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A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to a new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. This approach is based on multiple Fourier–Legendre series and multiple trigonometric Fourier series. The theorem on the mean-square convergent expansion for the iterated Stratonovich stochastic integrals of arbitrary multiplicity is formulated and proved under the condition of convergence of trace series. This condition has been verified for integrals of multiplicities 1 to 5 and complete orthonormal systems of Legendre polynomials and trigonometric functions in Hilbert space. The Hu–Meyer formula and multiple Wiener stochastic integral were used in the proof of the mentioned theorem. The rate of mean-square convergence of the obtained expansions is found. The results of the article can be applied to the numerical integration of Itô stochastic differential equations with non-commutative noise in the framework of the approach based on the Taylor–Stratonovich expansion.
Key words: iterated Stratonovich stochastic integral, iterated Itô stochastic

integral, Itô stochastic differential equation, generalized multiple Fourier series, multiple Fourier–Legendre series, multiple trigonometric Fourier series, mean-square convergence, expansion.

Contents

1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a non-decreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{w}_t^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{w}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as the Itô stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Itô SDEs are adequate mathematical models of dynamic systems of various physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions [2]-[13]. The most important feature of such expansions is a presence in them of the so-called iterated Itô and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Itô SDEs and have the following form

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t, T]$, $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\int \text{ and } \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively.

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in the classical Taylor–Ito and Taylor–Stratonovich expansions [2]-[8]. At the same time

$\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l=1,\ldots,k,\ \ q_1,\ldots,q_k=0,1,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ in the unified Taylor–Ito and Taylor–Stratonovich expansions [9], [10] (also see [11]-[13]).

Effective solution of the problem of mean-square approximation of iterated Stratonovich stochastic integrals (3) based on multiple Fourier–Legendre series and multiple trigonometric Fourier series composes the subject of the article.

Note that another approaches to the mean-square approximation of the iterated Itô and Stratonovich stochastic integrals (2) and (3) can be found in [2]-[5], [14]-[31].

2 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity k $(k \in \mathbb{N})$ Based on Generalized Multiple Fourier Series Converging in the Mean

The results of this section are auxiliary to the proof of the main result of this article (see Theorem 7 below).

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{4}$$

where $t_1,\ldots,t_k \in [t,T]$ $(k \ge 2)$, and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. It is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t, T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} \Delta\tau_j \to 0 \ \text{ if } \ N \to \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \tag{6}$$

Theorem 1 [11] (2006), [12]-[13], [32]-[52]. *Suppose that every* $\psi_l(\tau)$ $(l = 1, \ldots, k)$ *is a continuous nonrandom function on* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuous functions in the space* $L_2([t, T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \underset{p_1, \ldots, p_k \to \infty}{\text{l.i.m.}} J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k) p_1, \ldots, p_k},$$

where

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k) p_1, \ldots, p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} -$$

$$- \underset{N \to \infty}{\text{l.i.m.}} \sum_{(l_1, \ldots, l_k) \in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \Biggr), \tag{7}$$

$i_1, \ldots, i_k = 0, 1, \ldots, m$, l.i.m. *is a limit in the mean-square sense,* $J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)}$ *is defined by* (2),

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1\},$$

$$\mathrm{L}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},$$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $C_{j_k \ldots j_1}$ *is the Fourier coefficient* (5), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, \ 1, \ \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t, T]$*, which satisfies the condition* (6).

Let us consider the transformed particular cases of Theorem 1 (see (7)) for $k = 1, \ldots, 5$ [11] (the cases $k = 6, 7$ and the case of an arbitrary k ($k \in \mathbb{N}$) can be found in [12]-[13], [32]-[52])

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{8}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{9}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{10}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+ \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{11}$$

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$

$$
-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{12}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

The convergence with probability 1 (the cases of Legendre polynomials and trigonometric functions) [50] and convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) [13] are proved for the approximations $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p,\ldots,p}$ and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}$, correspondingly (see (7)). As it turned out, Theorem 1 remains valid for the systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ [11], [13]. Versions of Theorem 1 for iterated stochastic integrals with respect to martingale Poisson measures and for iterated stochastic integrals with respect to martingales are obtained in [13]. Another modification of Theorem 1 can be found in [13], where complete orthonormal with weight $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ were considered. Application of Theorem 1 and Theorem 2 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process is given in [39], [40].

Note that in [53], an analogue of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ was considered. Another proof of the mentioned analogue of Theorem 1 [53] can be found in [13] (Sect. 1.11), [44] (Sect. 15).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ using our notations [13] (Sect. 1.11), [44] (Sect. 15).

In order to do this, let us consider the unordered set $\{1, 2, \dots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \dots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \dots, q_{k-2r}}_{\text{part 2}}\}), \tag{13}$$

where $\{g_1, g_2, \dots, g_{2r-1}, g_{2r}, q_1, \dots, q_{k-2r}\} = \{1, 2, \dots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (13) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\dots,\{g_{2r-1},g_{2r}\}\},\{q_1,\dots,q_{k-2r}\}) \\ \{g_1,g_2,\dots,g_{2r-1},g_{2r},q_1,\dots,q_{k-2r}\}=\{1,2,\dots,k\}}} a_{g_1g_2,\dots,g_{2r-1}g_{2r},q_1\dots q_{k-2r}}. \tag{14}$$

Below there are several examples of sums of the form (14)

$$\sum_{\substack{(\{g_1,g_2\}) \\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\}) \\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\}) \\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\}) \\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} = a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$

$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\}) \\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$

$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Thus, we can formulate the following theorem.

Theorem 2 [13] (Sect. 1.11), [44] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(15)$$

that converges in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

The connection of the expression in parentheses on the right-hand side of (15) with Hermite polynomials is discussed in [13] (Sect. 1.10, 1.11), [44] (Sect. 14, 15). A similar result can be found in [53].

Note that Theorems 1 and 2 allow us to calculate exactly the mean-square approximation error for the approximations of iterated Itô stochastic integrals (2) of arbitrary multiplicity k.

Theorem 3 [13] (Sect. 1.12), [45] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p,\ldots,p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\},$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p,\ldots,p}$ *is the expression on the right-hand side of* (15) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ *for the case* $p_1=\ldots=p_k=p$; $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

3 Expansions of Iterated Stratonovich Stochastc Integrals of Multiplicities 1 to 4. Some Old Results

Let $\mathrm{M}_2([t,T])$ $(0\le t<T<\infty)$ be the class of random functions $\xi(\tau,\omega)\stackrel{\mathsf{def}}{=}\xi_\tau:[t,T]\times\Omega\to\mathbb{R}$, which satisfy the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau\in[t,T]$, ξ_τ is independent with increments $\mathbf{w}_{s+\Delta}-\mathbf{w}_s$ for $s\ge\tau$, $\Delta>0$, and

$$\int\limits_t^T \mathsf{M}\left\{(\xi_\tau)^2\right\}d\tau<\infty,\quad \mathsf{M}\left\{(\xi_\tau)^2\right\}<\infty\quad\text{for all}\quad \tau\in[t,T].$$

We introduce the class $\mathrm{Q}_4([t,T])$ of Itô processes $\eta_\tau^{(i)}$, $\tau\in[t,T]$, $i=1,\ldots,m$ of the form

$$\eta_\tau^{(i)}=\eta_t^{(i)}+\int\limits_t^\tau a_s ds+\int\limits_t^\tau b_s d\mathbf{w}_s^{(i)}\quad \text{w. p. 1},\tag{16}$$

where $(a_s)^4,(b_s)^4\in\mathrm{M}_2([t,T])$ and $\lim\limits_{s\to\tau}\mathsf{M}\bigl\{|b_s-b_\tau|^4\bigr\}=0$ for all $\tau\in[t,T]$. The second integral on the right-hand side of (16) is the Itô stochastic integral. Here and further, w. p. 1 means with probability 1.

Consider a function $F(x,\tau):\mathbb{R}\times[t,T]\to\mathbb{R}$ for fixed τ from the class $C_2(-\infty,\infty)$ consisting of twice continuously differentiable in x functions on the interval $(-\infty,\infty)$ such that the first two derivatives are bounded.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}}\sum_{j=0}^{N-1}F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)}+\eta_{\tau_{j+1}}^{(i)}\right),\tau_j\right)\left(\mathbf{w}_{\tau_{j+1}}^{(l)}-\mathbf{w}_{\tau_j}^{(l)}\right)\stackrel{\mathsf{def}}{=}\int\limits_t^{*T}F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}\tag{17}$$

is called [54] the Stratonovich stochastic integral with respect to the component $\mathbf{w}_\tau^{(l)}$ $(l=1,\ldots,m)$ of the multidimensional Wiener process $\mathbf{w}_\tau$, where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$, which satisfies the condition (6).

It is known [54] (also see [2]) that under proper conditions, the following relation between Stratonovich and Itô stochastic integrals holds

$$\int\limits_t^{*T} F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)} = \int\limits_t^{T} F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)} + \frac{1}{2}\mathbf{1}_{\{i=l\}}\int\limits_t^{T} \frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau \qquad (18)$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of the set A and $i,l=1,\ldots,m$.

A possible variant of conditions under which the formula (18) is correct, for example, consisits of the conditions $\eta_\tau^{(i)}\in \mathrm{Q}_4([t,T])$, $F(\eta_\tau^{(i)},\tau)\in \mathrm{M}_2([t,T])$, $F(x,\tau)\in C_2(-\infty,\infty)$ (for fixed τ), where $i=1,\ldots,m$.

As it turned out, approximations of the iterated Stratonovich stochastic integrals (3) are essentially simpler than the appropriate approximations of the iterated Itô stochastic integrals (2) based on Theorems 1 and 2. For the first time this fact was mentioned in [11] (2006).

According to the standard connection (18) between Itô and Stratonovich stochastic integrals, the iterated Itô and Stratonovich stochastic integrals (2) and (3) of first multiplicity are equal to each other w. p. 1. So, we begin the consideration from the multiplicity $k=2$ (the case $k=1$ is given by (8)).

The following three theorems adapt Theorems 1, 2 for the integrals (3) of multiplicities 2 to 4.

Theorem 4 [13], [32], [33], [48]. *Suppose that $\psi_1(\tau),\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of second multiplicity*

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \int\limits_t^{*T} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense is valid, where $i_1,i_2=0,1,\ldots,m$; another notations are the same as in Theorems 1, 2.

Theorem 5 [13], [32], [33], [48]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the* *space $L_2([t,T])$. Furthermore, let the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the functions $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{19}$$

that converges in the mean-square sense is correct, where $i_1, i_2, i_3 = 0, 1, \ldots, m$; another notations are the same as in Theorems 1, 2.

Theorem 6 [13], [32], [33], [48]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1 i_2 i_3 i_4)} = \int\limits_t^{*T} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}$$

the following expansion

$$I_{T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3, i_4 = 0, 1, \ldots, m$; another notations are the same as in Theorems 1, 2.

4 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$)

In this section, we prove the expansion of iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbb{N}$) under the condition of convergence of trace series.

Let us introduce some notations and formulate some auxiliary results. Consider the Fourier coefficient

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{20}$$

corresponding to the function (4), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k= \tag{21}$$

$$=\sqrt{T-t}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k=$$

$$=\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type (20) but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau)$, $\ldots$, $\psi_{l-2}(\tau)$, $\sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau)$, $\ldots$, $\psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (13))).

Let

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright j_m}\stackrel{\sf def}{=}$$

$$
\stackrel{\text{def}}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l) \times
$$

$$
\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2}) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_{l-2} dt_l t_{l+1} \ldots dt_k = \tag{22}
$$

$$
= \bar{C}_{j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1},
$$

i.e. $\bar{C}_{j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1}$ is again the Fourier coefficient of type (20) but with a new shorter multi-index $j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1$ and new weight functions $\psi_1(\tau), \ldots,$ $\psi_{l-2}(\tau)$, $\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau), \ldots, \psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (13))).

Denote

$$
\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\text{def}}{=}
$$

$$
\stackrel{\text{def}}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}. \tag{23}
$$

Introduce the following notation

$$
S_l\left\{\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}}\right\} \stackrel{\text{def}}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots
$$

$$
\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty} \sum_{j_{g_{2l-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Bigg|_{(j_{g_{2l}} j_{g_{2l-1}}) \curvearrowright (\cdot), j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}.
$$

Note that the operation S_l $(l = 1, 2, \ldots, r)$ acts on the value

$$
\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \tag{24}
$$

as follows: S_l multiplies (24) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$
\sum_{j_{g_{2l-1}}=p+1}^{\infty},
$$

and replaces

$$C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{25}$$

with

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{26}$$

Note that we write

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright j_m,j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright j_m,j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} =$$

$$= C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}, \quad \ldots$$

Since (26) is again the Fourier coefficient, then the action of superposition $S_l S_m$ on (25) is obvious. For example, for $r=3$

$$S_3 S_2 S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3 S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$
= \frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_5}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}}.
$$

Theorem 7 [48], [49]. *Assume that the continuously differentiable functions $\psi_l(\tau):[t,T]\to\mathbb{R}$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$
\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j=0}^{\infty}\int\limits_t^s \Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (27)
$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (27) *converges absolutely.*

2. *The estimates*

$$
\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \quad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},
$$

$$
\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}
$$

hold for all $s\in(t,T)$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$
\int\limits_t^T \Psi_1^2(\tau)d\tau<\infty, \quad \int\limits_t^T \left|\Psi_2(\tau)\right|d\tau<\infty.
$$

3. *The condition*

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0
$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (13)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ \ldots, \ r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} \stackrel{\mathsf{def}}{=} \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{28}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \tag{29}$$

that converges in the mean-square sense is valid, where

$$C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k \tag{30}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense*, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*)*, $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. The proof of Theorem 7 will consist of several steps.

Step 1. Let us find a representation of the random variable

$$\sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

Let us consider the following multiple stochastic integral

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_p;\ q\ne p;\ q,p=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}, \qquad (31)$$

where we assume that $\Phi(t_1,\ldots,t_k) : [t,T]^k \to \mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$, which satisfies the condition (6), $i_1,\ldots,i_k = 0,1,\ldots,m$.

The stochastic integral with respect to the scalar standard Wiener process $(i_1 = \ldots = i_k \ne 0)$ and similar to (31) was considered in [55] (1951) and is called the multiple Wiener stochastic integral [55]. Note that the following well known estimate

$$\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le C_k \int\limits_{[t,T]^k} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k \qquad (32)$$

is correct for the multiple Wiener stochastic integral, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (31) and C_k is a constant.

From the proof of Theorem 1 (see the proof of Theorem 5.1 in the monograph [11] (2006) in Russian or proof of Theorem 1.1 in the monograph [13] in English) it follows that (7) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \qquad (33)$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (31) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2).

Let us consider the following multiple stochastic integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}, \qquad (34)$$

where we assume that $\Phi(t_1,\ldots,t_k):[t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Another notations are the same as in (31).

The stochastic integral with respect to the scalar standard Wiener process ($i_1=\ldots=i_k\neq 0$) and similar to (34) (the function $\Phi(t_1,\ldots,t_k)$ is assumed to be symmetric on the hypercube $[t,T]^k$) has been considered in the literature (see, for example, Remark 1.5.7 [56]). The integral (34) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)}=J[\varphi_1]_{T,t}^{(i_1)}\ldots J[\varphi_k]_{T,t}^{(i_k)}\quad \text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k)=\varphi_1(t_1)\ldots\varphi_k(t_k)$ and

$$J[\varphi_l]_{T,t}^{(i_l)}=\int\limits_t^T\varphi_l(\tau)d\mathbf{w}_\tau^{(i_l)}\quad (l=1,\ldots,k).$$

Theorem 8 [13] (Sect. 1.9), [44] (Sect. 13). *Suppose that $\Phi(t_1,\ldots,t_k):[t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Furthermore, let $\{\phi_j(x)\}_{j=0}^\infty$ be a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j is continuous at the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity as well as $\phi_j(x)$ right-continuous at the interval $[t,T]$. Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\quad (35)$$

converging in the mean-square sense is valid, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (31),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient. Another notations are the same as in Theorems 1, 2.

Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})} dt_{s_l+1} d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (36)$$

where $(s_l,\ldots,s_1)\in \mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l}=\bigl\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;\ s_l,\ldots,s_1=1,\ldots,k-1\bigr\}, \qquad (37)$$

$l=1,2,\ldots,[k/2]$, $i_1,\ldots,i_k=0,1,\ldots,m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals (3) and (2) of arbitrary multiplicity k.

Theorem 9 [41] (1997) (also see [11]-[13], [32], [33]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals* (3) *and* (2) *is correct*

$$J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \qquad (38)$$

where $\sum\limits_{\emptyset}$ *is supposed to be equal to zero.*

Consider (35) for $\Phi(t_1,\ldots,t_k)=K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and without passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ (see the proof of Theorem 8 in [13] (Theorem 1.13) or [44] (Theorem 9))

$$
J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}-\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{39}
$$

w. p. 1, where $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (31), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral (34),

$$
K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{40}
$$

$$
K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}). \tag{41}
$$

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (39), we get w. p. 1 (see Theorems 1, 2)

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\underset{p\to\infty}{\text{l.i.m.}}\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J[K_{p\ldots p}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \quad (42)$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2).

If we prove that w. p. 1

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = -\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \quad (43)$$

then (see (42), (43), and Theorem 9)

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \quad (44)$$

w. p. 1, where notations in (44) are the same as in Theorem 9. Thus Theorem 7 will be proved.

From (39) we have that the multiple Stratonovich stochastic integral $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k is expressed as a finite linear combination of the multiple Wiener stochastic integral $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k and multiple Stratonovich stochastic integrals $J[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ of multiplicities $k-2,\ k-4,\ \ldots,\ k-2[k/2]$. By iteratively applying the formula (39) (also see

(9)–(12)), we obtain a representation of the multiple Stratonovich stochastic integral of multiplicity k as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{45}$$

w. p. 1, where $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$ are defined by the equalities (40), (41).

From (45) we have the following generalization of (47)–(50) (see below) for the case of an arbitrary k ($k\in\mathbb{N}$)

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad\text{w. p. 1.} \tag{46}$$

The formulas (45), (46) can be considered as new representations of the Hu–Meyer formula for the case of a multidimensional Wiener process [57] (also see [56], [58]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (40)).

For example, for $k=2,3,4,5$ we have from (46) w. p. 1

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} = J'[K_{p_1p_2}]_{T,t}^{(i_1i_2)} + \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}, \tag{47}$$

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)} +$$

$$+\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}]_{T,t}^{(i_3)} + \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}]_{T,t}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}\Biggr), \tag{48}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J'[K_{p_1p_2p_3p_4}]_{T,t}^{(i_1i_2i_3i_4)} +$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)} +$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+ \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{49}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} = J'[K_{p_1p_2p_3p_4p_5}]_{T,t}^{(i_1i_2i_3i_4i_5)} +$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_3i_4i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_4i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_2i_3i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2i_3i_4)} + \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_4i_5)} +$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_3i_5)}+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_3i_4)}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_1i_2i_5)}+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1i_2i_4)}+$$

$$+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$

$$\left.+\,\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right). \tag{50}$$

Further, we will use the representation (46) for $p_1=\ldots=p_k=p$, i.e.

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1} \ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{51}$$

Step 2. Let us prove that

$$\sum_{j_l=0}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} = 0 \tag{52}$$

or

$$\sum_{j_l=0}^{p} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} = - \sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}, \tag{53}$$

where $l-1 \geq s+1$, $p \in \mathbb{N}$.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau), \ldots, \psi_k(\tau)$. Therefore, sometimes in subsequent consideration we write $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$ for simplicity. Using the integration order replacement, we have

$$C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} =$$

$$= \int_t^T \phi_{j_k}(t_k) \ldots \int_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int_t^{t_{l+1}} \phi_{j_l}(t_l) \int_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int_t^{t_{s+1}} \phi_{j_l}(t_s) \int_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots$$

$$\ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \int_t^T \phi_{j_{s+1}}(t_{s+1}) \int_t^{t_{s+1}} \phi_{j_l}(t_s) \int_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s \times$$

$$\times \left(\int_{t_{s+1}}^T \phi_{j_{s+2}}(t_{s+2}) \ldots \int_{t_{l-2}}^T \phi_{j_{l-1}}(t_{l-1}) \int_{t_{l-1}}^T \phi_{j_l}(t_l) \int_{t_l}^T \phi_{j_{l+1}}(t_{l+1}) \ldots \right.$$

$$\left. \ldots \int_{t_{k-1}}^T \phi_{j_k}(t_k) dt_k \ldots dt_{l+1} dt_l dt_{l-1} \ldots dt_{s+2} \right) dt_{s+1} =$$

$$=\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s)\underbrace{\int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)}\,dt_s\times$$

$$\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)\underbrace{\int\limits_{t_l}^T \phi_{j_{l+1}}(t_{l+1})\ldots\int\limits_{t_{k-1}}^T \phi_{j_k}(t_k)dt_k\ldots dt_{l+1}}_{H_{j_k\ldots j_{l+1}}(t_l)}\times$$

$$\times\left(\underbrace{\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}}_{Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})}\,dt_l\right)dt_{s+1}=$$

$$=\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times$$

$$\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})dt_l dt_{s+1}. \qquad (54)$$

Applying the additive property of the integral, we obtain

$$Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})=\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}=$$

$$=\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3})\int\limits_t^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2})dt_{s+2}dt_{s+3}\ldots dt_{l-1}-$$

$$-\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3})dt_{s+3}\ldots dt_{l-1}\int\limits_t^{t_{s+1}} \phi_{j_{s+2}}(t_{s+2})dt_{s+2}=$$

$$\ldots$$

$$=\sum_{m=1}^{d} h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}),\quad d<\infty. \qquad (55)$$

Combining (54) and (55), we have

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= \sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=0}^{p}\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$
$$\left.\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\right). \qquad (56)$$

Using the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty}\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l =$$

$$= \int\limits_t^T \mathbf{1}_{\{\tau<t_{s+1}\}}G_{j_{s-1}\ldots j_1}(\tau)\cdot\mathbf{1}_{\{\tau>t_{s+1}\}}H_{j_k\ldots j_{l+1}}(\tau)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(\tau)d\tau = 0. \qquad (57)$$

From (56) and (57) we get

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= -\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$
$$\left.\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\right). \qquad (58)$$

Combining Condition 2 of Theorem 7 and (54)–(56), (58), we have

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$
= -\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.
$$

$$
\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\right) =
$$

$$
= -\sum_{j_l=p+1}^{\infty}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =
$$

$$
= -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}. \tag{59}
$$

The equality (59) implies (52), (53).

Step 3. Using Conditions 1 and 2 of Theorem 7, we obtain

$$
\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_l(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times\sum_{j_l=0}^{p}\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times\int\limits_t^{t_{l-1}}\psi_l(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_l(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times \sum_{j_l=0}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(t_l)\phi_{j_l}(t_l) \int\limits_t^{t_l} \psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times \int\limits_t^{t_{l-1}} \psi_l(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k -
$$

$$
-\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} =
$$

$$
= \frac{1}{2} C_{j_k\ldots j_1}\Bigg|_{(j_l j_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}. \tag{60}
$$

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (51), we have (see (33))

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +
$$

$$
+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{61}
$$

Taking into account (53) and (60), we obtain for $r=1$

$$
\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =
$$

$$
= -\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2>g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} +
$$

$$
+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}-
$$

$$
-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_{g_1}=p+1}^{\infty}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_{g_1}=p+1}^{\infty}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+
$$

$$
+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times\ J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}= \tag{62}
$$

$$
=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[g_1]}+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2(i_{q_1}\ldots i_{q_{k-2}})} \tag{63}
$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[g_1]}$ $(g_1=1,2,\ldots,k-1)$ is defined by (36),

$$
R_{T,t}^{(p)g_1,g_2(i_{q_1}\ldots i_{q_{k-2}})}=-\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}.
$$

Let us explain the transition from (62) to (63). We have for $g_2=g_1+1$

$$
\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0,j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\ \sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\zeta_{j_{m_1}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\ \sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\ J'[\phi_{j_{m_1}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(0i_{q_1}\ldots i_{q_{k-2}})}= \tag{64}
$$

$$
=\frac{1}{2}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[g_1]}\quad \text{w. p. 1}, \tag{65}
$$

where

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2},g_2=g_1+1}=
$$

$$
=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{g_1+3}}\psi_l(t_{g_1+2})\phi_{j_{g_1+2}}(t_{g_1+2})\int\limits_t^{t_{g_1+2}}\psi_{g_1+1}(t_{g_1})\psi_{g_1}(t_{g_1})\phi_{j_{m_1}}(t_{g_1})\times
$$

$$
\times \int\limits_{t}^{t_{g_1}} \psi_l(t_{g_1-1})\phi_{j_{g_1-1}}(t_{g_1-1}) \ldots \int\limits_{t}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_{g_1-1}dt_{g_1}dt_{g_1+2} \ldots dt_k,
$$

$$
\zeta_{j_{m_1}}^{(0)} = \int\limits_{t}^{T} \phi_{j_{m_1}}(\tau)d\mathbf{w}_{\tau}^{(0)} = \int\limits_{t}^{T} \phi_{j_{m_1}}(\tau)d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_{m_1} = 0 \\ & \\ 0 & \text{if } j_{m_1} \neq 0 \end{cases},
$$

$$
\phi_0(\tau) = \frac{1}{\sqrt{T-t}}.
$$

The transition from (64) to (65) is based on (33).

By Condition 3 of Theorem 7 we have (also see the property (32) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)g_1,g_2(i_{q_1}\ldots i_{q_{k-2}})}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\neq g_1,g_2}}^{p} \left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2}\right)^2 = 0,
$$

where constant K does not depend on p.

Thus

$$
\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =
$$

$$
= \frac{1}{2} \mathbf{1}_{\{g_2=g_1+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[g_1]} \quad \text{w. p. 1.}
$$

Involving into consideration the second pair $\{g_3, g_4\}$ (the first pair is $\{g_1, g_2\}$), we obtain by analogy with (62) for $r = 2$

$$
\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times
$$

$$
\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
= \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}} = \ i_{g_{2s}} \neq 0\}} \times
$$

$$
\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,g_3,g_4}}^{p} \left(\frac{1}{4} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)(j_{g_4} j_{g_3})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}} \prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} - \right.
$$

$$
-\frac{1}{2} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_4} j_{g_3})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}} \mathbf{1}_{\{g_4=g_3+1\}} -
$$

$$
-\frac{1}{2} \sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}} \mathbf{1}_{\{g_2=g_1+1\}} +
$$

$$
\left. + \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}} \right) J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} = \qquad (66)
$$

$$
= \frac{1}{4} \prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_2,s_1]} +
$$

$$
+ \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}} = \ i_{g_{2s}} \neq 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,g_3,g_4(i_{q_1}\ldots i_{q_{k-4}})} \qquad (67)
$$

w. p. 1, where $g_3 \stackrel{\mathsf{def}}{=} s_2$, $g_1 \stackrel{\mathsf{def}}{=} s_1$, $(s_2, s_1) \in \mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_2,s_1]}$ is defined by (36) and $\mathrm{A}_{k,2}$ is defined by (37),

$$
R_{T,t}^{(p)g_1,g_2,g_3,g_4(i_{q_1}\ldots i_{q_{k-4}})} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,g_3,g_4}}^{p} \left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\neq g_1,g_2,g_3,g_4} - \right.
$$

$$
\left. -S_1 \left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\neq g_1,g_2,g_3,g_4} \right\} - S_2 \left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\neq g_1,g_2,g_3,g_4} \right\} \right) \times
$$

$$
\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}.
$$

Let us explain the transition from (66) to (67). We have for $g_2 = g_1 + 1$, $g_4 = g_3 + 1$

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \frac{1}{4} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0(j_{g_4}j_{g_3})\curvearrowright 0,j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\zeta_0^{(0)}\zeta_0^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\ \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\ \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(00 i_{q_1}\ldots i_{q_{k-4}})} = \tag{68}
$$

$$
=\frac{1}{4}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_2,s_1]}\quad \text{w. p. 1.} \tag{69}
$$

The transition from (68) to (69) is based on (33).

Note that

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}$$

is the Fourier coefficient, where $g_2 = g_1 + 1$. Therefore, the value

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} =$$

$$= C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}(j_{g_3}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}$$

is determined recursively using (22) in an obvious way for $g_2 = g_1 + 1$ and $g_4 = g_3 + 1$.

By Condition 3 of Theorem 7 we have (also see the property (32) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)g_1,g_2,g_3,g_4(i_{q_1}\ldots i_{q_{k-4}})}\right)^2\right\} \le$$

$$\le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right)^2 + \right.$$

$$\left. + \left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2 + \left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2\right) = 0,$$

where constant K is independent of p.

Thus

$$\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} = \frac{1}{4} \prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_2,s_1]} \quad \text{w. p. 1},$$

where $g_3 \stackrel{\sf def}{=} s_2,\ g_1 \stackrel{\sf def}{=} s_1,\ (s_2,s_1) \in \mathrm{A}_{k,2},\ J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_2,s_1]}$ is defined by (36) and $\mathrm{A}_{k,2}$ is defined by (37).

Involving into consideration the third pair $\{g_6, g_5\}$ ($\{g_1, g_2\}$ is the first pair and $\{g_4, g_3\}$ is the second pair), we obtain by analogy with (66) for $r = 3$

$$\prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{3} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} =$$

$$= \prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \ne 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4,g_5,g_6}}^{p} \left(\frac{1}{2^3} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)(j_{g_4} j_{g_3})\curvearrowright(\cdot)(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \times\right.$$

$$\times \prod_{s=1}^{3} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} -$$

$$-\frac{1}{2^2} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_4} j_{g_3})\curvearrowright(\cdot)(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_4=g_3+1\}} \mathbf{1}_{\{g_6=g_5+1\}} -$$

$$-\frac{1}{2^2} \sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}} \mathbf{1}_{\{g_6=g_5+1\}} -$$

$$-\frac{1}{2^2} \sum_{j_{g_5}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)(j_{g_4} j_{g_3})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}} \mathbf{1}_{\{g_4=g_3+1\}} +$$

$$+\frac{1}{2} \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_6=g_5+1\}} +$$

$$+\frac{1}{2} \sum_{j_{g_5}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_4} j_{g_3})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_4=g_3+1\}} +$$

$$+\frac{1}{2} \sum_{j_{g_5}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}, j_{g_5} = j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}} -$$

$$
-\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}}\Biggr)\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=
$$

$$
=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_3,s_2,s_1]}+
$$

$$
+\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_5,g_6(i_{q_1}\ldots i_{q_{k-6}})}
$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i$; $i=1,2,3$, $(s_3,s_2,s_1)\in \mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_3,s_2,s_1]}$ is defined by (36) and $\mathrm{A}_{k,3}$ is defined by (37),

$$
R_{T,t}^{(p)g_1,g_2,\ldots,g_5,g_6(i_{q_1}\ldots i_{q_{k-6}})}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p}\Biggl(-\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}+
$$

$$
+S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}+S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}+
$$

$$
+S_3\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-
$$

$$
-S_3S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-S_3S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-
$$

$$
-S_2S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\Biggr)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}.
$$

By Condition 3 of Theorem 7 we have (also see the property (32) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty}{\sf M}\left\{\left(R_{T,t}^{(p)g_1,g_2,\ldots,g_5,g_6(i_{q_1}\ldots i_{q_{k-6}})}\right)^2\right\}\le
$$

$$\leq K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p} \left(\left(\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right)^2 + \right.$$

$$+\left(S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$+\left(S_3\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$+\left(S_3S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_3S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$\left. +\left(S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2\right) = 0,$$

where constant K does not depend on p.

Thus

$$\underset{p\to\infty}{\text{l.i.m.}} \prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}} =\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{3} \mathbf{1}_{\{j_{g_{2s-1}} =\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} = \frac{1}{2^3} \prod_{s=1}^{3} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_3,s_2,s_1]} \quad \text{w. p. 1},$$

where $g_{2i-1} \stackrel{\text{def}}{=} s_i$; $i = 1, 2, 3$, $(s_3, s_2, s_1) \in \mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_3,s_2,s_1]}$ is defined by (36) and $\mathrm{A}_{k,3}$ is defined by (37).

Repeating the previous steps, we obtain for an arbitrary r ($r = 1, 2, \ldots,$ $[k/2]$)

$$\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} =\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}} =\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} = \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} =\ i_{g_{2s}}\ne 0\}} \times$$

$$
\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$
+\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} = \tag{70}
$$

$$
=\frac{1}{2^r}\prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}+
$$

$$
+\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} \tag{71}
$$

w. p. 1, where $g_{2i-1} \stackrel{\mathsf{def}}{=} s_i$; $i=1,2,\ldots,r$; $r=1,2,\ldots,[k/2]$, $(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (36) and $\mathrm{A}_{k,r}$ is defined by (37),

$$
R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl((-1)^r \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} +
$$

$$
+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+
$$

$$
+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+
$$

$$
\ldots
$$

$$
+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \tag{72}
$$

Let us explain the transition from (70) to (71). We have for $g_2 = g_1 + 1$, $\ldots$, $g_{2r} = g_{2r-1} + 1$

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \frac{1}{2^r} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \frac{1}{2^r} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}\ldots\zeta_{j_{m_{2r-1}}}^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \frac{1}{2^r} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots 0 i_{q_1}\ldots i_{q_{k-2r}})} = \qquad (73)
$$

$$
= \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1.} \qquad (74)
$$

The transition from (73) to (74) is based on (33).

Note that

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}=C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}$$

is the Fourier coefficient, where $g_2=g_1+1.$ Therefore, the value

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}=$$

$$=C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d-1}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}$$

is determined recursively using (22) in an obvious way for $g_2=g_1+1,\ \ldots,$ $g_{2d}=g_{2d-1}+1$ and $d=2,\ldots,r.$

By Condition 3 of Theorem 7 we have (also see the property (32) of multiple Wiener stochastic integral)

$$\lim\limits_{p\to\infty}{\sf M}\left\{\left(R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\right)^2\right\}\le$$

$$\le K\lim\limits_{p\to\infty}\sum\limits_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\left(\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2+\right.$$

$$+\sum\limits_{l_1=1}^{r}\left(S_{l_1}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$+\sum\limits_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$\ldots$$

$$\left.+\sum\limits_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right)=0,$$

where constant K does not depend on $p.$

So we have

$$\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r}\prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \tag{75}$$

where $g_{2i-1} \overset{\text{def}}{=} s_i$; $i = 1,2,\ldots,r$; $r = 1,2,\ldots,[k/2]$, $(s_r,\ldots,s_1) \in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (36) and $\mathrm{A}_{k,r}$ is defined by (37).

Note that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \Biggl.\Biggr|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1} A_{g_1,g_3,\ldots,g_{2r-1}} =$$

$$= \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} A_{s_1,s_2,\ldots,s_r}, \tag{76}$$

where $A_{g_1,g_3,\ldots,g_{2r-1}}$, $A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1} = s_i$; $i = 1,2,\ldots,r$; $r = 1,2,\ldots,[k/2]$, $\mathrm{A}_{k,r}$ is defined by (37).

Using (61), (75), (76), and Theorem 9, we finally get

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{77}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (36). Theorem 7 is proved.

Let us make a number of remarks about Theorem 7. An expansion similar to (29) was obtained in [57], where the author used a definition of the Stratonovich stochastic integral, which differs from (17). The proof from [57] is somewhat simpler than the proof proposed in this article. However, in our proof, we essentially use the structure of the Fourier coefficients (30) corresponding to the kernel $K(t_1, \ldots, t_k)$ of the form (4). This circumstance actually made it possible to prove Theorem 7 using not the condition of finiteness of trace series, but using the condition of convergence to zero of explicit expressions for the remainders of the mentioned series. This leaves hope that it is possible to estimate the rate of convergence in Theorem 7 (see Theorems 14–17 below).

Note that under the conditions of Theorem 7 the sequential order of the series

$$\sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty}$$

in (23) is not important. We also note that the first and second conditions of Theorem 7 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ (see the proofs of Theorems 11–13 below). Note that in the proofs of Theorems 4–6, 11–13 the conditions of Theorem 7 are verified for various special cases of iterated Stratonovich stochastic integrals of multiplicities 2 to 5 with respect to components of the multidimensional Wiener process.

Taking into account the modification of Theorem 1 for the case of integration interval $[t,s]$ $(s \in (t,T])$ of iterated Itô stochastic integrals (see Theorem 1.11 in [13]), we can formulate an analogue of Theorem 7 for the case of integration interval $[t,s]$ $(s \in (t,T)$; the case $s = T$ is considered in Theorem 7) of iterated Stratonovich stochastic integrals of multiplicity k $(k \in \mathbb{N})$.

Denote

$$\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}(s)\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}(s)\Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}$$

and introduce the following notation

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\mathsf{def}}{=}\frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Bigg|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

where $l=1,2,\ldots,r,$

$$C_{j_k\ldots j_1}(s)\Bigg|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot)}$$

is defined by analogy with (21),

$$C_{j_k\ldots j_1}(s)=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k. \qquad (78)$$

Theorem 10 [13], [48], [49]. *Assume that the continuously differentiable functions $\psi_l(\tau):[t,T]\to\mathbb{R}$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (79)$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (79) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_\tau^s\phi_j(\theta)\Phi_2(\theta)d\theta\right|\le\frac{\Psi_2(s,\tau)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_{t}^{s}\Phi_2(\tau)\phi_j(\tau)\int\limits_{t}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_3(s)}{p^{\beta}}$$

hold for all s,τ *such that* $t<\tau<s<T$ *and for some* $\alpha,\beta>0,$ *where* $\Phi_1(\tau),$ $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T],$ $j,p\in\mathbb{N},$ *and*

$$\int\limits_{t}^{s}\left|\Psi_1(\tau)\Psi_2(s,\tau)\right|d\tau<\infty,\quad \int\limits_{t}^{s}\left|\Psi_3(\tau)\right|d\tau<\infty$$

for all $s\in(t,T).$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (13)*) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ \ldots,\ r\},$ $l_1>l_2>\ldots>l_d,$ $d=0,1,2,\ldots,r-1,$ *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0.$

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^{*}[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_{t}^{*s}\psi_k(t_k)\ldots\int\limits_{t}^{*t_2}\psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)},\tag{80}$$

the following expansion

$$J^{*}[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $s\in(t,T),$ $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (78), l.i.m. *is a limit in the mean-square sense,*

$i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(0)} = \tau$.

In Sect. 2.1.2 of the monograpth [13], the following formula is proved

$$\frac{1}{2} \int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1 = \sum_{j=0}^{\infty} C_{jj}, \tag{81}$$

where

$$C_{jj} = \int\limits_t^T \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1) dt_1 dt_2,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t, T]$.

Moreover [13] (Sect. 2.1.2), the following estimate

$$\left| \sum_{j=p+1}^{\infty} C_{jj} \right| \leq \frac{C}{p} \tag{82}$$

holds under the above assumptions, where constant C does not depend on p.

The relations (81) and (82) have been modified as follows [13] (Sect. 2.7, 2.9)

$$\frac{1}{2} \int\limits_t^s \psi_1(t_1)\psi_2(t_1) dt_1 = \sum_{j=0}^{\infty} C_{jj}(s), \tag{83}$$

$$\left| \sum_{j=p+1}^{\infty} C_{jj}(s) \right| \leq \frac{C}{p} \left(\frac{1}{\left(1 - z^2(s)\right)^{1/4}} + 1 \right), \tag{84}$$

where (83) holds for the case of Legendre polynomials or trigonometric functions and (84) holds for the case of Legendre polynomials, $s \in (t, T)$ (s is fixed, the case $s=T$ corresponds to (81) and (82)), constant C does not depend p, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$,

$$C_{jj}(s)=\int\limits_t^s \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2,$$

$$z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}. \tag{85}$$

For the trigonometric case, the estimate (84) is replaced by [13]

$$\left|\sum_{j=p+1}^{\infty} C_{jj}(s)\right|\le\frac{C}{p}, \tag{86}$$

where $s\in(t,T)$, constant C does not depend on p.

Note the well known estimate for the Legendre polynomials

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{87}$$

where $P_j(y)$ is the Legendre polynomial, constant K does not depend on y and j.

Using (87), we obtain the following useful estimates for the case of Legendre polynomials [13] (Sect. 1.7.2, 2.2.5)

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{88}$$

$$\left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{89}$$

$$\left|\int\limits_v^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+1\right), \tag{90}$$

where $j\in\mathbb{N}$, $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$, $v<x$, $\psi(\tau)$ is a continuously differentiable function at the interval $[t,T]$, constant C does not depend on j.

For the case of trigonometric functions, we note the following obvious estimates

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \qquad (91)$$

$$\left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \qquad (92)$$

$$\left|\int\limits_v^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \qquad (93)$$

where $j \in \mathbb{N}$, $x, v \in (t, T)$, $v < x$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t, T]$, constant C does not depend on j.

5 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3

In this section, we present a simple proof of an analogue of Theorem 5 based on Theorem 7. In this case, the conditions of Theorem 5 will be weakened.

First, we show that the equalities

$$\frac{1}{2}\int\limits_{t_1}^{t_2} \Phi_1(\tau)\Phi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2} \Phi_2(\tau)\phi_j(\tau)\int\limits_{t_1}^{\tau} \Phi_1(\theta)\phi_j(\theta)d\theta d\tau, \qquad (94)$$

$$\frac{1}{2}\int\limits_{t_1}^{t_2} \Phi_1(\tau)\Phi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2} \Phi_1(\theta)\phi_j(\theta)\int\limits_{\theta}^{t_2} \Phi_2(\tau)\phi_j(\tau)d\tau d\theta, \qquad (95)$$

hold for all t_1, t_2 such that $t \le t_1 < t_2 \le T$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space $L_2([t, T])$, the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t, T]$ and the series on the right-hand sides of (94), (95) converge absolutely.

From (83) we get

$$\frac{1}{2}\int_{t}^{t_1}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int_{t}^{t_1}\Phi_2(\tau)\phi_j(\tau)\int_{t}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau, \qquad (96)$$

$$\frac{1}{2}\int_{t}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int_{t}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int_{t}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau. \qquad (97)$$

Subtracting (96) from (97), we obtain

$$\frac{1}{2}\int_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int_{t}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau=$$

$$=\sum_{j=0}^{\infty}\int_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int_{t}^{t_1}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau+$$

$$+\sum_{j=0}^{\infty}\int_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int_{t_1}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau. \qquad (98)$$

Generalized Parseval's equality gives

$$\sum_{j=0}^{\infty}\int_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)d\tau\int_{t}^{t_1}\Phi_1(\theta)\phi_j(\theta)d\theta=$$

$$=\sum_{j=0}^{\infty}\int_{t}^{T}\mathbf{1}_{\{t_1<\tau<t_2\}}\Phi_2(\tau)\phi_j(\tau)d\tau\int_{t}^{T}\mathbf{1}_{\{\theta<t_1\}}\Phi_1(\theta)\phi_j(\theta)d\theta=$$

$$=\int_{t}^{T}\mathbf{1}_{\{t_1<\tau<t_2\}}\Phi_2(\tau)\mathbf{1}_{\{\tau<t_1\}}\Phi_1(\tau)d\tau=0. \qquad (99)$$

Combining (98) and (99), we obtain (94). The equality

$$\int_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int_{t_1}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau=\int_{t_1}^{t_2}\Phi_1(\theta)\phi_j(\theta)\int_{\theta}^{t_2}\Phi_2(\tau)\phi_j(\tau)d\tau d\theta$$

completes the proof of (95).

Theorem 11 [48], [49]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \qquad (100)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m,$

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$); *another notations are the same as in Theorem* 1.

Proof. As follows from Sect. 4 (see (81)–(84), (86), (88), (89), (91), (92)), Conditions 1 and 2 of Theorem 7 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ ($\alpha=1/2,\ \beta=1$). Let us verify Condition 3 of Theorem 7 for the iterated Stratonovich stochastic integral (100). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_1j_1}\right)^2=0, \qquad (101)$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1}\right)^2=0, \tag{102}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=0. \tag{103}$$

We have

$$\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_1j_1}\right)^2=$$

$$=\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{104}$$

$$=\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2\le \tag{105}$$

$$\le\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{106}$$

$$=\int\limits_t^T\psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3\le \tag{107}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{108}$$

if $p\to\infty$, where constant K does not depend on p.

Note that the transition from (104) to (105) is based on the estimate (84) for the polynomial case and its analogue (86) for the trigonometric case, the transition from (106) to (107) is based on the Parseval equality, and the transition from (107) to (108) is also based on the estimate (84) and its analogue (86) for the trigonometric case.

By analogy with the previous case we have

$$\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1}\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 = \tag{109}$$

$$= \sum_{j_1=0}^{p}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 \le \tag{110}$$

$$\le \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 =$$

$$= \int\limits_t^T \psi_1^2(t_1)\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_1(t_3)\phi_{j_3}(t_3)dt_3dt_2\right)^2 dt_1 \le \tag{111}$$

$$\le \frac{K}{p^2} \ \to\ 0 \tag{112}$$

if $p\to\infty$, where constant K is independent of p.

The transition from (109) to (110) is based on analogues of the estimates (84), (86) for the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2\right|$$

for the polynomial and trigonometric cases, the transition from (111) to (112) is also based on the mentioned analogues of the estimates (84), (86).

Further, we have

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1}\right)^2 =$$

$$
= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_3(t_3)\phi_{j_1}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 \right)^2 =
$$

$$
= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3) dt_3 dt_2 \right)^2 = \tag{113}
$$

$$
= \sum_{j_2=0}^{p} \left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3) dt_3 dt_2 \right)^2 \le \tag{114}
$$

$$
\le \sum_{j_2=0}^{\infty} \left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3) dt_3 dt_2 \right)^2 =
$$

$$
= \int\limits_t^T \psi_2^2(t_2) \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3) dt_3 \right)^2 dt_2. \tag{115}
$$

The transition from (113) to (114) is based on the estimates (88), (89) and its obvious analogues (91), (92) for the trigonometric case. However, the estimates (88), (89) cannot be used to estimate the right-hand side of (115), since we get the divergent integral. For this reason, we will obtain a new estimate based on the relation [13] (Sect. 2.2.5)

$$
\int\limits_t^x \psi(s)\phi_j(s) ds = \frac{\sqrt{T-t}\sqrt{2j+1}}{2} \int\limits_{-1}^{z(x)} P_j(y)\psi(u(y)) dy =
$$

$$
= \frac{\sqrt{T-t}}{2\sqrt{2j+1}} \Biggl((P_{j+1}(z(x)) - P_{j-1}(z(x)))\psi(x) -
$$

$$
- \frac{T-t}{2} \int\limits_{-1}^{z(x)} ((P_{j+1}(y) - P_{j-1}(y))\psi'(u(y)) dy \Biggr), \tag{116}
$$

where $x \in (t, T)$, $j \ge p+1$, $z(x)$ is defined by (85), $P_j(x)$ is the Legendre polynomial, ψ' is a derivative of the continuously differentiable function $\psi(s)$

with respect to the variable $u(y)$,

$$u(y)=\frac{T-t}{2}y+\frac{T+t}{2}.$$

From (87) and the estimate $|P_j(y)|\le 1$, $y\in[-1,1]$ we obtain

$$|P_j(y)|=|P_j(y)|^{\varepsilon}\cdot|P_j(y)|^{1-\varepsilon}\le|P_j(y)|^{1-\varepsilon}<\frac{C}{j^{1/2-\varepsilon/2}(1-y^2)^{1/4-\varepsilon/4}},\tag{117}$$

where $y\in(-1,1)$, $j\in\mathbb{N}$, and ε is an arbitrary small positive real number.

Combining (116) and (117), we have the following estimate

$$\left|\int\limits_t^s\psi_1(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right),\tag{118}$$

where $s\in(t,T)$, $z(s)$ is defined by (85), constant C does not depend on j.

Similarly to (118) we obtain

$$\left|\int\limits_s^T\psi_3(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right),\tag{119}$$

where $s\in(t,T)$, constant C is independent of j.

Combining (88) and (119), we have

$$\left|\int\limits_t^s\psi_1(\tau)\phi_j(\tau)d\tau\int\limits_s^T\psi_3(\tau)\phi_j(\tau)d\tau\right|<$$

$$<\frac{L}{j^{2-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+1\right),\tag{120}$$

where $s\in(t,T)$, $z(s)$ is defined by (85), constant L does not depend on j.

Observe that

$$\sum_{j=p+1}^{\infty}\frac{1}{j^{2-\varepsilon/2}}\le\int\limits_p^{\infty}\frac{dx}{x^{2-\varepsilon/2}}=\frac{1}{(1-\varepsilon/2)p^{1-\varepsilon/2}}.\tag{121}$$

Applying (120) and (121) to estimate the right-hand side of (115) gives

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{122}$$

if $p\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

The estimation of the right-hand side of (115) for the trigonometric case is carried out using the estimates (91), (92). At that we obtain the estimate (122) with $\varepsilon=0$. Theorem 11 is proved.

6 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4

Theorem 12 [48], [49]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = {\int\limits_t^{*T}} \psi_4(t_4)\ldots {\int\limits_t^{*t_2}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)} \tag{123}$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_4=0,1,\ldots,m,$

$$C_{j_4\ldots j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorem* 1.

Proof. As follows from Sect. 4 (see (81)–(84), (86), (88), (89), (91), (92)), Conditions 1 and 2 of Theorem 7 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ ($\alpha = 1/2,\ \beta = 1$). Let us verify Condition 3 of Theorem 7 for the iterated Stratonovich stochastic integral (123). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_3 j_1 j_1} \right)^2 = 0, \tag{124}$$

$$\lim_{p\to\infty} \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_1 j_2 j_1} \right)^2 = 0, \tag{125}$$

$$\lim_{p\to\infty} \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_3 j_2 j_1} \right)^2 = 0, \tag{126}$$

$$\lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4 j_2 j_2 j_1} \right)^2 = 0, \tag{127}$$

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2 j_3 j_2 j_1} \right)^2 = 0, \tag{128}$$

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_1} \right)^2 = 0, \tag{129}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2 j_1 j_2 j_1} \right)^2 = 0, \tag{130}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1} \right)^2 = 0, \tag{131}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=0, \tag{132}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0, \tag{133}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{134}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{135}$$

where we use the notation (21) in (133)–(135).

Applying arguments similar to those we used in the proof of Theorem 11, we obtain for (124)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{136}$$

$$=\sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2\le \tag{137}$$

$$\le\sum_{j_3,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{138}$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}} \psi_4^2(t_4)\psi_3^2(t_3)\times
$$

$$
\times \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \right)^2 dt_3dt_4 \le \tag{139}
$$

$$
\le \frac{K}{p^2} \ \to \ 0 \tag{140}
$$

if $p \to \infty$, where constant K is independent of p.

Note that the transition from (136) to (137) is based on the estimate (84) for the polynomial case and its analogue for the trigonometric case, the transition from (138) to (6) is based on the Parseval equality, and the transition from (6) to (140) is also based on the estimate (84) and its analogue for the trigonometric case.

Further, we have for (125)

$$
\sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4j_1j_2j_1} \right)^2 = \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{T} \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)\times \right.
$$

$$
\left. \times \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \right)^2 = \tag{141}
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{T} \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4 \right)^2 = \tag{142}
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\int\limits_t^{T} \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4 \right)^2 \le
$$

$$\leq \sum_{j_2,j_4=0}^{\infty}\left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2 =$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4^2(t_4)\psi_2^2(t_2)\times$$

$$\times\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2 dt_2dt_4 \leq$$

$$\leq \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{143}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The relation (143) was obtained by the same method as (140). Note that in obtaining (143) we used the estimates (90), (118) for the polynomial case and (91), (93) for the trigonometric case. We also used the integration order replacement in the iterated Riemann integrals (see (141), (142)).

Repeating the previous steps for (126) and (127), we get

$$\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2 = \sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =$$

$$=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 =$$

$$
= \sum_{j_2,j_3=0}^{p} \left(\int_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_3}^{T} \psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3 \right)^2 \le
$$

$$
\le \sum_{j_2,j_3=0}^{\infty} \left(\int_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_3}^{T} \psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<t_3\}}\psi_3^2(t_3)\psi_2^2(t_2)\times
$$

$$
\times \left(\sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_3}^{T} \psi_4(t_4)\phi_{j_1}(t_4)dt_4 \right)^2 dt_2dt_3 \le
$$

$$
\le \frac{K}{p^2} \ \to\ 0 \tag{144}
$$

if $p \to \infty$, where constant K does not depend on p;

$$
\sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1} \right)^2 = \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int_t^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)\times \right.
$$

$$
\left. \times \int_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)\times \right.
$$

$$
\left. \times \int_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4 \right)^2 =
$$

$$= \sum_{j_1,j_4=0}^{p} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)\times \right.$$

$$\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4 \right)^2 \le$$

$$\le \sum_{j_1,j_4=0}^{\infty} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)\times \right.$$

$$\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4 \right)^2 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_4\}}\psi_4^2(t_4)\psi_1^2(t_1)\times$$

$$\times \left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2 \right)^2 dt_1dt_4. \tag{145}$$

Note that, by virtue of the additive property of the integral, we have

$$\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2 = \tag{146}$$

$$= \sum_{j_2=p+1}^{\infty} \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3 -$$

$$- \sum_{j_2=p+1}^{\infty} \int\limits_{t}^{t_1} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3 -$$

$$- \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3 \int\limits_{t}^{t_1} \psi_2(t_2)\phi_{j_2}(t_2)dt_2. \tag{147}$$

However, all three series on the right-hand side of (147) have already been evaluated in (140) and (143). From (145) and (147) we finally obtain

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{148}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

In complete analogy with (143), we have for (128)

$$\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$\left.\times\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2\le$$

$$\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$
\times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2 \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_2}(t_4)dt_4 dt_1 dt_3 \Biggr)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_3\}} \psi_3^2(t_3)\psi_1^2(t_1) \times
$$

$$
\times \left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2 \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_2}(t_4)dt_4 \right)^2 dt_1 dt_3 \le
$$

$$
\le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \tag{149}
$$

if $p \to \infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

We have for (129)

$$
\sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_1} \right)^2 = \sum_{j_1,j_2=0}^{p} \Biggl(\sum_{j_3=p+1}^{\infty} \int\limits_{t}^{T} \psi_4(t_4)\phi_{j_3}(t_4) \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \times
$$

$$
\times \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \Biggr)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \Biggl(\sum_{j_3=p+1}^{\infty} \int\limits_{t}^{T} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_{j_2}(t_2) \times
$$

$$
\times \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \Biggr)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \Biggl(\int\limits_{t}^{T} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_{j_2}(t_2) \times
$$

$$
\times \sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \Biggr)^2 \le
$$

$$\leq \sum_{j_1,j_2=0}^{\infty}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^T \psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2\}}\psi_1^2(t_1)\psi_2^2(t_2)\times$$

$$\times\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3\right)^2 dt_2dt_1. \tag{150}$$

It is easy to see that the integral (see (150))

$$\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3$$

is similar to the integral from the formula (146) if in the last integral we substitute $t_4 = T$. Therefore, by analogy with (148), we obtain

$$\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 \leq \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{151}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

Now consider (130)–(132). We have for (130) (see **Step 2** in the proof of Theorem 7)

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \leq$$

$$\leq (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2. \tag{152}$$

Consider (128) and (149). We have

$$
\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2\Biggr|_{j_1=j_3} \le
$$

$$
\le \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \qquad (153)
$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (152) and (153), we obtain

$$
\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0
$$

if $p \to \infty$, where constant K_1 does not depend on p.

Similarly for (131) we have (see (127), (148))

$$
\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le
$$

$$
\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2, \qquad (154)
$$

$$
\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2\Biggr|_{j_1=j_4} \le
$$

$$
\le \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \qquad (155)
$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (154) and (155), we obtain

$$
\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0
$$

if $p \to \infty$, where constant K_1 does not depend on p.

Consider (132). Using (60), we get

$$\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{\infty} C_{j_3j_3j_1j_1} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1} =$$

$$= \frac{1}{2}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1}, \tag{156}$$

where (see (21))

$$C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2dt_3dt_4.$$

From the estimate (82) for the polynomial and trigonometric cases we get

$$\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{157}$$

where constant C is independent of p.

Further, we have (see (151))

$$\left(\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 =$$

$$= (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2\Biggl|_{j_1=j_2} \le$$

$$\le (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}}, \tag{158}$$

where constant K_1 does not depend on p.

Combining (156)–(158), we obtain

$$\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le \frac{K_2}{p^{1-\varepsilon}} \ \to\ 0$$

if $p \to \infty$, where constant K_2 does not depend on p.

Let us prove (133)–(135). It is not difficult to see that the estimate (157) proves (133).

Using the integration order replacement, we have

$$\sum_{j_1=p+1}^{\infty} C_{j_3 j_3 j_1 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} =$$

$$\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\psi_3(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \left(\psi_2(t_2) \int\limits_{t_2}^T \psi_4(t_4)\psi_3(t_4) dt_4 \right) \phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2, \quad (159)$$

$$\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1}\Bigg|_{(j_2 j_2)\curvearrowright(\cdot)} =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\psi_2(t_3) \int\limits_t^{t_3} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_3 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{t_4} \psi_3(t_3)\psi_2(t_3) dt_3 dt_1 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \left(\int\limits_t^{t_4} - \int\limits_t^{t_1} \right) \psi_3(t_3)\psi_2(t_3) dt_3 dt_1 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \left(\psi_4(t_4) \int\limits_t^{t_4} \psi_3(t_3)\psi_2(t_3) dt_3 \right) \phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_4 - \quad (160)$$

$$- \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \left(\psi_1(t_1) \int\limits_t^{t_1} \psi_3(t_3)\psi_2(t_3) dt_3 \right) \phi_{j_1}(t_1) dt_1 dt_4. \quad (161)$$

Applying the estimate (82) (polynomial and trigonometric cases) to the right-hand sides of (159)–(161), we get

$$\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Bigr|_{(j_3j_3)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{162}$$

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Bigr|_{(j_2j_2)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{163}$$

where constant C is independent of p. The estimates (162), (163) prove (134), (135). The relations (124)–(135) are proved. Theorem 12 is proved.

7 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5

Theorem 13 [48], [49]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{164}$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5 = 0,1,\ldots,m$,

$$C_{j_5\ldots j_1} = \int\limits_t^{T} \psi_5(t_5)\phi_{j_5}(t_5)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$); *another notations are the same as in Theorem* 1.

Proof. Note that in this proof we write k instead of 5 when this is true for an arbitrary k ($k \in \mathbb{N}$). As follows from Sect. 4 (see (81)–(84), (86), (88), (89), (91), (92)), Conditions 1 and 2 of Theorem 7 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$ ($\alpha = 1/2, \ \beta = 1$). Let us verify Condition 3 of Theorem 7 for the iterated Stratonovich stochastic integral (164). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}, j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2}} \right)^2 = 0, \tag{165}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}} \right)^2 = 0, \tag{166}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, g_2=g_1+1} \right)^2 = 0, \tag{167}$$

where $(\{g_1, g_2\}, \{g_3, g_4\}, \{q_1\})$ and $(\{g_1, g_2\}, \{q_1, q_2, q_3\})$ are partitions of the set $\{1, 2, \ldots, 5\}$ that is $\{g_1, g_2, g_3, g_4, q_1\} = \{g_1, g_2, q_1, q_2, q_3\} = \{1, 2, \ldots, 5\}$; braces mean an unordered set, and parentheses mean an ordered set.

Let us find a representation for $C_{j_k\ldots j_1}\Bigl|_{j_{g_1}=j_{g_2}, \ g_2>g_1+1}$ that will be convenient for further consideration. Using the integration order replacement in the Riemann integrals, we obtain

$$\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots$$
$$\ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l) dt_l \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l) dt_l \right) \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-1} dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-2} dt_{l-1} dt_{l+1} \ldots dt_k, \tag{168}$$

where $1 < l < k$ and $h_1(\tau), \ldots, h_k(\tau)$ are continuous functions on the interval $[t, T]$. By analogy with (168) we have for $l = k$

$$\int\limits_t^T h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-1} dt_l =$$

$$= \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^T h_l(t_l) dt_l dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right) \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1} \ldots dt_2 dt_1 -$$

$$- \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right) dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right) \int\limits_t^T h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1} -$$

$$- \int\limits_t^T h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}. \quad (169)$$

The formulas (168), (169) will be used further. Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau), \ldots, \psi_k(\tau)$. Therefore, sometimes in subsequent consideration we write $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$ for simplicity.

Let us continue the proof. Applying (168) to $C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$ (more precisely to $h_s(t_s) = \psi_s(t_s)\phi_{j_l}(t_s)$), we obtain for $l+1 \le k$, $s-1 \ge 1$, $l-1 \ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} = \quad (170)$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_{s-1}dt_s dt_{s+1} \ldots dt_{l-1}dt_l dt_{l+1} \ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$
\ldots \int_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \left(\int_t^{t_{s+1}} \phi_{j_l}(t_s) dt_s \right) \int_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \ldots
$$

$$
\ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k -
$$

$$
- \sum_{j_l=p+1}^{\infty} \int_t^T \phi_{j_k}(t_k) \ldots \int_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int_t^{t_{l+1}} \phi_{j_l}(t_l) \int_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots
$$

$$
\ldots \int_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \left(\int_t^{t_{s-1}} \phi_{j_l}(t_s) dt_s \right) \int_t^{t_{s-1}} \phi_{j_{s-2}}(t_{s-2}) \ldots
$$

$$
\ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-2} dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =
$$

$$
= \sum_{j_l=p+1}^{\infty} A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} - \sum_{j_l=p+1}^{\infty} B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}.
$$

Now we apply the formula (168) to

$$
A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} \quad \text{and} \quad B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}
$$

(more precisely to $h_l(t_l) = \psi_l(t_l)\phi_{j_l}(t_l)$). Then we have for $l+1 \le k$, $s-1 \ge 1$, $l-1 \ge s+1$

$$
\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} =
$$

$$
= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) \times
$$

$$
\times \prod_{\substack{g=1 \\ g \ne l,s}}^{k} \psi_g(t_g) \phi_{j_g}(t_g) dt_1 \ldots dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_{l+1} \ldots dt_k =
$$

$$
= \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_{l+1} j_{l-1} \ldots j_{s+1} j_{s-1} \ldots j_1} = \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne l,s}, \tag{171}
$$

where

$$F_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{172}$$

$$F_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{173}$$

$$F_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{174}$$

$$F_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau. \tag{175}$$

By analogy with (171) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty}C_{j_l j_{k-1}\ldots j_2 j_l}, \tag{176}$$

$$\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_2 j_l}\quad (l+1\le k), \tag{177}$$

$$\sum_{j_l=p+1}^{\infty}C_{j_l j_{k-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}\quad (s-1\ge 1). \tag{178}$$

Then we have for (176)–(178) (see (168), (169))

$$\sum_{j_l=p+1}^{\infty}C_{j_l j_{k-1}\ldots j_2 j_l}=\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{2}G_p^{(d)}(t_2,\ldots,t_{k-1})\prod_{g=2}^{k-1}\psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{k-1}, \tag{179}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_2j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} E_p^{(d)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=2\\ g\ne l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (180)$$

$$\sum_{j_l=p+1}^{\infty} C_{j_lj_{k-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} D_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})\times$$

$$\times \prod_{\substack{g=1\\ g\ne s}}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{k-1}, \qquad (181)$$

where

$$G_p^{(1)}(t_2,\ldots,t_{k-1}) = \mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^T \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$G_p^{(2)}(t_2,\ldots,t_{k-1}) = -\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(1)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(2)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{T} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$
D_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau.
$$

Now let us consider the value $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2=g_1+1}$. To do this, we will make the following transformations

$$
\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}}h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}}h_l(t_l)\int\limits_t^{t_l}h_l(t_{l-1})\int\limits_t^{t_{l-1}}h_{l-2}(t_{l-2})\ldots\int\limits_t^{t_2}h_1(t_1)dt_1\ldots
$$

$$
\ldots dt_{l-2}dt_{l-1}dt_ldt_{l+1}\ldots dt_k=
$$

$$
=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}}h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}}h_1(t_1)\int\limits_{t_1}^{t_{l+1}}h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}}h_{l-2}(t_{l-2})\times
$$

$$
\times\left(\int\limits_t^{t_{l+1}}-\int\limits_t^{t_{l-2}}\right)h_l(t_{l-1})\left(\int\limits_t^{t_{l+1}}-\int\limits_t^{t_{l-1}}\right)h_l(t_l)dt_ldt_{l-1}dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=
$$

$$
=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}}h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}}h_l(t_l)dt_l\int\limits_t^{t_{l+1}}h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l+1}}h_1(t_1)\times
$$

$$
\times\int\limits_{t_1}^{t_{l+1}}h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}}h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-
$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots$$

$$\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right)dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)\int\limits_t^{t_{l+1}} h_1(t_1)\times$$

$$\times\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k+$$

$$+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)\times$$

$$\times \int\limits_{t}^{t_{l+1}} h_{l-2}(t_{l-2}) \int\limits_{t}^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_{t}^{t_{2}} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k +$$

$$+ \int\limits_{t}^{T} h_k(t_k) \ldots \int\limits_{t}^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_{t}^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_{t}^{t_{l-2}} h_l(t_{l-1}) \int\limits_{t}^{t_{l-1}} h_l(t_l) dt_l dt_{l-1} \right) \times$$

$$\times \int\limits_{t}^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_{t}^{t_{2}} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k, \qquad (182)$$

where $l+1 \le k$, $l-2 \ge 1$, and $h_1(\tau), \ldots, h_k(\tau)$ are continuous functions at the interval $[t, T]$.

Applying (182) to $C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots\ldots j_1}$, we obtain for $l+1 \le k$, $l-2 \ge 1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots\ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} H_p^{(d)}(t_1, \ldots, t_{l-2}, t_{l+1}, \ldots, t_k) \times$$

$$\times \prod_{\substack{g=1\\ g \ne l-1, l}}^{k} \psi_g(t_g) \phi_{j_g}(t_g) dt_1 \ldots dt_{l-2} dt_{l+1} \ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{**(d)}_{j_k \ldots j_{l+1} j_{l-2} \ldots j_1} = \sum_{d=1}^{4} C^{**(d)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne l-1, l}, \qquad (183)$$

where

$$H_p^{(1)}(t_1, \ldots, t_{l-2}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1 < \ldots < t_{l-2} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_{t}^{t_{l+1}} \psi_l(\tau) \phi_{j_l}(\tau) d\tau \int\limits_{t}^{t_{l+1}} \psi_{l-1}(\tau) \phi_{j_l}(\tau) d\tau, \qquad (184)$$

$$H_p^{(2)}(t_1, \ldots, t_{l-2}, t_{l+1}, \ldots, t_k) =$$

$$= -\mathbf{1}_{\{t_1 < \ldots < t_{l-2} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_{t}^{t_{l+1}} \psi_l(\tau) \phi_{j_l}(\tau) d\tau \int\limits_{t}^{t_{l-2}} \psi_{l-1}(\tau) \phi_{j_l}(\tau) d\tau, \qquad (185)$$

$$H_p^{(3)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau, \qquad (186)$$

$$H_p^{(4)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-2}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau. \qquad (187)$$

By analogy with (183) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_l}, \qquad (188)$$

$$\sum_{j_l=p+1}^{\infty}C_{j_lj_lj_{k-2}\ldots j_1}. \qquad (189)$$

Then we have for (188), (189) (see (182) and its analogue for $t_{l+1}=T$)

$$\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_l}=\int\limits_{[t,T]^{k-2}}L_p(t_3,\ldots,t_k)\prod_{g=3}^{k}\psi_g(t_g)\phi_{j_g}(t_g)dt_3\ldots dt_k, \qquad (190)$$

$$\sum_{j_l=p+1}^{\infty}C_{j_lj_lj_{k-2}\ldots j_1}=\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}M_p^{(d)}(t_1,\ldots,t_{k-2})\prod_{g=1}^{k-2}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{k-2}, \qquad (191)$$

where

$$L_p(t_3,\ldots,t_k)=\mathbf{1}_{\{t_3<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(1)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(2)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(3)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(4)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau.$$

It is important to note that $C^{*(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$, $C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$ $(d=1,\ldots,4)$ are Fourier coefficients (see (171), (183)), that is, we can use Parseval's equality in the further proof.

Combining the equalities (171)–(175) (the case $g_2>g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 11, 12, we obtain for (171)

$$\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C^{*(d)}_{j_5\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2}\right)^2\le$$

$$\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C^{*(d)}_{j_5\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2}\right)^2=$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{\infty} \left(\int\limits_{[t,T]^3} \sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_5)\times \right.
$$

$$
\left. \times \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{5} \psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_5 \right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_5)\right)^2 \times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_5 \le
$$

$$
\le 4\sum_{d=1}^{4}\int\limits_{[t,T]^3} \left(F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_5)\right)^2 \times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_5 \le
$$

$$
\le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{192}
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (176)–(178) are considered analogously.

Absolutely similarly (see (192)) combining the equalities (183)–(187) (the case $g_2=g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 11, 12, we get for (183)

$$
\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2=g_1+1} \right)^2 =
$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2=g_1+1} \right)^2 =
$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{d=1}^{4} C_{j_5\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\ne g_1,g_2}\right)^2 \le
$$

$$
\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{\infty} \left(\sum_{d=1}^{4} C_{j_5\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\ne g_1,g_2}\right)^2 =
$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_5=0\\ q\ne g_1,g_2}}^{\infty} \Biggl(\int\limits_{[t,T]^3} \sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_5)\times
$$

$$
\times \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{5} \psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_5\Biggr)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_5)\right)^2 dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots\ldots dt_5 \le
$$

$$
\le 4\sum_{d=1}^{4}\int\limits_{[t,T]^3} \left(H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_5)\right)^2 dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_5 \le
$$

$$
\le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \tag{193}
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (188), (189) are considered analogously.

From (192), (193) and their analogues for the cases (176)–(178), (188), (189) we obtain

$$
\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2}}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{194}
$$

where constant K is independent of p. Thus the equality (165) is proved.

Let us prove the equality (166). Consider the following cases

1. $g_2>g_1+1,\ g_4=g_3+1$ 2. $g_2=g_1+1,\ g_4>g_3+1,$

$$3.\ g_2 > g_1+1,\ g_4 > g_3+1, \quad 4.\ g_2 = g_1+1,\ g_4 = g_3+1.$$

The proof for Cases 1–3 will be similar. Consider, for example, Case 2. Using (59), we obtain

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=0}^{p} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 \le \tag{195}$$

$$\le (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 =$$

$$= (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2\Biggl|_{j_{g_3}=j_{g_4}} \le$$

$$\le (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2. \tag{196}$$

It is easy to see that the expression (196) (without the multiplier $p+1$) is a particular case ($g_4 > g_3+1, g_2 = g_1+1$) of the left-hand side of (194). Combining (194) and (196), we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 \le$$

$$\le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0 \tag{197}$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider Case 4 ($g_2 = g_1 + 1,\ g_4 = g_3 + 1$). We have (see (60))

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 =$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\left(\sum_{j_{g_3}=0}^{\infty}-\sum_{j_{g_3}=0}^{p}\right) C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 =$$

$$=\sum_{j_{q_1}=0}^{p}\left(\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)} - \sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 \le$$

$$\le\frac{1}{2}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2 + \tag{198}$$

$$+\,2\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2. \tag{199}$$

An expression similar to (199) was estimated (see (195)–(197)). Let us estimate (198). We have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2 =$$

$$=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright 0}\right)^2 \le$$

$$\le(T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright j_{g_3}}\right)^2, \tag{200}$$

where the notations are the same as in the proof of Theorem 7.

The expression (200) without the multiplier $T-t$ is an expression of type (124)–(129) before passing to the limit $\lim\limits_{p\to\infty}$ (the only difference is the replacement of one of the weight functions $\psi_1(\tau),\ldots,\psi_4(\tau)$ in (124)–(129) by the product $\psi_{l+1}(\tau)\psi_l(\tau)$ ($l=1,\ldots,4$). Therefore, for Case 4 ($g_2 = g_1+1,\ g_4 = g_3+1$), we obtain the estimate

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1,g_2=g_1+1}\right)^2 \le$$

$$\le \frac{K}{p^{1-\varepsilon}}, \tag{201}$$

where constant K is independent of p.

The estimates (197), (201) prove (166). Let us prove (167). By analogy with (200) we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =$$

$$= (T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright 0,j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \le$$

$$\le (T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{g_1},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2. \tag{202}$$

Thus, we obtain the estimate (see (200) and the proof of Theorem 12)

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \le$$

$$\le \frac{K}{p^{2-\varepsilon}}, \tag{203}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The estimate (203) proves (167). Theorem 13 is proved.

8 Estimates for the Mean-Square Approximation Error of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k \in \mathbb{N}$)

In this section, we estimate the mean-square approximation error in Theorems 7, 10 for iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbb{N}$).

Theorem 14 [13], [48], [49]. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuously differentiable nonrandom function at the interval $[t, T]$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then the following estimates*

$$
\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le K_1 \times
$$

$$
\times\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\right)^2\right\}\right), \tag{204}
$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le K_2(s) \times
$$

$$
\times\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{s,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\right)^2\right\}\right) \tag{205}
$$

hold, where $s \in (t, T]$ (s is fixed), $p \in \mathbb{N}$, $i_1, \ldots, i_k = 1, \ldots, m$,

$$
R_{s,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} = R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\Bigg|_{T=s},
$$

$R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}$ *is defined by* (72), $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *and* $J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}$ *are iterated Stratonovich stochastic integrals* (28) *and* (80), $C_{j_k\ldots j_1}$ *and* $C_{j_k\ldots j_1}(s)$ *are*

Fourier coefficients (20) *and* (78), *constants* K_1, $K_2(s)$ *are independent of* p; *another notations are the same as in Theorems* 1, 7, 10.

Proof. Note that Conditions 1, 2 of Theorems 7, 10 are satisfied under the conditions of Theorem 14 (see (81)–(84), (86), (88)–(93)). Then from the proof of Theorem 7 we have that the expression (77) before passing to limit $\underset{p\to\infty}{\text{l.i.m.}}$ has the form

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}+$$

$$+\sum_{r=1}^{[k/2]}\Biggl(\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\Biggr), \qquad (206)$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the approximation for the iterated Itô stochastic integral (2), which is obtained using Theorems 1, 2, i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \qquad (207)$$

$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}$ is the approximation obtained using (207) for the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ (see (36)).

Using (206) and Theorem 9, we have

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}+$$

$$+\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-\right.$$

$$\left.-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-\right.$$

$$\left.-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} \qquad (208)$$

w. p. 1, where we denote $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$.

In [13] (Sect. 1.7.2, Remark 1.7) it is shown that under the conditions of Theorem 14 the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}\right)^2\right\}\le\frac{k!P_k(T-t)^k}{p} \qquad (209)$$

holds, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (2), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ has the form (207), $p\in\mathbb{N}$, $i_1,\ldots,i_k=1,\ldots,m$, constant P_k depends only on k.

Applying (209), we obtain the following estimates

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le\frac{C}{p},\tag{210}$$

$$\mathsf{M}\left\{\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)^2\right\}\le\frac{C}{p},\tag{211}$$

where $p\in\mathbb{N}$, constant C does not depend on p.

From (208), (210), (211) and the elementary inequality

$$(a_1+a_2+\ldots+a_n)^2\le n\left(a_1^2+a_2^2+\ldots+a_n^2\right),\quad n\in\mathbb{N}$$

we obtain (204).

The estimate (205) is obtained similarly to the estimate (204) using Theorem 1.11 in [13], Theorem 10 and the estimate [13] (Sect. 1.8.1, Remark 1.12)

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}\right)^2\right\}\le\frac{k!P_k(s-t)^k}{p},$$

where

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right),$$

where $s\in(t,T]$ (s is fixed), $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (78), $p\in\mathbb{N}$, $i_1,\ldots,i_k=1,\ldots,m$, constant P_k depends only on k; another notations are the same as in Theorems 2, 10. Theorem 14 is proved.

9 Rate of the Mean-Square Convergence for Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 5

In this section, we consider the rate of convergence for approximations of iterated Stratonovich stochastic integrals. It is easy to see that in Theorems 11–13 the second term in parentheses on the right-hand side of (204) is estimated for $k = 3, 4, 5$. Combining these results with Theorem 14, we obtain the following theorems.

Theorem 15 [13], [48], [49]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity* $J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ *defined by* (3) *the following estimate*

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p}$$

is fulfilled, where $p \in \mathbb{N}$, $i_1, i_2, i_3 = 1, \ldots, m$, *constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Theorem 16 [13], [48], [49]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity* $J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}$ *defined by* (3) *the following estimate*

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}}$$

holds, where $p \in \mathbb{N}$, $i_1, \ldots, i_4 = 1, \ldots, m$, *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal* *system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*; another notations are the same as in Theorem* 1.

Theorem 17 [13], [48], [49]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity* $J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}$ *defined by* (3) *the following estimate*

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}}$$

is valid, where $p\in\mathbb{N}$, $i_1,\ldots,i_5 = 1,\ldots,m$, *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*; another notations are the same as in Theorem* 1.

We should also note the following theorem for the case $k = 2$.

Theorem 18 [13] (Sect. 2.8.1). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of second multiplicity* $J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ *defined by* (3) *the following estimate*

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C}{p}$$

is fulfilled, where $p\in\mathbb{N}$, $i_1,i_2 = 1,\ldots,m$, *constant* C *is independent of* p*; another notations are the same as in Theorem* 1.

Note that the analogue of Theorem 18 for the case $k = 1$ follows from (209).

The expansion (29) for the narrow particular case $i_1 = \ldots = i_k \neq 0$ can be obtained under the condition of convergence of limiting traces [58] (Theorem 5.1), [59] (Theorem 4.1), [56] (Remark 1.5.7, Proposition 4.1.2) (the definition of limiting traces can be found in [59]).

10 Theorems 4–6, 11–13, 15–18 from Point of View of the Wong–Zakai Approximation

The iterated Itô stochastic integrals and solutions of Itô SDEs are complex and important functionals from the independent components $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ of the multidimensional Wiener process $\mathbf{w}_\tau, \tau \in [0, T]$. Let $\mathbf{w}_\tau^{(i)p}$ $(p \in \mathbb{N})$ be some approximation of $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$. Suppose that $\mathbf{w}_\tau^{(i)p}$ converges to $\mathbf{w}_\tau^{(i)}$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{w}_\tau^{(i)}$ by $\mathbf{w}_\tau^{(i)p}$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ of the multidimensional Wiener process $\mathbf{w}_\tau$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [60], [61], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to the iterated Itô stochastic integrals and solutions of Itô SDEs. The piecewise linear approximation as well as the regularization by convolution [60]-[62] relate to the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

It is well known that the following representation takes place [63], [64]

$$\mathbf{w}_\tau^{(i)} - \mathbf{w}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^\tau \phi_j(\theta) d\theta \ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(\theta) d\mathbf{w}_\theta^{(i)}, \tag{212}$$

where $t \ge 0$, $\tau \in [t, T]$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$, $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (212) converges for any $\tau \in [t, T]$ in the mean-square sense.

Let $\mathbf{w}_\tau^{(i)p} - \mathbf{w}_t^{(i)p}$ $(p \in \mathbb{N})$ be the mean-square approximation of the process $\mathbf{w}_\tau^{(i)} - \mathbf{w}_t^{(i)}$, which has the following form

$$\mathbf{w}_\tau^{(i)p} - \mathbf{w}_t^{(i)p} = \sum_{j=0}^{p} \int\limits_t^\tau \phi_j(\theta) d\theta \ \zeta_j^{(i)}. \tag{213}$$

From (213) we obtain

$$d\mathbf{w}_\tau^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau) \zeta_j^{(i)} d\tau. \tag{214}$$

Denote

$$d\mathbf{w}_\tau^{(0)p} = d\tau, \quad p \in \mathbb{N}. \tag{215}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{216}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t, T]$, $p_1, \ldots, p_k \in \mathbb{N}$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Let us substitute (214) and (215) into (216)

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{217}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$),

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient corresponding to the function (4).

For the particular case $p_1 = \ldots = p_k = p$, the formula (217) has the form

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_k}^{(i_k)p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}.$$

To best of our knowledge [60]-[62] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [62] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (213) do not satisfy the conditions of Definition 7.1 and Theorem 7.1 [62]. Also, these approximations of the Wiener process were not considered in [60], [61]. Therefore, the proof of analogue of Theorem 7.1 [62] for approximations of the Wiener process based on its series expansion (212) should be carried out separately.

From the other hand, Theorems 4–6, 11–13, 15–18 and Theorems 1, 2 ($k = 1$) from this article can be considered as the proof of the Wong–Zakai approximation based on the iterated Riemann–Stieltjes integrals (216) of multiplicities 1 to 5 and the Wiener process approximation (213) on the base of its series expansion. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 4–6, 11–13, 15–18 and Theorems 1, 2 ($k = 1$)) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (212), (213), and Theorems 4–6, 11–13, 15–18) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

The Wong–Zakai approximation is widely used to approximate stochastic integrals and SDEs. In particular, the Wong–Zakai approximation can be used to approximate the iterated Stratonovich stochastic integrals in the context of numerical integration of Itô SDEs in the framework of the approach based on the Taylor–Stratonovich expansion [2]-[13].

For example, the authors of the works [2](Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [22] (pp. 438-439), [31] (pp. 263-264) use the Wong–Zakai approximation within the frames of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process. However, in these works there is no rigorous proof of convergence for approximations of the mentioned stochastic integrals.

From the other hand, the theory constructed in this article (also see Chapters 1 and 2 of the monograph [13]) can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 5 based on the Wiener process series expansion (212) using Legendre polynomials and trigonometric functions.

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