = \alpha(\mathcal{A}_0\delta t + \sum_{j=1}^3 \mathcal{A}_j(v_j\delta t)) = \alpha\mathcal{A}_j\delta s^j \tag{24}$$

where δs^j are the infinitesimal changes of the coordinates of the barycenter of the particle, i.e., the time-space changing along particle's trajectory. Note that the same result we obtain from the Euler-Lagrange equation of motion obtained from the Lagrangian $\mathcal{L}_{gauge}(\Psi,\overline{\Psi})$ in (21), because $\frac{\partial\mathcal{L}_{gauge}}{\partial\overline{\Psi}} = \frac{\hbar}{\mathbf{1}_\Phi}(\frac{\delta\theta}{\delta t} - \alpha(\mathcal{A}_0 + \sum_{j=1}^3 v_j\mathcal{A}_j)\Psi = \sum_{j=0}^3 \partial_j\frac{\partial\mathcal{L}_{gauge}}{\partial(\partial_j\overline{\Psi})} = 0$.

This result shows that in this mathematical framework, the gauge field properties are not determined in all time-space points, but exclusively in the time-space points of the particle's barycenter, that is, on its trajectory $\overrightarrow{\mathbf{r}_T}(t)$. Let us show that it is indeed so.

Indeed, let us consider the invariant free Lagrangian $\mathcal{L}^I_{free}$ in (18) and show that it corresponds to the general Lagrangian $\mathcal{L}$ restricted only on the particle's trajectory time-space points. Thus, we have to show that analogously to $\mathcal{L} = 0$, $\mathcal{L}^I_{free} = 0$. It is enough to verify that from (24) and (21), $\mathcal{L}_{gauge} = 0$, so that from (19), $\mathcal{L}^I_{free} = \mathcal{L} - \mathcal{L}_{gauge} = 0 - 0 = 0$. Let us show now that $\mathcal{L}^I_{free}$ is just the reduction of $\mathcal{L}$ to the time-space points on the particle's trajectory. In fact, in Lagrangian $\mathcal{L}^I_{free}$ given by (18), in the place of $\overrightarrow{\mathbf{w}}$ we have the speed of barycenter $\overrightarrow{\mathbf{v}}$, and what we need is only to show that the component $\overline{\Psi}\Psi$, $-(E - pv) + \alpha(\mathcal{A}_0 + \sum_{j=1}^3 \mathcal{A}_j v_j)$, of the Lagrangian $\mathcal{L}^I_{free}$, is equal to $-\frac{\partial \varphi_T}{\partial t}|_{t'} \equiv -\frac{\partial \varphi_T}{\partial t}$.

In effect, it holds from (20), so that the Lagrangian $\mathcal{L}^I_{free}$, for the infinitesimal phase transformations, can be equivalently written as follows,

$$\mathcal{L}^I_{free}(\Psi, \overline{\Psi}) = \frac{\hbar}{\mathbf{1}_\Phi}\left(-\frac{\partial \varphi_T}{\partial t}\overline{\Psi}\Psi + \frac{i}{2}(\overline{\Psi}\partial_0\Psi - \Psi\partial_0\overline{\Psi} + \sum_{j=1}^{3} v_j(\overline{\Psi}\partial_j\Psi - \Psi\partial_j\overline{\Psi}))\right) \quad (25)$$

which is a reduction of the general Lagrangian for accelerated particles $\mathcal{L}$ in (17) only to the time-space points of the particle's trajectory, where $w_j = v_j$, $j = 1, 2, 3$.

Note that this is a general result for any possible combination of the external forces (electromagnetic, weak and strong), so that the parameter α do not depend on the particular case of the gauge fields. Let us now determine this parameter and, after that, the values that must have the general gauge 4-vector field $\mathbf{A}$ on the trajectory of the particle.

Let us consider the changing of the field $\delta\Psi$ on the trajectory points $(t, \overrightarrow{\mathbf{r}_T})$ in (20) caused by an infinitesimal local complex-phase transformation $\delta\theta = \delta\theta_R + i\delta\theta_I$, so that

$$\begin{aligned}
&\delta\Psi|_{(t,\overrightarrow{\mathbf{r}_T})} = \mathrm{e}^{i\delta\theta}\Psi - \Psi \\
&\approx (1 + i\delta\theta)\Psi - \Psi = i\delta\theta\Psi = i\alpha(\mathcal{A}_0\delta t + \textstyle\sum_{j=1}^3 \mathcal{A}_j(v_j\delta t))\Psi \qquad \text{from (24)} \\
&= i\alpha \textstyle\sum_{j=0}^3 \mathcal{A}_j(\overrightarrow{\mathbf{r}_4})\delta l_j \Psi|_{(t,\overrightarrow{\mathbf{r}_T})},
\end{aligned}$$

with generally complex components $\mathcal{A}_j$. Thus, we obtained a general result of the covariant derivatives in equation (13), restricted only to the time-space points of the particle's trajectory. With this, we concluded consistently the introduction of the gauge field. Let us determine its values on the particle's trajectory. Thus for $\delta t \to 0$, we obtain

$$\frac{d\Psi}{dt}|_{(t,\overrightarrow{\mathbf{r}_T})} = lim_{\delta t \to 0}\frac{\delta\Psi}{\delta t} = i\alpha(\mathcal{A}_0 + \sum_{j=1}^{3} v_j\mathcal{A}_j)\Psi|_{(t,\overrightarrow{\mathbf{r}_T})} \quad (26)$$

So, we can use the total derivative of Ψ on the particle's barycenter time-space points $\mathbf{r}_T(t) = (t, \overrightarrow{\mathbf{r}_T})$, that is, $\mathbf{r}_T(t) = t\mathbf{e}_0 + \overrightarrow{\mathbf{r}_T}(t)$, with $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}_T}) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}_T})$ and from $\frac{d\varphi_T}{dt} = \frac{E + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}}}{\hbar}$ and

$$\begin{aligned}
\frac{d\Phi(t, \overrightarrow{\mathbf{r}})}{dt} &= \frac{\partial \Phi}{\partial t} + \textstyle\sum_{j=1}^{3} \frac{\partial \Phi}{\partial q_j}\frac{dq_j}{dt} = \frac{\partial \Phi}{\partial t} - \overrightarrow{\mathbf{w}}\nabla\Phi \\
&= (\overrightarrow{\mathbf{w}}\nabla\Phi - \tfrac{\nabla\cdot\overrightarrow{\mathbf{w}}}{2}\Phi) - \overrightarrow{\mathbf{w}}\nabla\Phi \qquad\qquad \text{from (6)} \\
&= -\tfrac{\nabla\cdot\overrightarrow{\mathbf{w}}}{2}\Phi = -\tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}\Phi,
\end{aligned}$$

we obtain

$$\frac{d\Psi}{dt}|_{(t,\overrightarrow{\mathbf{r}}_T)} = (\Phi\frac{d\ \mathrm{e}^{-i\varphi_T}}{dt} + \mathrm{e}^{-i\varphi_T}\frac{d\Phi}{dt})_{(t,\overrightarrow{\mathbf{r}}_T)} = -(i\frac{E + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}}}{\hbar} + \frac{\nabla\cdot\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}_T)}{2})\Psi|_{(t,\overrightarrow{\mathbf{r}}_T)} \tag{27}$$

Consequently, from (26) and (27), we may determine the gauge field (only on the points of the particle's trajectory $\mathbf{r}_T(t) = t\mathbf{e}_0 + \overrightarrow{\mathbf{r}_T}(t)$), i.e., $\mathbf{r}_T(t) = (t, \overrightarrow{\mathbf{r}}_T)$, as follows:

$$\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g = -\frac{1}{\alpha}(\frac{E + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}}}{\hbar} - i\frac{\nabla\cdot\overrightarrow{\mathbf{u}}(\mathbf{r}_T)}{2}) \tag{28}$$

where $\overrightarrow{\mathcal{A}}_g =_{def} \mathcal{A}_1\mathbf{e}_1 + \mathcal{A}_2\mathbf{e}_2 + \mathcal{A}_3\mathbf{e}_3$ is the vector component of the gauge 4-potential $\mathbf{A}^\mu = \mathcal{A}_0\mathbf{e}_0 + \overrightarrow{\mathcal{A}}_g$ and $\mathcal{A}_0$ is its scalar (time) component. So we can chose a natural choice (by dividing terms with particle's speed and that without)

$$\mathcal{A}_0 = -\frac{1}{\alpha}(\frac{E}{\hbar} - i\frac{\nabla\cdot\overrightarrow{\mathbf{u}}(\mathbf{r}_T)}{2}), \qquad \overrightarrow{\mathcal{A}}_g = \frac{\overrightarrow{\mathbf{p}}}{\alpha\hbar} \tag{29}$$

It seems a natural choice that the scalar component $\mathcal{A}_0$ of the gauge 4-potential determines particle's total energy (or Hamiltonian), while the vector gauge potential $\overrightarrow{\mathcal{A}}_g$ determines particle's canonical momentum.

In fact, the imaginary component of the scalar gauge potential $Im(\mathcal{A}_0)$ is proportional to the divergence of the rest-mass energy flux $\nabla\cdot\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}_T) = \nabla\cdot(\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}_T) + \overrightarrow{\mathbf{v}}(t)) = \nabla\cdot\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}_T)$ which is responsible for the change of the particle's shape Φ during interaction of the particle with intermediary bosons (gauge field). It is easy to verify (we can extend its validity from the particle's barycenter also for any point $\overrightarrow{\mathbf{r}}$ from the fact that $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ is different from zero for the points inside particle's body) that we obtain:

$$Im(\mathcal{A}_0) = \frac{\nabla\cdot\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})}{2\alpha} = -\frac{1}{\alpha}\frac{d\ln\Phi(t, \overrightarrow{\mathbf{r}})}{dt} = -\frac{1}{2\alpha}\frac{d\ln\Phi_m(t, \overrightarrow{\mathbf{r}})}{dt} \tag{30}$$

That is, the total time derivative of particle's rest-mass energy density Φ_m is determined by the imaginary component of scalar gauge potential $\mathcal{A}_0$. With this generalization, we can define the gauge 4-potential also for the positions different from the particle's barycenter $(t, \overrightarrow{\mathbf{r}}_T)$, from (29)

$$\mathcal{A}_0(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}}) = -\frac{1}{\alpha}(\frac{E(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}})}{\hbar} - i\frac{\nabla \cdot \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})}{2}), \qquad \overrightarrow{\mathcal{A}}_g(\overrightarrow{\mathbf{p}}) = \frac{\overrightarrow{\mathbf{p}}}{\alpha\hbar} \quad (31)$$

from the fact that the canonical momentum $\overrightarrow{\mathbf{p}}$ is time-dependent, while the total energy E is equal to the Hamiltonian. In the case when $\Psi|_{(t,\overrightarrow{\mathbf{r}_T})} = \sqrt{1_\Phi \delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_T})}\mathrm{e}^{-i\varphi_T}$ is the field of a stable boson, where $\varphi_T = 0$, we have that $\frac{d\varphi_t}{dt} = 0$ and $\overrightarrow{\mathbf{u}} = 0$ and hence $\nabla \cdot \overrightarrow{\mathbf{u}} = 0$, so that equation (28) is equal to zero which is satisfied when all components $\mathcal{A}_j = 0$, that is for the zero 4-vector potential $\mathbf{A}^\mu = 0$, as expected in the flat Minkowski time-space (there is no any interaction of this massless boson with another bosons during its stable Dirac function state when propagates with the speed of light in the vacuum along the GR geodesics). Only if this boson becomes unstable massive boson (by the presence of another material objects on its trajectory) then it can have an interaction with another particles and change its trajectory, as it happens for the fermions.

Now, from (24) and (31), we obtain that $\delta\theta_I = Im(\alpha\mathcal{A}_0)\delta t$, that is

$$\delta\theta_I = \frac{\nabla \cdot \overrightarrow{\mathbf{u}}}{2}\delta t = -\frac{d\ln\Phi}{dt}\delta t \quad (32)$$

so that for $\delta t \mapsto 0$, $\delta\theta_I = -\frac{\ln\Phi|_{t+\delta t}) - \ln\Phi|_t)}{\delta t}\delta t = -\ln\frac{\Phi|_{t+\delta t}}{\Phi|_t}$ we obtain the result in (22). So by substitution of (32) into (24) by using (31), we obtain

$$\delta\theta_R = -\frac{E + \overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}}}{\hbar}\delta t = -\frac{d\varphi_T}{dt}\delta t \quad (33)$$

and finally, we provide an infinitesimal localy symmetry transformation by

$$\delta\theta = \delta\theta_R + i\delta\theta_I = -\frac{E + \overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}}}{\hbar}\delta t + i\frac{\nabla \cdot \overrightarrow{\mathbf{u}}}{2}\delta t = -\frac{d\varphi_T}{dt}\delta t - i\frac{d\ln\Phi}{dt}\delta t = -i\frac{d\ln\Psi}{dt}\delta t \quad (34)$$

Remark: In our working framework, where we determine the (composed) gauge field only from the particle's behaviour (dominantly from its trajectory), it is an expected result, because based only on the observation of the particle's trajectory, mathematically we are not able to determine the values of the gauge filed in the time-space points different from the particle's trajectory. That is, we are able to derive acting forces on the particle only in the time-space points of the particle's trajectory, that is in the barycenter of the particle.

But this is not a particular side-effect of this completion of the Quantum Mechanics, based on the field Ψ of the matter/energy-density wave-packet of an elementary particle, but a fundamental new result: it creates a strong continuity between a 'macroscopic' classical mechanics and 'microscopic' Quantum Mechanics. This continuity is based on the fact that gauge fields acts on the particle's barycenter with the produced forces which determine and modify the original free particle trajectory.

It is well known that the standard Quantum Mechanics and its gauge theory is based on the energy considerations and wavefunctions based on the Hamiltonian energy-levels, as in Schrödinger, Klein-Gordon or Dirac equations, and on their Lagrangian densities which define the interactions of a free particle with external forces by an *explicit* introduction of the particular gauge field.

The second new result of this dual approach used in this completion of QM is that here the general Lagrangian density of the accelerated particle's wave-packet *contains implicitly* the gauge field, whose effects on the particle are visible in the behavior of the general Lagrangian density, that is, in its Euler-Lagrangian equation of motion. This is a similar approach to that used by Johannes Kepler, for example, to establish that the orbit of a planet is an ellipse with the Sun at one of the two foci, and after that by Newton to explain that the 'gauge field' that determines such orbits must satisfy his own laws of motion and law of universal gravitation (the direction of the acceleration is towards the Sun and the magnitude of the acceleration is in inverse proportion to the square of the distance from the Sun).
□

Let us show how the parallel transport at finite distances works, along a particle's path (of the particle barycenter) from the initial point $\mathbf{r}_T = (t, \overrightarrow{\mathbf{r}_T}(t))$, at the time-instance t, into the final point $\mathbf{r}'_T = (t', \overrightarrow{\mathbf{r}_T}(t'))$ at the time instance $t' = t + \triangle t$, for any finite time interval $\triangle t > 0$. From the fact that the parallel transport along a finite particle's *trajectory* $\gamma(\mathbf{r}_4, \mathbf{r}'_4)$ is given by (16), it may be integrated and hence, from (24), we have that

$$\Psi|_{\mathbf{r}'_T} = \mathrm{e}^{i\theta}\Psi|_{\mathbf{r}_T} = \mathrm{e}^{i\alpha\int_\gamma \mathcal{A}_k(z)dz^k}\Psi|_{\mathbf{r}_T} \tag{35}$$

where the local phase transformation θ is obtained by the path integral:

$$\theta = \alpha\int_{\gamma(\mathbf{r}_T,\mathbf{r}'_T)} \mathcal{A}_k(z)dz^k = \alpha\int_t^{t'} (\mathcal{A}_0(\tau, \overrightarrow{\mathbf{r}_T}(\tau))) + \sum_{j=1}^{3} v_j(\tau)\mathcal{A}_j(\tau, \overrightarrow{\mathbf{r}_T}(\tau)))d\tau \tag{36}$$

Consequently, by the path integration of the phase along the particle's trajectory, we obtain the following result:

Proposition 1 *The gauge 4-vector potential of the particle's filed $\Psi = \Phi e^{-i\varphi_T}$ satisfies:*

$$\frac{d}{dt}\ln\Psi|_{(t,\overrightarrow{\boldsymbol{r}_T}(t))} = i\alpha(\mathcal{A}_0(t,\overrightarrow{\boldsymbol{r}_T}(t))) + \sum_{j=1}^{3} v_j(t)\mathcal{A}_j(t,\overrightarrow{\boldsymbol{r}_T}(t))) \tag{37}$$

A 'parallel transport' (35) of this particle's filed along a finite curve of its barycenter $\gamma(\boldsymbol{r}_T, \boldsymbol{r}'_T)$, from the initial point $\boldsymbol{r}_T = (t, \overrightarrow{\boldsymbol{r}_T}(t))$ into the final point $\boldsymbol{r}'_T = (t', \overrightarrow{\boldsymbol{r}_T}(t'))$, where $t' = t + \triangle t$ for a finite time interval $\triangle t > 0$, produces the following local complex-phase transformation (36):

$$\theta = -\varphi_T|_{\boldsymbol{r}_T}^{\boldsymbol{r}'_T} - i\ln\Phi|_{\boldsymbol{r}_T}^{\boldsymbol{r}'_T} = -\varphi_T|_{\boldsymbol{r}'_T} + \varphi_T|_{\boldsymbol{r}_T} - i\ln\Phi(\boldsymbol{r}'_T) + i\ln\Phi(\boldsymbol{r}_T) = -i\ln\Psi|_{\boldsymbol{r}_T}^{\boldsymbol{r}'_T} \tag{38}$$

Proof: From (26) we obtain the equation (37). Then, from (36),

$\theta = \alpha\int_{\gamma(\mathbf{r}_T,\mathbf{r}'_T)}(\mathcal{A}_0 + \sum_{j=1}^{3} v_j\mathcal{A}_j)dt = \int_{\gamma(\mathbf{r}_T,\mathbf{r}'_T)}(-i\frac{d}{dt}\ln\Psi)dt = -i\ln\Psi|_{\mathbf{r}_T}^{\mathbf{r}'_T}$

$= -i\ln\frac{\Phi(\mathbf{r}'_T)e^{-i\varphi_T|_{\mathbf{r}'_T}}}{\Phi(\mathbf{r}_T)e^{-i\varphi_T|(\mathbf{r}_T)}} = -\varphi_T|_{\mathbf{r}'_4} + \varphi_T|_{\mathbf{r}_4} - i(\ln(\Phi(\mathbf{r}'_T) - \ln\Phi(\mathbf{r}_T)).$

In effect, for the infinitesimal $\triangle t = \delta t$ we have that

$\frac{\delta\theta}{\delta t} = -\frac{\varphi_T|_{t+\delta t} - \varphi_T|_t}{\delta t} - i\frac{\ln\Phi|_{t+\delta t} - \ln\Phi_t}{\delta t} = -\frac{d\varphi_T}{dt} - i\frac{d}{dt}(\ln\Phi)|_{(t,\overrightarrow{\mathbf{r}_T}(t))}$

$= -\frac{d\varphi_T}{dt} - i(\frac{1}{\Phi}\frac{d\Phi}{dt})|_{(t,\overrightarrow{\mathbf{r}_T}(t))} = -\frac{E + \overrightarrow{\mathbf{v}}\overrightarrow{\mathbf{p}}}{\hbar} - i(\frac{1}{\Phi}(-\frac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}\Phi))$

$= \alpha(-\frac{1}{\alpha}(\frac{E}{\hbar} - \frac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}) - \overrightarrow{\mathbf{v}}(\frac{\overrightarrow{\mathbf{p}}}{\alpha\hbar}))$

$= \alpha(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}_g})$, in accordance with (24).

□

Notice that this result (38) can be obatiend directly by integration of infinitesimal local symmetry transformation (34) in time, iver particle's trajectory γ from t to $t' = t + \triangle t$.

It is easy to verify that the results of this proposition are in according with the gauge theory. In fact from (35) for the 'parallel transformation', we have

$\Psi|_{\mathbf{r}'_4} = e^{i\theta}\Psi|_{\mathbf{r}_4} = e^{i\alpha\int_\gamma \mathcal{A}_k(z)dz^k}\Psi|_{\mathbf{r}_4} = e^{i(-i\ln(\frac{\Psi|\mathbf{r}'_4}{\Psi|\mathbf{r}_4}))}\Psi|_{\mathbf{r}_4} = e^{\ln(\frac{\Psi|\mathbf{r}'_4}{\Psi|\mathbf{r}_4})}\Psi|_{\mathbf{r}_4}$

$= (\frac{\Psi|_{\mathbf{r}'_4}}{\Psi|_{\mathbf{r}_4}})\Psi|_{\mathbf{r}_4}.$

Moreover, the interaction of a particle with an external field boson cnages only locally particle's density Φ mainly in particle's surface, so that the density in particle's barycenter (which is maximal possible density Φ_∞ does not change during the interactions (emission/absorbtion) of bosons, and the speed of this invariant particle's denisity in its barucenter is equal to particle's speed $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{v}}$.

Hence, in particle's barycenter $\mathbf{r}_T = (t, \overrightarrow{\mathbf{r}}_T)$,

$$\ln \frac{\Phi|_{\mathbf{r}'_T}}{\Phi|_{\mathbf{r}_T}} \approx 0 \tag{39}$$

and practically alway just equal to zero. So, we have from (38) that (for $\partial_0 = \frac{\partial}{\partial t}$),

$$\begin{aligned}
\partial_0\theta &= -(\partial_0\varphi_T)_{\mathbf{r}'_T} + (\partial_0\varphi_T)_{\mathbf{r}_T} - i(\tfrac{1}{\phi}\tfrac{\partial\Phi}{\partial t})_{\mathbf{r}'_T} + i(\tfrac{1}{\phi}\tfrac{\partial\Phi}{\partial t})_{\mathbf{r}_T} \\
&= -\tfrac{E}{\hbar}|_{\mathbf{r}'_T} + \tfrac{E}{\hbar}|_{\mathbf{r}_T} - i(\tfrac{1}{\phi}(\overrightarrow{\mathbf{w}}\nabla\phi - \tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}\Phi))_{\mathbf{r}'_T} + i(\tfrac{1}{\phi}(\overrightarrow{\mathbf{w}}\nabla\phi - \tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}\Phi))_{\mathbf{r}_T} \\
&= -(\tfrac{E}{\hbar} - i\tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2})_{\mathbf{r}'_T} + (\tfrac{E}{\hbar} - i\tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2})_{\mathbf{r}_T} - i\overrightarrow{\mathbf{v}}\nabla(\ln\tfrac{\Phi|_{\mathbf{r}'_T}}{\Phi|_{\mathbf{r}_T}}) \\
&\approx -(\tfrac{E}{\hbar} - i\tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2})_{\mathbf{r}'_T} + (\tfrac{E}{\hbar} - i\tfrac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2})_{\mathbf{r}_T} \text{ , from (39), that is}
\end{aligned}$$

$$\partial_0\theta = \alpha\mathcal{A}'_0 - \alpha\mathcal{A}_0 \tag{40}$$

Similarly,

$$\begin{aligned}
\nabla\theta &= -(\nabla\varphi_T)_{\mathbf{r}'_T} + (\nabla\varphi_T)_{\mathbf{r}_T} - i\nabla(\ln\tfrac{\Phi|_{\mathbf{r}'_T}}{\Phi|_{\mathbf{r}_T}}) = -(\nabla\varphi_T)_{\mathbf{r}'_T} + (\nabla\varphi_T)_{\mathbf{r}'_T}, \text{ from (39)} \\
&= -(\tfrac{\partial\varphi_T}{\partial\overrightarrow{\mathbf{r}}_T})_{\mathbf{r}'_T} + (\tfrac{\partial\varphi_T}{\partial\overrightarrow{\mathbf{r}}_T})_{\mathbf{r}'_T}, \text{ because on trajectory } \overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_T \\
&= \tfrac{\overrightarrow{\mathbf{p}}}{\hbar}|_{\mathbf{r}'_T} - \tfrac{\overrightarrow{\mathbf{p}}}{\hbar}|_{\mathbf{r}_T}, \text{ that is}
\end{aligned}$$

$$\nabla\theta = \alpha\overrightarrow{\mathcal{A}}'_g - \alpha\overrightarrow{\mathcal{A}}_g \tag{41}$$

That is, we obtained the satisfaction of the the gauge transformation in (15),

$\mathcal{A}_j \quad\mapsto\quad \mathcal{A}'_j = \mathcal{A}_j + \frac{1}{\alpha}\partial_j\theta$, for $j = 0, 1, 2, 3$.

So, we obtain that

$$\mathbf{A}^\mu(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}}) = -\frac{1}{\alpha}(\frac{E(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}})}{\hbar} - i\frac{\nabla\cdot\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})}{2})\mathbf{e}_0 + \frac{\overrightarrow{\mathbf{p}}}{\alpha\hbar} \tag{42}$$

is well defined gauge. By considering that on the particle's trajectory γ we have that $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{v}}$, the equation (36) is the line integral of the gauge field $\mathbf{A}^\mu$ done on the trajectory $\mathbf{r}_T = t\mathbf{e}_0 + \overrightarrow{\mathbf{r}_T}(t)$ and hence $d\mathbf{r}_T = dt\mathbf{e}_0 + d\overrightarrow{\mathbf{r}_T}(t) = (\mathbf{e}_0 + \overrightarrow{\mathbf{v}}(t))dt$,

$$\theta = \alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu} \tag{43}$$

where for $d\mathbf{r}_T = d\mathbf{r}_T^\mu = d(t'\mathbf{e}_0 + \overrightarrow{\mathbf{r}}_T(t)) = dt'\mathbf{e}_0 + d\overrightarrow{\mathbf{r}}_T(t) = (\mathbf{e}_0 + \overrightarrow{\mathbf{v}})dt' = (dt, v_1dt, v_2dt, v_3dt)$, it corresponding dual 4-vector is $d\mathbf{r}_{T\mu} = d(t'\mathbf{e}_0 - \overrightarrow{\mathbf{r}}_T(t)) = dt'\mathbf{e}_0 - d\overrightarrow{\mathbf{r}}_T(t) = (\mathbf{e}_0 - \overrightarrow{\mathbf{v}})dt' = (dt, -v_1dt, -v_2dt, -v_3dt)$.

We may consider $\hbar\alpha\mathbf{A}^\mu$ as the force (resulting of all quantum forces) acting on the particle at a given point in space (the particle's barycenter $\mathbf{r}_T(t)$), the line integral $\hbar\theta$ above is the work done on the particle when it propagates along a trajectory-path $\gamma(\mathbf{r}_T, \mathbf{r}'_T)$.

For any gauge field's effect $\theta = \theta_R + i\theta_I$ in (43) of the initial state of the particle $\Psi|_{t_0}$ along its trajectory (path γ) up to the time $t \geq t_0$, we obtain for any Minkowski time-space point $(t_0, \overrightarrow{\mathbf{r}})$ such that $\Phi(t_0, \overrightarrow{\mathbf{r}}) \neq 0$:

$$\mathrm{e}^{i\theta}\Psi|_{t_0} = \mathrm{e}^{i\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu}}\Psi|_{t_0} = \mathrm{e}^{i\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu}}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T|_{t_0}}$$
$$= \mathrm{e}^{i(\theta_R + iIm(\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu}))}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T|_{t_0}} = \mathrm{e}^{-Im(\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu})}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)}$$
$$= \mathrm{e}^{-\alpha\int_{t_0}^t Im(\mathcal{A}_0)dt')}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)} = \mathrm{e}^{\alpha\int_{t_0}^t \frac{\nabla\cdot\overrightarrow{\mathbf{u}}(t', \overrightarrow{\mathbf{r}})}{2\alpha}dt')}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)}$$
$$= \mathrm{e}^{\alpha\int_{t_0}^t \frac{\nabla\cdot\overrightarrow{\mathbf{u}}(t', \overrightarrow{\mathbf{r}})}{2\alpha}dt')}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)}$$
$$= \mathrm{e}^{\int_{t_0}^t \frac{d\ln\Phi(t', \overrightarrow{\mathbf{r}})}{dt'}dt')}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)} \qquad \text{from (30)}$$
$$= \mathrm{e}^{\ln\Phi(t, \overrightarrow{\mathbf{r}}) - \ln\Phi(t_0, \overrightarrow{\mathbf{r}})}\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)}$$
$$= (\tfrac{\Phi(t, \overrightarrow{\mathbf{r}})}{\Phi(t_0, \overrightarrow{\mathbf{r}})})\Phi(t_0, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)} = \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\varphi_T|_{t_0} - \theta_R)}$$
$$= \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T|_t} \qquad \text{from (38), } \theta_R = -\varphi_T|_t + \varphi_T|_{t_0},$$
$$= \Psi|_t,$$

with, for particle's trajectory γ from $(t_0, \overrightarrow{\mathbf{r}}_{t_0})$ to $(t, \overrightarrow{\mathbf{r}}_T)$,

$$\theta_R((t, \overrightarrow{\mathbf{r}}_T); (t_0, \overrightarrow{\mathbf{r}}_{t_0})) = Re(\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu}) = \int_\gamma(\tfrac{E(t', \overrightarrow{\mathbf{r}}_T, \overrightarrow{\mathbf{p}})}{\hbar}\mathbf{e}_0 - \tfrac{\overrightarrow{\mathbf{p}}}{\hbar})d\mathbf{r}_{T\mu}$$
$$= \int_\gamma(\tfrac{E(t', \overrightarrow{\mathbf{r}}_T, \overrightarrow{\mathbf{p}})}{\hbar}\mathbf{e}_0 - \tfrac{\overrightarrow{\mathbf{p}}}{\hbar})(\mathbf{e}_0 - \overrightarrow{\mathbf{v}})dt' = -\int_{t_0}^t \tfrac{E(t', \overrightarrow{\mathbf{r}}_T, \overrightarrow{\mathbf{p}}) - pv}{\hbar}dt', \text{ and, from above,}$$

$$\theta_I((t, \overrightarrow{\mathbf{r}}_T); (t_0, \overrightarrow{\mathbf{r}}_{t_0})) = Im(\alpha\int_\gamma \mathbf{A}^\mu d\mathbf{r}_{T\mu}) = -\ln(\frac{\Phi(t, \overrightarrow{\mathbf{r}}_T)}{\Phi(t_0, \overrightarrow{\mathbf{r}}_{t_0})}) \tag{44}$$

As expected, we obtained the gauge theory result, of the particle's internal *local symmetry* transformation, $\Psi|_t = \mathrm{e}^{i\theta}\Psi|_{t_0}$ where $\theta = \theta_R((t, \overrightarrow{\mathbf{r}}_T); (t_0, \overrightarrow{\mathbf{r}}_{t_0})) + i\theta_I((t, \overrightarrow{\mathbf{r}}_T); (t_0, \overrightarrow{\mathbf{r}}_{t_0}))$.

Geodesic configurations. In this particular cases we have that $D_j\Psi = 0$. Let us consider the cases when $j = 1, 2, 3$, for which on the trajectory with coordinates q_T^j with j-th momentum component p_j, so indeed is satisfied this condition,

$$D_j\Psi = (\partial_j - i\alpha\mathcal{A}_j)\Psi = \partial_j\Psi - i\tfrac{p_j}{\hbar}\Psi = \tfrac{\partial}{\partial q_T^j}\Psi - i\tfrac{p_j}{\hbar}\Psi = i\tfrac{p_j}{\hbar}\Psi - i\tfrac{p_j}{\alpha\hbar}\Psi = 0$$

on the *free* particle's trajectory, from $\Psi = \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T(t, \overrightarrow{\mathbf{r}}_T)}$, with

$$\varphi_T = \tfrac{1}{\hbar}(\overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{r}}_T(t) - \overrightarrow{\mathbf{r}}_T(0)) + Et) = \tfrac{1}{\hbar}(-\textstyle\sum_{j=1}^3 p_j(q_T^j(t) - q_T^j(0)) + Et).$$

By considering thatin geodesic configuration we have a propagation of a free particle, when $\omega_p \equiv \frac{\partial \varphi_T}{\partial t} = \frac{E+\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}}}{\hbar}$, we obtain that on particles trajectory

$D_0\Psi = (\partial_0 - i\alpha\mathcal{A}_0)\Psi = (-i\omega_p\Psi + \overrightarrow{\mathbf{w}}\nabla\Psi - \frac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2}\Psi) + i(\frac{E}{\hbar} - i\frac{\nabla\cdot\overrightarrow{\mathbf{u}}}{2})\Psi$

$= -i\omega_p\Psi + \overrightarrow{\mathbf{w}}\nabla\Psi + i\frac{E}{\hbar}\Psi = -i\omega_p\Psi + \overrightarrow{\mathbf{v}}\frac{\partial\Psi}{\partial\overrightarrow{\mathbf{r}}_T} + i\frac{E}{\hbar}\Psi$, because in barycenter $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{v}}$

$= -i\omega_p\Psi + \overrightarrow{\mathbf{v}}(i\frac{\overrightarrow{\mathbf{p}}}{\hbar}) + i\frac{E}{\hbar}\Psi = i(-\omega_p + \frac{E+\overrightarrow{\mathbf{v}}\,\overrightarrow{\mathbf{p}}}{\hbar})\Psi = 0\cdot\Psi = 0$, as expected.

□

Conclusions: We have seen that the general Lagrangian density of a particle is the addition of a Lagrangian density of this particle in its perfect stationary (free particle) state and a total gauge field (as a combination of a number of existent quantum fields), which can be observed on the particles barycenter (trajectory). That is, like in the standard quantum gauge field theory where the particles are considered as point-like objects. We are able to use the standard gauge field theory also in the constructive way (as in the standard quantum mechanics) by considering a Lagrangian density $\mathcal{L}_{free}$ of the free particle (that propagates with a constant speed) by adding to it a specific gauge Lagrangian.

In both cases, the point-like approximation for the particles is well suited for the interactions of a particle with the gauge fields. Thus, this new theory of QM (the IQM theory), based on the non point-like massive particle that propagates as wave-packets of a given invariant amount of matter, distributed in a small but finite volume, is a *conservative extension* of the standard point-like based quantum mechanics for the particles. But for the high-energy physics, where the annihilation and creation of the new particles are dominant, the interactions of the non point-like particles introduce the direct collisions of their matter-densities, which produce the forces that are not based only on the bosons (also in their unstable states when they have the properties of the 'virtual particles') of the gauge fields.

A local symmetry can be connected also to indeterminism because of the time dependence of its transformation parameter. For example, the time dependence of a transformation for a local symmetry will ensure that a transformation can have the initial and final configurations invariant, while transforming the history between those two points. Thus for the local $U(1)$ symmetry with $\theta(t, \overrightarrow{\mathbf{r}})$, this can be theoretically achieved if the parameter $\theta(t, \overrightarrow{\mathbf{r}})$ vanishes at the initial and final times, t_i and t_f, for every $\overrightarrow{\mathbf{r}}$, while it has non-zero value at at least one point $\overrightarrow{\mathbf{r}}$ for a time t, for which $t_i < t < t_f$. Since such a transformation is a symmetry of the theory, the transformed history will extremize the action for the same initial and final configurations and hence a theory with a local symmetry can thus be indeterministic.

Notice that this is not possible in our case, because from equation (38) for a limit $\triangle t \mapsto 0$ around this time-instance t we always have that $lim_{\triangle t \mapsto 0}\theta = 0$. That is, we can not have at t the value of $\theta \neq 0$ if the initial $\Phi(\mathbf{r}_4)$ and final $\Phi(\mathbf{r}'_4)$ configurations are the same, so that $\ln(\frac{\Phi(\mathbf{r}'_4)}{\Phi(\mathbf{r}_4)}) = 0$. Consequently, in our case there is no more than one realizable history for a given initial and final configuration of the field Ψ, and hence our system is *deterministic* and these local symmetries must be necessarily the *gauge symmetries*.

4 Example: Aharonov-Bohm Effect

The most commonly described case, sometimes called the Aharonov-Bohm solenoid effect, takes place when the wave function of a charged particle passing around a long solenoid experiences a phase shift as a result of the enclosed magnetic field, despite the magnetic field being negligible in the region through which the particle passes and the particle's wavefunction being negligible inside the solenoid.

It is sometimes called the Ehrenberg-Siday-Aharonov-Bohm effect, first predicted the effect in 1949 by Werner Ehrenberg and Raymond Siday [1]. Aharonov-Bohm effect was first described in 1959 in an article [2], written by Yakir Aharonov and his doctoral advisor David Bohm and received various responses. The Aharonov-Bohm effect is important conceptually because it bears on three issues apparent in the recasting of (Maxwell's) classical electromagnetic theory as a gauge theory, which before the advent of quantum mechanics could be argued to be a mathematical reformulation with no physical consequences. In fact, when developing their idea, Aharonov and Bohm consulted experimental physicist Robert G.Chambers and in their article, they described the experiment which had to be carried out to prove their theory. Only a year later [3], in 1960, Chambers performed the proposed experiment and proved that effect does exist. In following years, effect was confirmed by more and more precise experiments.

The Aharonov-Bohm effect shows that the local electric $\overrightarrow{E}$ and magnetic $\overrightarrow{B}$ *forces* do not contain full information about the electromagnetic field, and the electromagnetic 4-potential vector, $(\frac{\phi(t,\overrightarrow{\mathbf{r}})}{c}, \overrightarrow{A}(t, \overrightarrow{\mathbf{r}}))$ in (8) with (9), must be used instead. By Stokes' theorem, the magnitude of the Aharonov-Bohm effect can be calculated using the electromagnetic fields alone, or using the 4-potential alone. But when using just the electromagnetic fields, the effect depends on the field values in a region from which the test particle is excluded. In contrast, when using just the electromagnetic four-potential, the effect only depends on the potential in the region where the test particle is allowed. Therefore, one must either abandon the principle of locality, which most physicists (and me) are reluctant to do, or accept that the electromagnetic 4-potential offers a more complete description of electromagnetism than the electric and magnetic forces can.

My opinion [17] is that the electric and magnetic forces are *only derived effects* of the fundamental electromagnetic phenomena based on the 4-potentials (which are the statistical results of the interaction produced by high number of photons as intermediators between charged massive particles). This vision of the electromagnetic phenomena is compatible with the vision of gravitation phenomena based on the fundamental concept of the time-space curvature, that is, on the gravitational potential (metrics), where the *gravitational acceleration force* $\overrightarrow{\mathbf{g}}$ is a derived effect (in the case of the Newton's gravitation approximation, this force is just equal to the gradient of the gravitational potential, similarly to the electric force which is proportional to the gradient of the scalar potential ϕ in (9). So, from the fact that $\overrightarrow{E}$ and $\overrightarrow{B}$ are derived forces, we can also have the 3-D regions where we have the photons (thus 4-potential different from zero) but with zero electromagnetic fields $\overrightarrow{E} = \overrightarrow{B} = 0$, as in the case of the scalar Tesla's waves (composed by massive photons) considered in [17].

Let us now consider a long solenoid, carrying magnetic field $\overrightarrow{B}$. If solenoid is ideal (i.e. infinitely long and with a perfectly uniform current distribution), the field $\overrightarrow{B}$ inside is uniform and the field outside is zero. We will use cylindrical coordinate system (r, θ, z) with z axis in the middle of solenoid and pointing in direction of magnetic field.

Thus, we consider the case when *outside* the solenoid $\phi = 0$ while, from (9), $\overrightarrow{A}$ satisfies condition that $\overrightarrow{E} = -\nabla\phi - \frac{\partial\overrightarrow{A}}{\partial t} = 0 - \frac{\partial\overrightarrow{A}}{\partial t} = 0$. That is, in this cylindrical system $\overrightarrow{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$ is static and depends only on the distance r from the center of the solenoid, such that $\overrightarrow{B} = \nabla \times \overrightarrow{A} = 0$. The model for the Aharonov-Bohm effects, with thin solenoid with radius r_0, in order to obtain the constant magnetic force $\overrightarrow{B}$ inside the infinite long solenoid and zero outside this solenoid is provided by the potential vector filed $\overrightarrow{A}$ with

$A_r = A_z = 0$ and

$$A_\theta(r) \equiv \begin{cases} \frac{\Phi_B}{2\pi r_0^2} r & \text{, if } \; 0 \le r \le r_0 \\ \frac{\Phi_B}{2\pi r} & \text{, if } \; r > r_0 \end{cases} \tag{45}$$

where Φ_B is total magnetic flux through the solenoid. That is, $A_\theta(r)$ iz zero in the center of the solenoid and grows linearly up to the max value $\frac{\Phi_B}{2\pi r_0}$ and then decreases again to zero for $r \to \infty$. Notice that for $\overrightarrow{A}(r, \theta) = A_\theta(r)\mathbf{e}_\theta$, holds the Lorenz gauge (also the Coulomb gauge-transverse gauge) condition, when

$$\nabla \cdot \overrightarrow{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} + \frac{1}{c^2} \cdot 0 = 0 \tag{46}$$

Then, from $\nabla \times \overrightarrow{A} = (\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z})\mathbf{e}_r + (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\mathbf{e}_\theta + \frac{1}{r}(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta})\mathbf{e}_z$, we obtain for the magnetic force

$$\overrightarrow{B} = \nabla \times \overrightarrow{A} = \begin{cases} \frac{\Phi_B}{\pi r_0^2}\mathbf{e}_z & \text{, if } \; 0 \le r \le r_0 \\ 0 & \text{, if } \; r > r_0 \end{cases} \tag{47}$$

In the case when we send a coherent beam of electrons toward this solenoid, these massive bosons around solenoid interact with the electrons of this beam by changing their direction of propagation, momentum and energy (Hamiltonian is changed by the 4-potential, that is by the interaction with massive bosons). This model can be used as the physical explanation of the Aharonov-Bohm effect, based on the massive photons (the intermediate bosons of the electromagnetic 4-potential field) interacting with the coherent beam of electrons (see figure 1). Thus, the electrons passing near to the solenoid have the interactions with these massive photons, so that these interactions change the trajectory, momentum vector, and energy (Hamiltonian) of each electron in the beam and explain the interference between left and right (around the solenoid) beams of electrons. The right beam trajectory is parallel to $\overrightarrow{A}$ (at the point of closest approach), while the left beam trajectory is antiparallel. Even in the absence of any Lorentz force on the electrons, these interactions produce a phase shift between the complex wave-packets Ψ of the electrons in the right w.r.t. the left beams.

For any contracted curve around the solenoid C with the surface S, from the Stokes theorem we have that (consider that we adopted positive-time Minkowski time-space signature $(+, -, -, -)$, so that $\sum_{i=1}^3 A_i dq_i = -\overrightarrow{A} d\overrightarrow{\mathbf{r}}$), from (47),

$$\oint_C (-\overrightarrow{A} d\overrightarrow{\mathbf{r}}) = -\int_S (\nabla \times \overrightarrow{A}) d\overrightarrow{S} = \int_S (-\overrightarrow{B} d\overrightarrow{S}) = \Phi_B \tag{48}$$

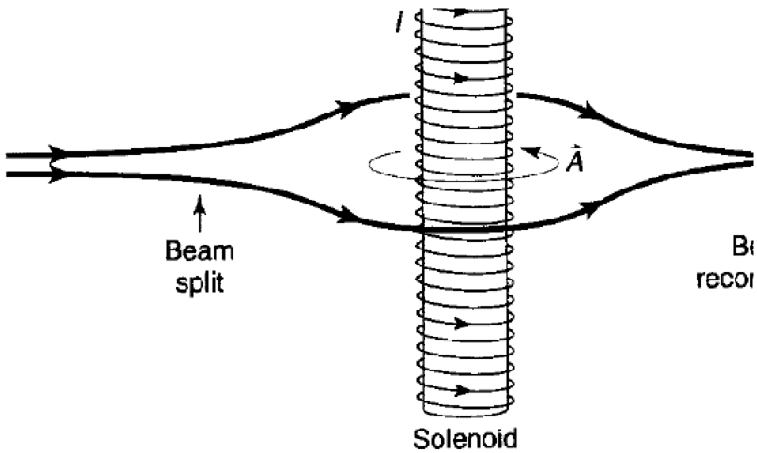

Figure 1: The Aharonov-Bohm

where $\Phi_B \neq 0$ is the total magnetic flux inside this solenoid. Electromagnetic theory implies that an electron with *negative* electric charge $\mathbf{e}$ travelling along some path γ, composed by particle's trajectory from time 0 to time $t > 0$, in a region with zero magnetic field $\overrightarrow{B}$, but non-zero $\overrightarrow{A}$, acquires a phase shift $\triangle\, \theta_A$, given in SI units by

$$\triangle\, \theta_A = -\frac{\mathbf{e}}{\hbar}\int_\gamma \overrightarrow{A} d\overrightarrow{\mathbf{r}} = -\frac{\mathbf{e}}{\hbar}\int_{\overrightarrow{\mathbf{r}}_T(0)}^{\overrightarrow{\mathbf{r}}_T(t)} \overrightarrow{A} d\overrightarrow{\mathbf{r}} = -\frac{\mathbf{e}}{\hbar}\int_0^t \overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(\tau))\overrightarrow{\mathbf{v}}(\tau)d\tau \qquad (49)$$

where $d\overrightarrow{\mathbf{r}} \equiv \overrightarrow{\mathbf{v}}(\tau)d\tau$ is an infinitesimal trajectory length of this electron. That is, in our new Individual-particle QM (IQM) theory [5, 6, 7], we have that the complex wave-packet of a free individual electron (when the solenoid is off) $\Psi_{off} = \Phi_0(t, \overrightarrow{\mathbf{r}})\ \mathrm{e}^{-i\varphi_T^0|_t}\ \mathrm{e}^{-i\alpha^H}$, where $\Phi_m = \Phi_0^2(t, \overrightarrow{\mathbf{r}})$ has a hydrostatic equilibrium spherically symmetric energy-density [10], during this non-relativistic propagation,

$$\varphi_T^0|_t = \frac{E_0 - p_0 v_0}{\hbar} t = \frac{m_0 c^2 + \frac{1}{2} m_0 \overrightarrow{\mathbf{v}}_0^2}{\hbar} t \qquad (50)$$

where m_0 is electron's rest-mass and v_0 (with $\overrightarrow{\mathbf{v}}_0^2 = -v_0^2$) is a constant speed of this free electron along the axis x (the direction of the propagation of the beam of electrons). In the case when the solenoid is on, this electron is not more a free particle, so that it changes its de Broglie pilot phase φ_T by the amount $\triangle\, \theta_A$ in (49). This is expected just because the interactions of massive photons emitted by solenoid (by generating scalar waves above with 4-potential components ϕ and $\overrightarrow{A}$) with an electron, changes its energy, momentum and trajectory, by generating the new pilot-wave phase of this electron $\varphi_T|_t$, such that, from (49) and (50),

$$\theta_R = -\varphi_T|_t = -\varphi_T^0|_t + \triangle\, \theta_A \approx -\frac{1}{\hbar}\int_0^t (m_0c^2 + \frac{1}{2}m_0\overrightarrow{\mathbf{v}}^2(\tau) + \mathbf{e}\overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(\tau))\overrightarrow{\mathbf{v}}(\tau))d\tau \tag{51}$$

by considering that $\|\overrightarrow{\mathbf{v}}\| \approx v_0$ and $d\overrightarrow{\mathbf{r}} = d\overrightarrow{\mathbf{v}}(\tau)d\tau$. However, we can not use the simple method used for the Schrödinger wavefunctions in SQM, where only this phase changing of the wavefunction under influence of $\overrightarrow{A}$ is considered, i.e., by[4] $\psi' = \psi\ \mathrm{e}^{i\theta_R}$ because, in statistic quantum mechanics, the electron is considered as a point-like particle for which only the phase is changed.

Magnetic Aharonov-Bohm effect: Let us now consider the phase-shift Aharonov-Bohm effects on the left and right beams of electrons in Figure 1: two electrons (one in left and other in the right beam) traveling *together* (in parallel) with the same velocity $\overrightarrow{\mathbf{v}}$ pass by the solenoid on opposite sides. At the point of closest approach to the solenoid, one is traveling parallel to $\overrightarrow{A}(r,\theta) = A_\theta(r)\mathbf{e}_\theta$ on its side, the other antiparallel. So, the pilot-wave of each wave-packet (Ψ_1 and Ψ_2, relatively) accumulates a phase shift $\triangle\, \theta_A$ given by (49) as it traverses the vector potential at trajectory points $\overrightarrow{\mathbf{r}}_T(t) = (r,\theta)$.
The scalar product $\overrightarrow{A}\overrightarrow{\mathbf{v}}$ in equation (49) implies that the phase changes are of opposite sign for electrons passing the solenoid on opposite sides. So, the two individual electrons Ψ_1 and Ψ_2 on their different paths (γ_1 and γ_2, respectively) past a long solenoid enclosing the magnetic flux Φ_B accumulate a phase difference

$$\triangle\varphi = -\frac{\mathbf{e}}{\hbar}\int_{\gamma_1}\overrightarrow{A}d\overrightarrow{\mathbf{r}} - (-\frac{\mathbf{e}}{\hbar}\int_{\gamma_2}\overrightarrow{A}d\overrightarrow{\mathbf{r}}) \approx -\frac{\mathbf{e}}{\hbar}\oint_C \overrightarrow{A}d\overrightarrow{\mathbf{r}} = \frac{\mathbf{e}}{\hbar}\Phi_B \tag{52}$$

known as the magnetic Aharonov-Bohm phase shift. We consider that from

[4]Note that such a transformation from the Schrödinger wavefunction ψ (a state vector in Hilbert space) when the solenoid is off to the state $\psi' = \psi\mathrm{e}^{i\theta_R}$ when the solenoid is on, from my point of view seems non correct, because we know that each state in SQM remains equal if we change only the phase of it. Consequently, in the SQM, we have that both ψ and ψ' correspond to the *same quantum state* of the electron. This is in contradiction with the fact that the state ψ of the electron, when the solenoid is off, must be different from the state ψ' when the solenoid is on, and hence they would have different trajectories depending if solenoid is off or on. So, also the probability density to find this electron (described by these two wavefunctions) at a given point, instead to be equal, $\overline{\psi'}\psi' = \overline{\psi}\mathrm{e}^{-i\theta_R}\psi\mathrm{e}^{i\theta_R} = \overline{\psi}\psi$, must be *different* (consider that $\psi(t,\overrightarrow{\mathbf{r}})$ and $\psi'(t,\overrightarrow{\mathbf{r}})$ are distributed theoretically in whole space), that is, for the modules of these two complex time-space functions has to different ($|\psi(t,\overrightarrow{\mathbf{r}})| \neq |\psi'(t,\overrightarrow{\mathbf{r}})|$ and not only their phases). Thus, it has to be provided a different mathematical elaboration of the Aharonov-Bohm quantum experiments in the statistical quantum theory (SQM).

$t = 0$ (when the beam of electrons is split in the left and right-hand side components) to $t > 0$ (when the left and right beams are recombined, see figure 1) these two considered electrons traveling together defined two paths γ_1 and γ_2, respectively, such that $\gamma_1 + \gamma_2$ defines approximatively the whole closed loop C around this solenoid.
So, if we change magnetic field in solenoid, we change phase difference between beams and interference pattern will shift. Aharonov and Bohm suggested how it could be measured by heaving electron waves traversing the two paths interfere with each other. Achieving interference patterns requires that the initial electron beam be adequately coherent and that its splitting maintain that coherence. The former is accomplished by having a well-collimated beam with narrow spreads in angle and particle energy, and later requires something like a diffraction grating or an electromagnetic biprism.

4.1 Unified IQM Gauge Theory for Aharonov-Bohm effects

By considering that IQM theory is classical deterministic part of the quantum theory, also in classical electrodynamics theory this Aharonov-Bohm effect there exists, and not only in the statistical quantum mechanics based on the probabilistic wavefunction (with the ensemble interpretation). In our case, in IQM theory, an individual electron is not point-like but is composed by its 3-D body of rest-mass density $\Phi_m(t, \overrightarrow{\mathbf{r}}) = \Phi^2(t, \overrightarrow{\mathbf{r}})$. During the interaction with the field $\overrightarrow{A}$ (that is, with massive photons which are the intermediate bosons of this field), the rest-mass energy density of the electron is changed as well, so that for the modified electron's wave-packet $\Psi' = \Phi'(t, \overrightarrow{\mathbf{r}})\, \mathrm{e}^{-i\varphi'_T}\, \mathrm{e}^{-i\alpha^H} \neq \Psi\, \mathrm{e}^{i\theta_R}$, because $\Phi'(t, \overrightarrow{\mathbf{r}}) \neq \Phi(t, \overrightarrow{\mathbf{r}})$. In fact, the interaction of an individual massive particle (an electron in our case) and external fields ($\overrightarrow{A}$ in our case) in the IQM theory is defined in the unified gauge theory [7], based on the local symmetry (a kind of localization of the internal phase transformations for the electrons's accelerated complex wave-packet Ψ).
So, we will use now this Aharonov-Bohm experiment as an example, to which we can apply this new gauge theory, with the following property:
Let us consider the evolution of the particle's wave-packet $\Psi \equiv \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T^H}$, from initial time t to another $\Psi' \equiv \Phi(t + \delta t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T^H|_{(t+\delta t)}}$ at a time-instance $t + \delta t > t$, in a given external field (a combination of basic forces). We would like to express this evolution as the result of the local phase transformation $\delta\theta(t, \overrightarrow{\mathbf{r}})$, that is, by $\Psi \mapsto \Psi' = \mathrm{e}^{i\delta\theta(t, \overrightarrow{\mathbf{r}})}\Psi$. We assume that δt is an infinitesimal amount of time during which a particle has the interaction with only one boson

of the external field. Thus, after this interaction, this particle changed its velocity, momentum, energy, and hence its phase for a small amount $\delta\theta$. So, this local phase transformation $\theta(t, \overrightarrow{\mathbf{r}}) = \theta_R(t, \overrightarrow{\mathbf{r}}) + i\theta_I(t, \overrightarrow{\mathbf{r}})$ *is complex*, differently from the SQM theory for the point-like particles (where it is a real value).
In the gauge theory of the IQM presented in Section 3, we derived the gauge Lagrangian density with intermediators of the 4-potential $\mathbf{A}^\mu = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4)$ representing a total external field (as a combination of the basic quantum forces), by (21), $\mathcal{L}_{gauge} = \frac{\hbar}{\mathbf{1}_\Phi}(\frac{\delta\theta}{\delta t} - \alpha(\mathcal{A}_0 + \sum_{j=1}^{3} v_j\mathcal{A}_j))\overline{\Psi}\Psi$, which is invariant for the local phase transformations, where in our case $\alpha = \frac{\mathbf{e}}{\hbar}$ is the *coupling constant*. We demonstrated that the total 4-potential on the massive particle's trajectory (here the trajectory of an electron), that is, for the points $\mathbf{r}_T(t) = t\mathbf{e}_0 + \overrightarrow{\mathbf{r}_T}(t))$, determines particle's velocity $\overrightarrow{\mathbf{v}}$, canonical momentum $\overrightarrow{\mathbf{p}}$ and total energy E (i.e., the Hamiltonian), from (29):

$$\mathcal{A}_0 = -\frac{1}{\alpha}(\frac{E(\mathbf{r}_T, \overrightarrow{\mathbf{p}})}{\hbar} - i\frac{\nabla\cdot\overrightarrow{\mathbf{u}}(\mathbf{r}_T)}{2}), \qquad \overrightarrow{\mathcal{A}}_g = \frac{\overrightarrow{\mathbf{p}}}{\alpha\hbar} \tag{53}$$

Note that the scalar component $\mathcal{A}_0$ is a complex component, while other $\mathcal{A}_j$ are real components of the gauge 4-potential $\mathbf{A}^\mu = \mathcal{A}_0\mathbf{e}_0 + \overrightarrow{\mathcal{A}}_g)$.
Remark: In fact, in our case from (28), we are interested for the real component $Re(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g)$ of the total 4-potential which generates the modification $\delta\theta_R$ of electron's pilot-wave phase (the imaginary components of $\mathcal{A}_0$, defines the complex phase component $\delta\theta_I$) so that, from (24) and (51)

$Re(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g) = \frac{1}{\alpha}\frac{\delta\theta_R}{\delta t} = \frac{1}{\alpha\delta t}(-\frac{1}{\hbar}(m_0c^2 + \frac{1}{2}m_0\overrightarrow{\mathbf{v}}^2(t) + \mathbf{e}\overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(t))\overrightarrow{\mathbf{v}}(t))\delta t$, that is,

$$Re(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g) = -\frac{1}{\mathbf{e}}(m_0c^2 + \frac{1}{2}m_0\overrightarrow{\mathbf{v}}^2(t) + \mathbf{e}\overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(t))\overrightarrow{\mathbf{v}}(t)) \tag{54}$$

Note that for $\mathbf{v}_4 = (v_0, v_1, v_2, v_3) = (c, \overrightarrow{\mathbf{v}})$, and solenoid's 4-potential (8), $\mathbf{A}_4 = (A_0, A_1, A_2, A_3) = (\frac{\phi}{c}, \overrightarrow{A})$, we obtain,

$$-\mathbf{A}_4\mathbf{v}_4 = -\phi - \overrightarrow{A}\overrightarrow{\mathbf{v}} = -\phi + \sum_{i=1}^{3} A_i v_i \tag{55}$$

Thus, in our case with $A_0 = \frac{\phi}{c} = 0$, from general equation (54), we obtain this specific equation

$$Re(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g) = -\frac{1}{\mathbf{e}}(m_0c^2 + \frac{1}{2}m_0\overrightarrow{\mathbf{v}}^2 + \mathbf{e}\mathbf{A}_4\mathbf{v}_4) \tag{56}$$

and from this we see the fundamental physical difference between the general IQM gauge 4-potential $\mathbf{A}^{\mu}$ and electromagnetic 4-potential $\mathbf{A}_4$: when the electromagnetic 4-potential is zero, the gauge 4-potential becomes constant (different from zero) and continues to govern particle's inertial propagation as well. That is, the gauge 4-potential always determine particle's pilot-wave phase also in the intervals of time when particle is free.

Thus, from equation (38) in Proposition 1, $\delta\theta_R = -\varphi_T|_{t+\delta t} + \varphi_T|_t$, we obtain the changing of electron's pilot-wave phase φ_T, for an infinitesimal amount of time δt,

$$\varphi_T|_{t+\delta t} = \varphi_T|_t + \frac{1}{\hbar}(m_0c^2 + \frac{1}{2}m_0\overrightarrow{\mathbf{v}}^2 + \mathbf{e}\mathbf{A}_4\mathbf{v}_4)\delta t \tag{57}$$

caused by the electromagnetic vector field $\overrightarrow{A}$, generated by the solenoid, as expected.

Moreover, we note the fundamental difference from the *mathematical* total gauge *complex* 4-potential $\mathbf{A}^{\mu} = (\mathcal{A}_0, \overrightarrow{\mathcal{A}}_g)$ and the *physical* solenoid's *real* 4-potential (8), $\mathbf{A}_4 = (\frac{\phi}{c}, \overrightarrow{A})$.

$\square$

If we consider now the whole process of the interaction of an electron with massive photons of the field $\overrightarrow{A}$, along electron's path (on its trajectory) from time 0 to a time $t > 0$, then the sum θ_R of all infinitesimal changes $\delta\theta_R(t)$ of the phase along electron's trajectory $\overrightarrow{\mathbf{r}}_T(t)$ is equal to the integral in (51). Let us how that this integral corresponds to the path-integral of the gauge 4-potential for computation of the total particle's pilot wave phase θ_R. In fact, from (51) we have that

$$\begin{aligned}
&\theta_R = -\varphi_T^0|_t + \triangle\,\theta_A = -\varphi_T^0|_t - \tfrac{\mathbf{e}}{\hbar}\textstyle\int_\gamma \overrightarrow{A}d\overrightarrow{\mathbf{r}} \\
&\approx -\tfrac{1}{\hbar}\textstyle\int_0^t (m_0c^2 + \tfrac{1}{2}m_0\overrightarrow{\mathbf{v}}^2(\tau) + \mathbf{e}\overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(\tau))\overrightarrow{\mathbf{v}}(\tau))d\tau \\
&= \tfrac{\mathbf{e}}{\hbar}\textstyle\int_0^t -\tfrac{1}{\mathbf{e}}(m_0c^2 + \tfrac{1}{2}m_0\overrightarrow{\mathbf{v}}^2(\tau) + \mathbf{e}\overrightarrow{A}(\overrightarrow{\mathbf{r}}_T(\tau))\overrightarrow{\mathbf{v}}(\tau))d\tau \\
&= \alpha\textstyle\int_0^t Re(\mathcal{A}_0 - \overrightarrow{\mathbf{v}}\overrightarrow{\mathcal{A}}_g)d\tau \qquad\qquad \text{from (54)} \\
&= \alpha\textstyle\int_0^t Re(\mathcal{A}_0 + \overrightarrow{\mathcal{A}}_g)((\mathbf{e}_0 - \overrightarrow{\mathbf{v}}))d\tau \\
&= \alpha\textstyle\int_\gamma Re(\mathbf{A}^{\mu})d\overrightarrow{\mathbf{r}}_{T\mu} \qquad \text{from } d\overrightarrow{\mathbf{r}}_{T\mu} = (\mathbf{e}_0 - \overrightarrow{\mathbf{v}})\delta\tau \\
&= Re(\alpha\textstyle\int_\gamma \mathbf{A}^{\mu}d\overrightarrow{\mathbf{r}}_{T\mu})
\end{aligned}$$

That is, we obtain the following result of the path integration if we are using the electromagnetic vector field $\overrightarrow{A}$ or the gauge 4-potential $\mathbf{A}^{\mu}$:

$$\theta_R = -\varphi_T^0|_t - \frac{\mathbf{e}}{\hbar}\int_\gamma \overrightarrow{A}d\overrightarrow{\mathbf{r}} = Re(\alpha\int_\gamma \mathbf{A}^{\mu}d\overrightarrow{\mathbf{r}}_{T\mu}) \tag{58}$$

where $-\varphi_T^0|_t$ is the path integration (50) when the solenoid is 'off'. With this we concluded the translation from the classical electromagnetic framework into the global gauge IQM theory for individual electrons of the Aharonov-Bohm effects.

Consequently, when the solenoid is active (solenoid is 'on'), we have the transformation from the wave-packet of an electron $\Psi = \Phi(t, \overrightarrow{\mathbf{r}})\ \mathrm{e}^{-i\varphi_T}$ into the modified (by the potential vector $\overrightarrow{A}(t, \overrightarrow{\mathbf{r}})$) wave-packet $\Psi' = \Phi'(t, \overrightarrow{\mathbf{r}})\ \mathrm{e}^{-i\varphi'_T} = \Phi'(t, \overrightarrow{\mathbf{r}})\ \mathrm{e}^{-i(\varphi_T - \theta_A)}$, which is shifted for the phase θ_A given by (49).

4.2 Quantum Forces in the Aharonov-Bohm Effect

Based on the results in previous section, we can explain the unified gauge theory, based on the the total gauge 4-potential $\mathcal{A}_j$, $0 \le j \le 3$, in this example of Aharonov-Bohm experiments:

Corollary 1 *The velocity-dependent potential energy U of the non-relativistic electron in the Aharonov-Bohm experiments is given by*

$$U = -\boldsymbol{e}Re(\mathcal{A}_0 - \overrightarrow{\boldsymbol{v}}\overrightarrow{\mathcal{A}}_g) - \frac{m_0\overrightarrow{\boldsymbol{v}}^2}{2} \tag{59}$$

where $\mathcal{A}_0$ is the time and $\overrightarrow{\mathcal{A}}_g$ vector coponents of the gauge 4-potential.

Proof: It is enough to demonstrate that the electron's potential energy U, used in the Lagrangian $L = T - U$, where $T = \frac{m_0 v^2}{2} = -\frac{m_0}{2}\overrightarrow{\mathbf{v}}(t)\overrightarrow{\mathbf{v}}(t)$ is electron's kinetic energy for the velocity $\overrightarrow{\mathbf{v}}(t) = \frac{d}{dt}\overrightarrow{\mathbf{r}}_T(t)$, is equal, from from (56) and(55), to

$$U = \mathbf{e}(\phi + \overrightarrow{A}\overrightarrow{\mathbf{v}}) + m_0c^2 = \mathbf{e}(\phi - \sum_{i=1}^{3} A_i v_i) + m_0c^2 \tag{60}$$

In the SI system, the Lorentz force, acting on an electron during Aharonov-Bohm experiments, is equal to

$$\overrightarrow{F} = \mathbf{e}(\overrightarrow{E} + \overrightarrow{\mathbf{v}} \times \overrightarrow{B}) = \mathbf{e}(-\nabla\phi - \frac{\partial\overrightarrow{A}}{\partial t} + \overrightarrow{\mathbf{v}} \times (\nabla \times \overrightarrow{A})) \tag{61}$$

Let us, for example, compute the first component of the force $\overrightarrow{F}$, with

$(\overrightarrow{\mathbf{v}} \times (\nabla \times \overrightarrow{A}))_1 = v_2(\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}) - v_3(\frac{\partial A_1}{\partial q_3} - \frac{\partial A_3}{\partial q_1})$

$= \frac{\partial}{\partial q_1}(\sum_{i=1}^{3} A_i v_i) - \frac{dA_1}{dt} + \frac{\partial A_1}{\partial t} = \frac{\partial}{\partial q_1}(-\overrightarrow{A}\overrightarrow{\mathbf{v}}) - \frac{dA_1}{dt} + \frac{\partial A_1}{\partial t}$

since

$\frac{\partial}{\partial q_1}(-\overrightarrow{A}\overrightarrow{\mathbf{v}}) = v_1\frac{\partial A_1}{\partial q_1} + v_2\frac{\partial A_2}{\partial q_2} + v_3\frac{\partial A_3}{\partial q_3}$,
and
$\frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + v_1\frac{\partial A_1}{\partial q_1} + v_2\frac{\partial A_1}{\partial q_2} + v_3\frac{\partial A_1}{\partial q_3}$,
so we obtain,
$F_1 = \mathbf{e}[-\frac{\partial\phi}{\partial q_1} - \frac{\partial A_1}{\partial t} + \frac{\partial}{\partial q_1}(-\overrightarrow{A}\overrightarrow{\mathbf{v}}) - \frac{dA_1}{dt} + \frac{\partial A_1}{\partial t}]$
$= \mathbf{e}[-\frac{\partial}{\partial q_1}(\phi + \overrightarrow{A}\overrightarrow{\mathbf{v}}) + \frac{d}{dt}\frac{\partial}{\partial v_1}(\phi + \overrightarrow{A}\overrightarrow{\mathbf{v}})] = -\frac{\partial U}{\partial q_1} + \frac{d}{dt}\frac{\partial U}{\partial v_1}$
and hence:

$$\overrightarrow{F} = -\nabla U + \frac{d}{dt}(\frac{\partial U}{\partial\overrightarrow{\mathbf{v}}}) \tag{62}$$

Let us show that U is electron's potential, which can be used to define the electron's Lagrangian in classical electrodynamics,

$$L = T - U = -\frac{m_0\overrightarrow{\mathbf{v}}^2}{2} - \mathbf{e}(\phi + \overrightarrow{A}\overrightarrow{\mathbf{v}}) - m_0c^2 = \frac{m_0v^2}{2} - m_0c^2 - \mathbf{e}\phi + \mathbf{e}\sum_{i=1}^{3}A_iv_i \tag{63}$$

which is a non-relativistic case when the external field energy-potential $L_e = -\mathbf{e}(\phi + \overrightarrow{A}\overrightarrow{\mathbf{v}})$ depends also on the particle's velocity.
It can be shown by using the Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}} = 0 \tag{64}$$

where, for the space coordinate q, $\dot{q}$ denotes the velocity, and here rewritten in the full vectorial form:
$0 = \frac{d}{dt}\frac{\partial L}{\partial\overrightarrow{\mathbf{v}}} - \nabla L = \frac{d}{dt}\frac{\partial}{\partial\overrightarrow{\mathbf{v}}}(-\frac{m_0\overrightarrow{\mathbf{v}}^2}{2} - U) - \nabla(-\frac{m_0\overrightarrow{\mathbf{v}}^2}{2} - U)$
$= \frac{d}{dt}(m_0\overrightarrow{\mathbf{v}} - \frac{\partial U}{\partial\overrightarrow{\mathbf{v}}}) + \nabla U = m_0\frac{d\overrightarrow{\mathbf{v}}}{dt} - (-\nabla U + \frac{d}{dt}(\frac{\partial U}{\partial\overrightarrow{\mathbf{v}}}))$.
Thus, from the Newton's law $\overrightarrow{F} = m_0\frac{d\overrightarrow{\mathbf{v}}}{dt}$, we obtain the correct Lorentz force in (62), so that indeed U is electron's potential. We know the *canonical* momentum from classical mechanics, from (63),

$$\overrightarrow{\mathbf{p}} \equiv \frac{\partial L}{\partial\overrightarrow{\mathbf{v}}} = m_0\overrightarrow{\mathbf{v}} + \mathbf{e}\overrightarrow{A} \tag{65}$$

where $m_0\overrightarrow{\mathbf{v}}$ is the kinetic momentum, that is,

$$\overrightarrow{\mathbf{v}} = \frac{1}{m_0}(\overrightarrow{\mathbf{p}} - \mathbf{e}\overrightarrow{A}) \tag{66}$$

so that the Hamiltonian (in our Minkowski time-space signature $(+,-,-,-)$) is given by the Legendre transformation $H = -\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}} - L = -\frac{m_0\overrightarrow{\mathbf{v}}^2}{2} + \mathbf{e}\phi + m_0c^2$ and, from (66),

$$H = -\frac{1}{2m_0}(\overrightarrow{\mathbf{p}} - \mathbf{e}\overrightarrow{A})^2 + \mathbf{e}\phi + m_0c^2 \tag{67}$$

Hence, the unified gauge theory of IQM is in perfect accordance with classical electrodynamics. Note that in the de Broglie pilot-wave phase, the total energy is equal to the Hamiltonian and the momentum is canonical.
□
In addition, the classical Maxwell equations are derived from QED scattering process while both classical electromagnetic fields and potentials serve as mathematical tools that are constructed to approximate the interactions among elementary particles described by QED physics, i.e., neither classical fields nor potentials represent any real entities of nature.
The vector potentials $(\phi, \overrightarrow{A})$ as well as EM field are nothing but mathematical idealisations that approximate the interactions among electrons which are mediated by the photons, as it has been shown in [17], the EM field will come into play in classical theory since they comprise major parts of the Lorentz force equation. Now it is clear that, in the Aharonov-Bohm effect, it does not matter whether the EM fields are zero or not in the region where the beam of electrons can enter, the underlying mechanism is the interactions between this beam of electrons and the field $\overrightarrow{A}$, while such interactions are mediated by massive photons. Moreover, the Maxwell's EM fields are not produced directly by electrons or any hypothetical magnetic monopoles, they serve as calculation tools and emerge in the classical world because of the coupling effect between charged particles with photons. We can regard the classical four-potential $(\phi, \overrightarrow{A})$ as well as EM field as mathematical statistical manifestations of photons. There is no any 'nonlocal feature' in an Aharonov-Bohm experiment: its effect can be interpreted as the manifestation of massive photons propagating between electrons in the framework of the IQM theory.
As it was shown in [17], the *electromagnetic* phenomena and forces are achieved trough the basic phenomena of *emission and absorption* of the long/short-range photons by the charged particles. So, also Lorentz force is the statistical effect of this phenomena of *emission and absorption* of the long/short-range photons by the charged particle.
Thus, we can have the particular cases when the 4-potential field $(\phi, \overrightarrow{A})$ there exists, mediated by massive and slow photons, in which the Maxwell's electromagnetic fields $\overrightarrow{E}$ and $\overrightarrow{B}$ are zero and the Lorentz force consequently is zero, but still there is another force that these massive photons generate on an electron, based on their mutual collisions in which these massive photons transfer their momentum to this electron.
It is just the case of the Aharonov-Bohm effect, because for $\phi = \frac{\partial \overrightarrow{A}}{\partial t} = \nabla \times \overrightarrow{A} = 0$ we obtain, from (61), that the Lorentz force $\overrightarrow{F}$ is zero. However, in the late

1990's, a quantum force was predicted for the Aharonov-Bohm physical system by Shelankov [18], and elucidated by Berry [19]. That this (non-Lorentz) force is indeed present is demonstrated a number of times [20]. For experiments with electron beams, the presence of a longitudinal force along the beam can lead to time delays, while a transverse force leads to deflections. The absence of a longitudinal force, as made apparent by the absence of electron time delays, has been investigated more recently. The deflection, another indicator of force, is accompanied by a characteristic asymmetry in the electron diffraction pattern. In order to understand the underlying physical nature of these non-Lorentz forces, we have to analyze the consequences of the Maxwell equations in the case of the Aharonov-Bohm effects.

So, if we replace the equations (9) into the ordinary Maxwell equations for the electric and magnetic fields $\overrightarrow{E}$ and $\overrightarrow{B}$, we obtain the following equations for the electromagnetic 4-potential in the region outside the solenoid for $r > r_0$, where $\rho = 0$ and current $\overrightarrow{j} = 0$:

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \overrightarrow{A}) = 0, \quad (-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\overrightarrow{A} + \nabla(\nabla \cdot \overrightarrow{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}) = 0 \quad (68)$$

From the fact that, in our case, the voltage potential is $\phi = 0$, we have no any scalar wave for ϕ, differently from the case analyzed in [17] for Tesla's scalar waves. So, we need to investigate if we have the vector $\overrightarrow{A}$ (radiation) waves. Notice that, in our case, it holds not only Coulomb gauge potential condition $\nabla \cdot \overrightarrow{A} = 0$ but also the Lorentz gauge potential condition $\nabla \cdot \overrightarrow{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$. Thus, the system of two equations (68) is reduced only to the vector-potential wave equation:

$$(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\overrightarrow{A} = 0 \quad (69)$$

Hence, if it would be a non trivial identity with zero, then we would have that the vector potential field $\overrightarrow{A}$ is mediated by the long-range photons which are emitted from the accelerated electrons in the solenoid (we recall that also if the speed of free electrons in the solenoid's current is constant, from the fact that their orbit in the solenoid is a circle, all electrons in the solenoid's current have a constant acceleration with direction toward the center of the solenoid) and then become massless photons able to propagate by the speed of light as a cylindrical wave.

Let us show that this *is not* the case here, so that the mediated bosons for the field $\overrightarrow{A}$ are massive and slow *short-range* photons [17], which generate the electrostatic field of each electron (in its rest-frame).

So, let us show that (69) is not the wave equation of massless photon's propagation with the speed of light. Let us control its first term with Laplacian operator, given in the solenoids cylindrical coordinate system with the center in the center of solenoid,

$$\nabla^2\overrightarrow{A} = (\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta})\mathbf{e}_r + (\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta})\mathbf{e}_\theta + \nabla^2 A_z\mathbf{e}_z \quad (70)$$

so, from (45), we obtain:

$$\nabla^2\overrightarrow{A} = (\nabla^2 A_\theta - \tfrac{A_\theta}{r^2})\mathbf{e}_\theta = ((\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial A_\theta}{\partial r}) + \tfrac{1}{r^2}\tfrac{\partial^2 A_\theta}{\partial\theta^2} + \tfrac{\partial^2 A_\theta}{\partial z^2}) - \tfrac{A_\theta}{r^2})\mathbf{e}_\theta$$
$$= (\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial A_\theta}{\partial r}) - \tfrac{A_\theta}{r^2})\mathbf{e}_\theta = (\tfrac{\partial^2 A_\theta}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial A_\theta}{\partial r} - \tfrac{A_\theta}{r^2})\mathbf{e}_\theta = \mathbf{e}_\theta\tfrac{\Phi_B}{2\pi}(\tfrac{\partial^2}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial}{\partial r} - \tfrac{1}{r^2})\tfrac{1}{r}$$
$$= 0.$$

Also the second time-derivation component of the wave-equation (69) is equal to zero, because the vector $\overrightarrow{A}$ is time-invariant, and hence this wave equation does not represent the propagation of the massless photons with the speed of light. Thus, the intermediators for the vector-potential field $\overrightarrow{A}$ can only be the massive and slow short-range photons, irradiated by electrons in the solenoid. Let us show that the non-wave vector potential $\overrightarrow{A}(r,\theta) = A_\theta(r)\mathbf{e}_\theta$, given in this cylindrical coordinate system with the center in the center of the solenoid, outside this solenoid (for $r > r_0$, where r_0 is the radius of the solenoid) is really the statistical consequence of the short-range photons generated by the free electrons inside this solenoid that propagate with a velocity $\overrightarrow{\mathbf{v}}_s = v_s\mathbf{e}_\theta$, with a constant speed v_s. We consider that the natural interference of the free electrons inside the solenoid (their mutual repulsion) can be neglected w.r.t. the external source-filed that generates the electric current inside this solenoid. Thus, the density of the free electrons in this solenoid is constant, so that for any small angle $\delta\theta = \frac{\pi}{n}$, where $n >> 1$, the number of free electrons in this solenoid's section is $N > 1$. We denote the electrostatic potential generated by a free electron (w.r.t. its rest-frame) in the solenoid's section at the angle $\theta+\alpha$ by $\phi_{+\alpha}$, while that in the section at the angle $\theta-\alpha$ (symmetrically to the observed angle θ) by $\phi_{-\alpha}$.

From the fact that all free electrons in the solenoid's current move with a constant speed $\overrightarrow{\mathbf{v}}_s = v_s\mathbf{e}_\theta$, there is no any electrostatic potential ($\phi = 0$) generated by them (in our frame with cylindrical coordinate system), but only the vector potential $\overrightarrow{A}$ generated by superposition of all moving electrostatic potentials of each individual free electron of the current. We consider that $r - r_0 >> d$, where d is diameter of the wire of the solenoid, so that all N free electrons in a given section large the infinitesimal angle $\delta\theta = \frac{\pi}{n}$, for $n >> 1$, are practically at the same distance from the observed point (r,θ), and that inside this sector $\delta\theta$ all N free electrons practically propagate linearly with constant speed v_s. Hence, from (72)[5], for the discrete set of angles $\alpha = j\delta\theta$, $0 \leq j \leq 2n$, so that $0 \leq \alpha \leq 2\pi$ covers the whole solenoid, we obtain:

$$\overrightarrow{A}(r,\theta) = \sum_{j=0}^{2n} \frac{(N\phi_{+\alpha})}{c^2} v_s \mathbf{e}_{+\alpha} = \frac{Nv_s}{c^2}\sum_{j=0}^{n}(\phi_{+\alpha}\mathbf{e}_{+\alpha} + \phi_{-\alpha}\mathbf{e}_{-\alpha}) \tag{73}$$

Consequently, from the fact that the electrons with $\phi_{+\alpha}$ and $\phi_{-\alpha}$ are at the same distance $\rho = \sqrt{(r-r_0)^2 + r_0^2\sin^2\alpha}$ from the observed point (r,θ), both of them we will denote by the same Coulomb potential $\phi(r,\alpha) = k_e\frac{\mathbf{e}}{\rho}$ and, from the fact that $\mathbf{e}_{+\alpha} + \mathbf{e}_{+\alpha} = 2\cos\alpha\mathbf{e}_\theta$, so that for any $0 \leq \alpha = j\delta\theta \leq \pi$, $\phi_{+\alpha}\mathbf{e}_{+\alpha} + \phi_{-\alpha}\mathbf{e}_{-\alpha} = \phi(r,\alpha)(\mathbf{e}_{+\alpha} + \mathbf{e}_{+\alpha}) = 2\phi(r,\alpha)\cos\alpha\mathbf{e}_\theta$, the (73) reduces to

$$\overrightarrow{A}(r,\theta) = (\frac{2Nv_s}{c^2}\sum_{j=0}^{n}\phi(r,\alpha)\cos\alpha)\ \mathbf{e}_\theta = A_\theta(r)\mathbf{e}_\theta \tag{74}$$

That is, we demonstrated that the vector potential $\overrightarrow{A}$ has no radial component, as expected, and $A_\theta(r) = \frac{2Nv_s}{c^2}\sum_{j=0}^{n}\phi(r,\alpha)\cos\alpha$ corresponds to that defined by the magnetic flux in (45) for $r > r_0$.
Note that each short-range photon has an angular speed as well caused by the angular speed of the free electrons in the solenoid. So, the cloud of short-distance photons emitted/absorbed by the free electrons in the solenoid has an radial movement from and toward (returning phase) the center of solenoid, but also this rotation around the solenoid which generates a vortex of short-range photons. The radial movement of the photons of this dense cloud has no effect to the electrons in the beams, because they have impact on them from both sides (emitted/returned photons) with the same radial momentum quantity. But the angular momentum of the vortex of the short-range photons in this vortex is only in one direction: so its effects on the left-hand and right-hand

[5] The vector-potential is obtained [17] as the effect of the movement of the electrostatic scalar field ϕ defined in the rest-frame,

$$\overrightarrow{A}_1 = \frac{\phi}{c^2}\overrightarrow{\mathbf{v}} \tag{71}$$

that is, as the flux of the electrostatic scalar field (defined in a rest-frame). So, the direction of $\overrightarrow{A}$ is equal to the direction of the velocity $\overrightarrow{\mathbf{v}}$. If we consider a general case, that the current $\overrightarrow{j}$ is composed by $n > 1$ charged particles, each one with its electrostatic potential ϕ_j moving with the speed $\overrightarrow{\mathbf{v}}_j$, by superposition we obtain that the resulting 3-D vector potential generated by this current is equal to

$$\overrightarrow{A} = \sum_{j=1}^{n}\frac{\phi_j}{c^2}\overrightarrow{\mathbf{v}}_j \tag{72}$$

that is, $\overrightarrow{A}$ is a result of moving of the electrostatic potentials of the charged particles.

sides of the electron-beams around the solenoid are very different, and this explains the Aharonov-Bohm effect.
So, we have the collisions of the two fluids: the vortex fluid of the massive short-range photons emitted from the solenoid (intermediators for the electromagnetic vector-potential $\overrightarrow{A}$) and the fluid of the coherent beam of the electrons outside the solenoid. In the two-fluid hydrodynamics the superfluid (Magnus force is not the only force on the vortex transverse to its velocity) there is also a transverse force between the vortex and quasiparticles moving with respect to the vortex. The transverse force from rotons was found by Lifshitz and Pitaevskij [21] from the quasiclassical scattering theory and also Iordanskij [22] revealed the transverse force from phonons. Then it was demonstrated that the Lifshitz-Pitaevskij force for rotons and the Iordanskij force for phonons originate from interference between quasiparticles which move past the vortex on the left and on the right sides with *different phase shifts*, like in the Aharonov-Bohm effect. So, let us consider in details such effects of the collisions between the vortex of massive low-speed short-range photons around the solenoid and an individual electron passing near to this solenoid vortex.
When the solenoid is 'on', the speed of the electron is $\overrightarrow{\mathbf{v}}(t) = v_x(t)\mathbf{e}_1 + v_y(t)\mathbf{e}_2$, where $v_x(t) \approx v_0$ is electron's longitudinal speed and $v_y(t)$ its time-dependent transverse speed when the solenoid is 'on'. Consequently, we have the transverse force $F = m_0 \frac{dv_y(t)}{dt}$ acting on an individual electron (caused by the collisions with the massive photons generated by the accelerated electrons in the solenoid). This transverse is asymmetric (w.r.t. the left/right electron-beam components around the solenoid) and experimentally demonstrated [18, 19, 20]. The word *force* is a label to describe a transfer of momentum between two objects: from the massive photons which creates a vortex around the solenoid and an individual electron which passes close to this solenoid.
In hydrodynamics, such a flow pattern is called a vortex tube, a vortex line, or simply a vortex. Thus, as noted in [23], the Aharonov-Bohm effect of the transverse force is similar to the classical hydrodynamics Magnus force (The Iordanskij force [22] is related to the acoustic Aharonov-Bohm effect). The analogy between wave scattering by vortex and electron scattering by the magnetic-flux tube (the Aharonov-Bohm effect) was studied in classical hydrodynamics for water surface waves (the acoustic Aharonov-Bohm effect).
The scattering of the electrons from the vortex of massive photons (quantum intermediators of the electromagnetic potential vector $\overrightarrow{A}$), emitted from the solenoid (by generating a kind of a superfluid), is analog to the scattering of phonons by the vortex in the hydrodynamics [23]. In fact, the formula (56) in

[23], representing the quasiclassical solution ϕ of the sound equation, is formally similar to the rest-mass energy wave packet Ψ of an electron in the IQM, where $\hbar\omega$ corresponds to the electrons total energy and $\hbar\overrightarrow{\mathbf{k}}$ to its momentum; The variation of the action δS, due to interaction with the circular velocity from the vortex along quasiclassical trajectories, corresponds to the variation of the phase $\hbar\delta\theta_R$ used here in our IQM theory of Aharonov-Bohm effects:

"*Using this solution in the momentum balance one obtains the equation of vortex motion, which contains the Iordanskii force. The momentum transfer responsible for the Iordanskii force occurs at small scattering angles where a phenomenon analogous to the Aharonov-Bohm effect is important: an interference between the waves* [here: the electron's rest-mass energy density wave-packets] *on the left and on the right from the vortex with different phase shifts after interaction with the vortex.*"
From page 3 in [23].
However, the relationship with Aharonov-Bohm effects presented in [23] (Section 7) is given in the statistical Schrödinger equation SQM framework, while here this work is done in the IQM classical deterministic mechanics framework.

References

[1] Ehrenberg W. and Siday R., The Refractive Index in Electron Optics and the Principles of Dynamics, Proceedings of the Physical Society, Series B, 62, 1949

[2] Aharonov Y. and Bohm D, Significance of electromagnetic potentials in the quantum theory, Physical Revie 115, 485, 1959

[3] Chamners R.G., Shift of an electron interference pattern by enclosed magnetic flux, Phys. Rev. Lett. 5, 3, 1960

[4] Majkić Z., Partial Differential Equations for Wave Packets in the Minkowski 4-dimensional Spaces, Differencialnie uravnenia i protsesy upravlenia, no.1(2011), http://www.math.spbu.ru/diffjournal/pdf/madjkic.pdf, 2011

[5] Majkić Z., Completion and Unification of Quantum Mechanics with Einstein's GR Ideas, Part I: Completion of QM, Nova Science Publishers, New York, ISBN:978-1-53611-946-6, July, 2017

[6] Majkić Z., Completion and Unification of Quantum Mechanics with Einstein's GR Ideas, Part II: Unification with GR, Nova Science Publishers, New York, ISBN:978-1-53611-947-3, September, 2017

[7] Majkić Z., Completion and Unification of Quantum Mechanics with Einstein's GR Ideas, Part III: Advances, Revisions and Conclusions, Nova Science Publishers, New York, ISBN:978-1-53617-200-3, November, 2019

[8] Osakabe N., at all. Experimental confirmation of Aharonov-Bohm effect using a toroidal magnetic field confined by a superconductor, Physical Review A. 34 (2), 1986

[9] Majkić Z., Double-slit Experiment: a Test for Individual Particles Completion of Quantum Mechanics, Differencialnie uravnenia i protsesy upravlenia, no.2(2019), Publisher: Mathematics and Mechanics Faculty of Saint-Petersburg State University, Russia, http://www.math.spbu.ru/diffjournal/pdf/madjkic.pdf, 2019

[10] Majkić Z., Hydrodynamic equilibrium and stability for particle's energy-density wave-packets: State and revision, Differencialnie uravnenia i protsesy upravlenia, no.3(2018), Publisher: Mathematics and Mechanics Faculty of Saint-Petersburg State University, Russia, http://www.math.spbu.ru/diffjournal/pdf/madjkic.pdf, 2018

[11] Majkić Z., Schrödinger Equation and Wave Packets for Elementary Particles in the Minkowski Spaces, Differencialnie uravnenia i protsesy upravlenia, no.3(2011), Publisher: Mathematics and Mechanics Faculty of Saint-Petersburg State University, Russia, http://www.math.spbu.ru/diffjournal/pdf/madjkic2.pdf, 2011

[12] Majkić Z., Differential Equations for Elementary Particles: Beyond Duality, LAP LAMBERT Academic Publishing, Saarbr*ü*cken, Germany, 2013

[13] Jackson J.D., Classical Electrodynamics, (3rd ed.). Wiley, 1998

[14] Adelberger E., Dvali G., Gruzinov A., Photon Mass Bound Destroyed by Vortices, Phys.Rev.Lett.98:010402, 2007

[15] Gibs P., Is The Speed of Light Constant?, in Carlip, S. Usenet Physics FAQ. University of California, Riverside. http://math.ucr.edu/home/baez/physics/Relativity/SpeedOfLight /speed_of_light.html, 1997

[16] Sidharth B., The Thermodynamic Universe, World Scientific. p. 134. ISBN 9812812342, http://books.google.com/books?id=OUfHR36wSfAC&pg=PA134], 2008

[17] Majkić Z., Derivation of Electromagnetism from Quantum Theory of Photons: Tesla Scalar Waves, E-Journal Differential Equations and Control Processes, N.3, 2020, Publisher: Mathematics and Mechanics Faculty of Saint-Petersburg State University, Russia, 2020

[18] Shelankov A.L., Magnetic force exerted by the Aharonov-Bohm line, Europhys. Lett. 43, 623, 1998

[19] Berry M.V.J., Aharonov-Bohm beam deflection: Shelankov's formula, exact solution, asymptotics and an optical analogue, J.Phys.A: Math.Gen. 32, 5627, 1999

[20] Becker M, and Guzzinati G. and Armand B. and Verbeeck J. and Batelaan H., Observation of Quantum Forces in the Aharonov-Bohm effect, arXiv: 1705.07139, May, 2017

[21] Lifshitz E.M. and Pitaevskij L.P., Zh. Eksp. Teor. Fiz. 33, 535 (1957), [Sov. Phys.-JETP 6, 418], 1958

[22] Iordanskij S.V., Zh. Eksp. Teor. Fiz. 49, 225 (1965), [Sov. Phys.-JETP 22, 160], 1966

[23] Sonin E.B., Magnus force and Aharonov-Bohm effect in superfluids, arXiv:cond-mat 0104221, 2001

[24] Hehl F.W. and von der Heyde P. and Kerlick G.D. and J.M.Nester J.M., General relativity with spin and torsion: foundations and prospects, Rev. Mod. Phys. 48, N0.3, 1976