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A new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The case of arbitrary complete orthonormal systems in Hilbert space

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Abstract. The article is devoted to the development of a new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. This approach was proposed by the author in 2022 and is based on generalized multiple Fourier series in complete orthonormal systems of functions in Hilbert space. In the previous parts of this work, expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 6 were obtained. At that, the expansions were constructed using two specific bases in Hilbert space. More precisely, Legendre polynomials and the trigonometric Fourier basis were used. In this paper, expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4 are obtained on the base of arbitrary complete orthonormal systems of functions in Hilbert space. Sufficient conditions for the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity are formulated in terms of trace series. The results of the article will be useful for construction of strong numerical methods

with orders 1.0, 1.5 and 2.0 (based on the Taylor–Stratonovich expansion) for Itô stochastic differential equations with non-commutative noise.
Key words: iterated Stratonovich stochastic integral, iterated Itô stochastic integral, Itô stochastic differential equation, multidimensional Wiener process, generalized multiple Fourier series, mean-square convergence, expansion.

Contents

1 Introduction

Let (Ω, F, P) be a complete probability space, let $\{\text{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$ and has independent components $\mathbf{w}_t^{(i)}$ $(i = 1, \ldots, m)$. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \sum_{j=1}^m \int\limits_0^t B_j(\mathbf{x}_\tau, \tau) d\mathbf{w}_\tau^{(j)}, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \quad (1)$$

Here $\mathbf{x}_t$ is the n-dimensional stochastic process satisfying (1). The functions $\mathbf{a}, B_j : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral in (1) is the Itô stochastic integral. Further, $\mathbf{x}_0$ is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$ (M denotes a mathematical expectation). We also assume that $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

Consider the following families of iterated Itô and Stratonovich stochastic integrals:

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (2)$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (3)$$

where $\psi_1(\tau), \ldots, \psi_k(\tau) : [t,T] \to \mathbb{R}$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\mathbf{w}_\tau^{(0)} = \tau$,

$$\int \text{ and } \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively.

It is well known that the stochastic integrals (2) and (3) play an important role when solving Itô SDEs numerically using Taylor–Itô and Taylor–Stratonovich expansions [2]-[14]. From the other hand, Itô SDEs have many applications, which explains the relevance of the problem of their numerical solution [2]-[13].

Note that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$, $i_1,\ldots,i_k=0,1,\ldots,m$ (the case of classical Taylor–Itô and Taylor–Stratonovich expansions) [2]-[8] and $\psi_l(\tau)\equiv (t-\tau)^{q_l}$, $q_l=0,1,\ldots$ $(l=1,\ldots,k)$, $i_1,\ldots,i_k=1,\ldots,m$ (the case of unified Taylor–Ito and Taylor–Stratonovich expansions) [9]-[14].

This article is Part III of the work devoted to a new approach to the series expansion and mean-square approximation of iterated Stratonovich stochastic integrals (3) ([15] and [16] are Part I and Part II of the mentioned work, respectlively).

We also note other approaches to the mean-square approximation of iterated Itô and Stratonovich stochastic integrals (2) and (3) [2]-[5], [17]-[36].

2 Preliminary Results

2.1 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity k $(k\in\mathbb{N})$ Based on Generalized Multiple Fourier Series Converging in the Mean

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Define the following function (Volterra–type kernel) on the hypercube $[t,T]^k$:

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{4}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system (CONS) of functions in the space $L_2([t,T])$. It is well known that the generalized multiple

Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K-K_{p_1\ldots p_k}\right\|_{L_2([t,T]^k)}=0,$$

where

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2},$$

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l),$$

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient.

Consider the partition $\{\tau_j\}_{j=0}^N$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

The following theorem marked the beginning of a systematic study of the problem of strong approximation of iterated Itô and Stratonovich stochastic integrals (2) and (3) that have been most fully studied to date in [14].

Theorem 1 [11] (2006), [12]-[16], [37]-[52]. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous nonrandom functions on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a CONS of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (2), $i_1,\ldots,i_k=0,1,\ldots,m$, l.i.m. *is a limit in the mean-square sense,*

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1, \ldots, l_k): \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},$$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $C_{j_k \ldots j_1}$ is the Fourier coefficient (5), $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, \ 1, \ \ldots, m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t, T]$ *satisfying the condition* (6).

A number of generalizations and modifications of Theorem 1 can be found in [14], Chapter 1 (see also bibliography therein).

Let us consider corollaries from Theorem 1 (see (7)) for $k = 1, \ldots, 5$ [11]

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{8}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{9}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1, p_2, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{10}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1 \ldots i_4)} = \underset{p_1, \ldots, p_4 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_4=0}^{p_4} C_{j_4 \ldots j_1} \Biggl(\prod_{l=1}^4 \zeta_{j_l}^{(i_l)} -$$

$$- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$

$$- \mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$

$$- \mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$

$$+ \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} + \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} +$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (11)
$$

$$
J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \qquad (12)
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider a generalization of (8)–(12) to the case $k\in\mathbb{N}$ and also to the case of an arbitrary CONS in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [14] (Sect. 1.11, 1.14), [43] (Sect. 15, 18), [44]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$. *Then the following expansion*:

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \quad (13)$$

that converges in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x*, the sum in the second line of the formula* (13) *means the sum with respect to all possible permutations of the set*

$$(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}), \quad (14)$$

braces mean an unordered set, and parentheses mean an ordered set, $\{g_1,g_2,\ldots, g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$; $\prod\limits_{\emptyset}\stackrel{\sf def}{=}1$, $\sum\limits_{\emptyset}\stackrel{\sf def}{=}0$; *another notations are the same as in Theorem* 1.

2.2 Stratonovich Stochastic Integral

Let $\mathrm{M}_2([t,T])$ $(0\le t<T<\infty)$ be the class of random functions $\xi(\tau,\omega)\stackrel{\sf def}{=}\xi_\tau:[t,T]\times\Omega\to\mathbb{R}$, which satisfy the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau\in[t,T]$, ξ_τ is independent with increments $\mathbf{w}_{s+\Delta}-\mathbf{w}_s$ for $s\ge\tau$, $\Delta>0$, and

$$\int\limits_t^T \mathsf{M}\left\{(\xi_\tau)^2\right\}d\tau<\infty,\quad \mathsf{M}\left\{(\xi_\tau)^2\right\}<\infty\ \text{ for all }\ \tau\in[t,T].$$

We introduce the class $\mathrm{Q}_4([t,T])$ of Itô processes $\eta_\tau^{(i)}$, $\tau\in[t,T]$, $i=1,\ldots,m$ of the form

$$\eta_\tau^{(i)}=\eta_t^{(i)}+\int\limits_t^\tau a_s ds+\int\limits_t^\tau b_s d\mathbf{w}_s^{(i)}\quad \text{w. p. 1}, \quad (15)$$

where $(a_s)^4,(b_s)^4\in\mathrm{M}_2([t,T])$ and $\lim\limits_{s\to\tau}\mathsf{M}\bigl\{|b_s-b_\tau|^4\bigr\}=0$ for all $\tau\in[t,T]$. The second integral on the right-hand side of (15) is the Itô stochastic integral. Here and further, w. p. 1 means with probability 1.

Let $C^{2,1}(\mathbb{R},[t,T])$ $(t\ge 0)$ be the space of functions $F(x,\tau):\mathbb{R}\times[t,T]\rightarrow\mathbb{R}$ with the following property: these functions are twice differentiable in x and have one derivative in τ. Moreover, all these derivatives are uniformly bounded.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}}\ \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)}+\eta_{\tau_{j+1}}^{(i)}\right),\tau_j\right)\left(\mathbf{w}_{\tau_{j+1}}^{(l)}-\mathbf{w}_{\tau_j}^{(l)}\right)\stackrel{\sf def}{=}\int\limits_t^{*T} F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}\qquad(16)$$

is called [53] the Stratonovich stochastic integral with respect to the component $\mathbf{w}_\tau^{(l)}$ $(l=1,\ldots,m)$ of the multidimensional Wiener process $\mathbf{w}_\tau$, where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition (6).

It is known [53] (also see [2]) that under proper conditions, the following relation between Stratonovich and Itô stochastic integrals holds:

$$\int\limits_t^{*T} F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}=\int\limits_t^{T} F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}+\frac{1}{2}\mathbf{1}_{\{i=l\}}\int\limits_t^{T}\frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau\qquad(17)$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of the set A and $i,l=1,\ldots,m$.

A possible variant of conditions under which the formula (17) is correct, for example, consists of the conditions $\eta_\tau^{(i)}\in\mathrm{Q}_4([t,T])$, $F(\eta_\tau^{(i)},\tau)\in\mathrm{M}_2([t,T])$, $F(x,\tau)\in C^{2,1}(\mathbb{R},[t,T])$, where $i=1,\ldots,m$.

Note that if $F(x,\tau)=F_1(x)F_2(\tau)$, then the smoothness condition $F(x,\tau)\in C^{2,1}(\mathbb{R}\times[t,T])$ can be weakened. Namely, it suffices to replace the condition with respect to τ by continuity with respect to this variable.

In Sect. 3.3, we will also consider another definition of the Stratonovich stochastic integral.

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k $(k\in\mathbb{N})$ Under the Condition on Trace Series

In this section, we recall Theorem 3 (see below) from [15] (Part I of this work) on the expansion of iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k $(k\in\mathbb{N})$ and introduce some notations.

Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have (compare with (14))

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{18}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

Let us call (18) a partition of the set $\{1,2,\ldots,k\}$. Further, we will consider sums with respect to all possible partitions (18) (also see (13)).

Consider the Fourier coefficient

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{19}$$

corresponding to the Volterra–type kernel (4), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS in the space $L_2([t,T])$. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k=$$

$$=\sqrt{T-t}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k, \tag{20}$$

i.e. (20) is again the Fourier coefficient of type (19) but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau)$, $\ldots$, $\psi_{l-2}(\tau)$,

$\sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau)$, $\ldots$, $\psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (18))).

Let

$$C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}\Biggr|_{(j_l j_l)\curvearrowright j_m} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k = \tag{21}$$

$$= \bar{C}_{j_k\ldots j_{l+1}j_m j_{l-2}\ldots j_1} \quad (j_m = 0, 1, 2, \ldots),$$

i.e. $\bar{C}_{j_k\ldots j_{l+1}j_m j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type (19) but with a new shorter multi-index $j_k\ldots j_{l+1}j_m j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau)$, $\ldots$, $\psi_{l-2}(\tau)$, $\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau)$, $\ldots$, $\psi_k(\tau)$ (also we suppose that $\{l-1, l\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (18))).

Let

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}. \tag{22}$$

Introduce the following notation:

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}}\right\} \stackrel{\sf def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}.$$

Note that the operation S_l $(l = 1, 2, \ldots, r)$ acts on the value

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{23}$$

as follows: S_l multiplies (23) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{24}$$

with

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{25}$$

Note that we write

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} =$$

$$= C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}, \quad \ldots$$

Since (25) is again the Fourier coefficient, then the action of superposition S_lS_m on (24) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$
=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},
$$

$$
S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.
$$

Theorem 3 [15] (also see [14], [16], [49]-[51]). *Assume that the continuously differentiable functions $\psi_1(\tau),\ldots,\psi_k(\tau):[t,T]\to\mathbb{R}$ and the CONS $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions ($\phi_0(x)=1/\sqrt{T-t}$) in $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$
\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (26)
$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (26) *converges absolutely.*

2. *The estimates*

$$
\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},
$$

$$
\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_2(s)}{p^{\beta}}
$$

hold for all $s\in(t,T)$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$
\int\limits_t^T\Psi_1^2(\tau)d\tau<\infty,\quad \int\limits_t^T\left|\Psi_2(\tau)\right|d\tau<\infty.
$$

3. *The condition*

$$\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\}\right)^2 = 0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (18)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} \stackrel{\mathsf{def}}{=} \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{27}$$

the following expansion:

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \tag{28}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^{T} \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1\ldots dt_k \tag{29}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense*, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*)*, $\mathbf{w}_\tau^{(0)} = \tau$.

Futher, we will see that Condition 1 of Theorem 3 is fulfilled.

2.4 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6. The Case of Legendre Polynomials and Trigonometric Fourier Basis

In this section, we recall several theorems on the expansion of iterated Stratonovich stochastic integrals (3) of multiplicities 3 to 6 that we obtained in [15], [16] (Parts I and II of this work) using Theorem 3. In addition, we recall the expansion of integrals (3) of multiplicity 2 (old result) [14] (Sect. 2.1.2, 2.8.1).

Theorem 4 [14] (Sect. 2.1.2, 2.8.1). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ $(i_1,i_2 = 0,1,\ldots,m)$ *defined by* (3) *the following relations*:

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{30}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C}{p} \tag{31}$$

are fulfilled, where $i_1,i_2 = 0,1,\ldots,m$ *in* (30) *and* $i_1,i_2 = 1,\ldots,m$ *in* (31), *constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Note that an analogue of Theorem 4 for the case $k=1$ follows from (8).

Theorem 5 [14], [15], [49]-[51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ $(i_1,i_2,i_3 = 0,1,\ldots,m)$ *defined by* (3) *the following relations*:

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{32}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{33}$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (32) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (33)*, constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Theorem 6 [14], [15], [49]-[51]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a CONS of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integrals* $J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}$, $J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}$ $(i_1,\ldots,i_5=0,1,\ldots,m)$ *defined by* (3) *the following relations*:

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\quad(k=4,5), \tag{34}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}\quad(k=4,5) \tag{35}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (34) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (35), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of CONS of Legendre polynomials in* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of CONS of trigonometric functions in* $L_2([t,T])$; *another notations are the same as in Theorem* 1.

Theorem 7 [14], [16], [49]-[51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{36}$$

the following expansion:

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorem 1*.*

2.5 Connection Between Iterated Stratonovich and Itô Stochastic Integrals of Arbitrary Multiplicity k $(k\in\mathbb{N})$

Introduce the following notations:

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]}\ \stackrel{\rm def}{=}\ \prod_{q=1}^{l}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}}\ \times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)},\qquad (37)$$

where $(s_l,\ldots,s_1)\in\mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l}=\bigl\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;\ s_l,\ldots,s_1=1,\ldots,k-1\bigr\},\qquad (38)$$

$l=1,2,\ldots,[k/2]$, $i_1,\ldots,i_k=0,1,\ldots,m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals (3) and (2) of arbitrary multiplicity k $(k\in\mathbb{N})$.

Theorem 8 [54] (1997) (also see [11]-[14], [37], [52]). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous nonrandom functions at the interval* $[t,T]$. *Then, the following relation between iterated Stratonovich and Itô stochastic integrals* (3) *and* (2) *is correct*:

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \tag{39}$$

where $i_1,\ldots,i_k = 0,1,\ldots,m$ *and* $\sum\limits_{\emptyset}$ *is supposed to be equal to zero.*

Note that the condition of continuity of the functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ is related to the definition (16) of the Stratonovich stochastic integral that we use (see [14], [52] for details).

2.6 Multiple Wiener Stochastic Integral With Respect to Components of a Multidimensional Wiener Process

For further consideration, we will need the multiple Wiener stochastic integral with respect to components of a multidimensional Wiener process (generalization of the multiple stochastic integral from Itô's famous work [55] (1951)).

Consider the following step function on the hypercube $[t,T]^k$:

$$\Phi_N(t_1,\ldots,t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{40}$$

where $a_{l_1\ldots l_k} \in \mathbb{R}$ and such that $a_{l_1\ldots l_k} = 0$ if $l_p = l_q$ for some $p \neq q$,

$$\mathbf{1}_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A \\ 0 & \text{otherwise} \end{cases},$$

$N = 2,3,\ldots,$ $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$ satisfying the condition (6).

Let us define the multiple Wiener stochastic integral for $\Phi_N(t_1,\ldots,t_k)$ [55]

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} \stackrel{\mathsf{def}}{=} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{41}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

It is known (see [56], Lemma 9.6.4) that for any $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ there exists a sequence of step functions $\Phi_N(t_1,\ldots,t_k)$ of the form (40) such that

$$\lim_{N\to\infty}\int\limits_{[t,T]^k}\left(\Phi(t_1,\ldots,t_k)-\Phi_N(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k=0. \tag{42}$$

We will define the multiple Wiener stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the formula [55] (see [14], Sect. 1.11 for details)

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\stackrel{\text{def}}{=}\underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{43}$$

where $\Phi_N(t_1,\ldots,t_k)$ is an arbitrary function of the form (40) satisfying the condition (42), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

We note the following estimate for the multiple Wiener stochastic integral:

$$\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le C_k\left\|\Phi\right\|^2_{L_2([t,T]^k)}, \tag{44}$$

where $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$, the constant C_k depends only on k.

In [14] (Sect. 1.11) or [52] (Sect. 1.11) the following equality:

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad\text{w. p. 1} \tag{45}$$

is proved, where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (43) and

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Itô stochastic integral.

Using (45) and Theorem 5 from [44], we obtain the following theorem.

Theorem 9 [14] (Sect. 1.14), [44]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$. *Then the following representation*:

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \qquad (46)$$

is valid w. p. 1, *where* $i_1,\ldots,i_k=0,1,\ldots,m$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (43), $[x]$ *is an integer part of a real number* x, $\prod\limits_{\emptyset}\stackrel{\sf def}{=}1$, $\sum\limits_{\emptyset}\stackrel{\sf def}{=}0$; *another notations are the same as in Theorems* 1, 2.

Combining Theorems 2 and 9 we get the following theorem.

Theorem 10 [44]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. *Then the following equality*:

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} \qquad (47)$$

is valid w. p. 1, *where* $i_1,\ldots,i_k=0,1,\ldots,m$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (43) *and* $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *has the form* (2); *another notations are the same as in Theorems* 1, 2.

3 Main Results

3.1 Generalizations of Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$) Under the Condition on Trace Series

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \stackrel{\mathsf{def}}{=} \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}, \qquad (48)$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero; another notations are the same as in Theorem 8 (see Sect. 2.5).

Theorem 11 [14], [49]-[51]. *Suppose that the CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x) = 1/\sqrt{T-t})$ *in* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$ $(\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2, 3, \ldots, k))$ *are such that the following condition*:

$$\lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_q=0}^{p_q} \ldots \sum_{j_k=0}^{p_k} \Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \times$$

$$\times \Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}} \ldots \sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \Biggr)^2 = 0 \qquad (49)$$

is fulfilled for all $r = 1, 2, \ldots, [k/2]$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (48) *the following expansion*:

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \qquad (50)$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (29), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. Let us find a representation of the expression

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

From (46) we obtain w. p. 1

$$\prod_{l=1}^k\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}-$$

$$-\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}. \tag{51}$$

By iteratively applying the formula (51) (also see (8)–(12)), we obtain the following representation of the product $\prod\limits_{l=1}^k\zeta_{j_l}^{(i_l)}$ as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k :

$$\prod_{l=1}^k\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad\text{w. p. 1}, \tag{52}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$.

Multiplying both sides of the equality (52) by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get w. p. 1

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad\hbox{w. p. 1.}\qquad(53)$$

Implementing the passage to the limit $\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}$ in (53) using Theorem 10, we obtain w. p. 1

$$\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}.\qquad(54)$$

Without loss of generality, let us temporarily set $p_1=\ldots=p_k=p$. We have

$$\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr) \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} +
$$

$$
+ \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} = \qquad (55)
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr) \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} +
$$

$$+\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}\quad \text{w. p. 1},\tag{56}$$

where $g_{2i-1}\stackrel{\text{def}}{=} s_i,\ i=1,2,\ldots,r,\ r=1,2,\ldots,[k/2],\ (s_r,\ldots,s_1)\in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (37) and $\mathrm{A}_{k,r}$ is defined by (38), $g_1,g_2,\ldots,$ g_{2r-1},g_{2r} as in (18), $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$; another notations are the same as above.

Let us explain the transition from (55) to (56). We have for $g_2=g_1+1$, $\ldots,\ g_{2r}=g_{2r-1}+1$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}\ldots\zeta_{j_{m_{2r-1}}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$= \frac{1}{2^r} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots0 i_{q_1}\ldots i_{q_{k-2r}})} = \tag{57}$$

$$= \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \tag{58}$$

where notations as the same as in (56). The transition from (57) to (58) is based on Theorem 10.

Using the estimate (44), we obtain that the condition

$$\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2 = 0 \tag{59}$$

implies that

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} = 0, \tag{60}$$

where $r = 1, 2, \ldots, [k/2]$. Obviously, we can omit the condition $p_1 = \ldots = p_k = p$ in the above consideration.

Further, note that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \Biggl|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1} A_{g_1,g_3,\ldots,g_{2r-1}} =$$

$$= \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} A_{s_1,s_2,\ldots,s_r}, \qquad (61)$$

where $A_{g_1,g_3,\ldots,g_{2r-1}}$, $A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1} = s_i$, $i = 1, 2, \ldots, r$, $r = 1, 2, \ldots, [k/2]$, $\mathrm{A}_{k,r}$ is defined by (38).

Let us return again to the condition $p_1, \ldots, p_k \to \infty$ instead of the condition $p_1 = \ldots = p_k = p \to \infty$. Using (56), (59), (60) with obvious changes and (54), (61), we have

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (37) and $\mathrm{A}_{k,r}$ is defined by (38). Theorem 11 is proved.

Now suppose that $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous functions at the interval $[t, T]$. Then by Theorem 8 we have

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1},$$

where $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Stratonovich stochastic integral (3) and $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the sum of iterated Itô stochastic integrals defined by (48).

Thus, we obtain the following theorem.

Theorem 12 [14], [49]-[51]. *Assume that the continuous functions* $\psi_1(\tau)$, $\ldots,\psi_k(\tau)$ *at the interval* $[t,T]$ *and the CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *are such that the following condition*:

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=0 \qquad (62)$$

is satisfied for all $r=1,2,\ldots,[k/2]$. *Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity* k

$$J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion:

$$J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (29), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $\mathbf{w}_{\tau}^{(0)}=\tau$.

Note that the condition of continuity of the functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ is related to the definition (16) of the Stratonovich stochastic integral that we use. Theorem 12 can be generalized (at least for $k=2$ (see Sect. 3.4)) to the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ if instead of the definition (16) we use another definition of the Stratonovich stochastic integral (see the definition (101) below).

Theorem 13 [14], [49]-[51]. *Suppose that the CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)$ $=1/\sqrt{T-t})$ *in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ $(\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k))$ *are such that the condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0 \qquad (63)$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (18)*) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (48) *the following expansion:*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (29), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $\mathbf{w}_{\tau}^{(0)}=\tau$; *another notations are the same as in Theorem* 3.

Proof. Step 1. First, we prove that

$$\sum_{j_l=0}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=0 \qquad (64)$$

or

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}, \tag{65}$$

where $l-1 \ge s+1$.

Our proof of (64) will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we set $\psi_1(\tau),\ldots,\psi_k(\tau) \equiv 1$ for simplicity.

Using Fubini's Theorem, we have (see [15] (Part I of this work) for details)

$$C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s dt_{s+1}\ldots dt_{l-1} dt_l dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1\ldots dt_{s-1} dt_s \times$$

$$\times \left(\int\limits_{t_{s+1}}^T \phi_{j_{s+2}}(t_{s+2})\ldots \int\limits_{t_{l-2}}^T \phi_{j_{l-1}}(t_{l-1}) \int\limits_{t_{l-1}}^T \phi_{j_l}(t_l) \int\limits_{t_l}^T \phi_{j_{l+1}}(t_{l+1})\ldots \right.$$

$$\left. \ldots \int\limits_{t_{k-1}}^T \phi_{j_k}(t_k) dt_k\ldots dt_{l+1} dt_l dt_{l-1}\ldots dt_{s+2} \right) dt_{s+1} =$$

$$= \int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \underbrace{\int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1\ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)} dt_s \times$$

$$\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) \underbrace{\int\limits_{t_l}^{T} \phi_{j_{l+1}}(t_{l+1}) \ldots \int\limits_{t_{k-1}}^{T} \phi_{j_k}(t_k) dt_k \ldots dt_{l+1}}_{H_{j_k \ldots j_{l+1}}(t_l)} \times$$

$$\times \left(\underbrace{\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1}}_{Q_{j_{l-1} \ldots j_{s+2}}(t_l, t_{s+1})} dt_l \right) dt_{s+1} =$$

$$= \int\limits_{t}^{T} \phi_{j_{s+1}}(t_{s+1}) \int\limits_{t}^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1} \ldots j_1}(t_s) dt_s \times$$

$$\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k \ldots j_{l+1}}(t_l) Q_{j_{l-1} \ldots j_{s+2}}(t_l, t_{s+1}) dt_l dt_{s+1}. \qquad (66)$$

Applying the additive property of the integral, we obtain

$$Q_{j_{l-1} \ldots j_{s+2}}(t_l, t_{s+1}) =$$

$$= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1} =$$

$$= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3}) \int\limits_{t}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} dt_{s+3} \ldots dt_{l-1} -$$

$$- \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3}) dt_{s+3} \ldots dt_{l-1} \int\limits_{t}^{t_{s+1}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} =$$

$$\ldots$$

$$= \sum_{m=1}^{d} h^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_l) q^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_{s+1}), \quad d < \infty. \qquad (67)$$

Combining (66) and (67), we have

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= \sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}) \sum_{j_l=0}^{p}\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1}\ldots j_1}(t_s) dt_s \times\right.$$

$$\left.\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k\ldots j_{l+1}}(t_l) h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l) dt_l dt_{s+1}\right). \qquad (68)$$

Applying the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty}\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1}\ldots j_1}(t_s) dt_s \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k\ldots j_{l+1}}(t_l) h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l) dt_l =$$

$$= \int\limits_t^T \mathbf{1}_{\{\tau<t_{s+1}\}} G_{j_{s-1}\ldots j_1}(\tau)\cdot \mathbf{1}_{\{\tau>t_{s+1}\}} H_{j_k\ldots j_{l+1}}(\tau) h^{(m)}_{j_{l-1}\ldots j_{s+2}}(\tau) d\tau = 0. \qquad (69)$$

From (68) and (69) we get

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= -\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}) \sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1}\ldots j_1}(t_s) dt_s \times\right.$$

$$\left.\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k\ldots j_{l+1}}(t_l) h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l) dt_l dt_{s+1}\right). \qquad (70)$$

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\Phi_1(\tau)$, $\Phi_2(\tau)\in L_2([t,T])$. Then we have

$$\sum_{j=0}^{\infty}\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau \int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le$$

$$\leq \frac{1}{2}\sum_{j=0}^{\infty}\left(\left(\int\limits_t^T \mathbf{1}_{\{\tau<s\}}\phi_j(\tau)\Phi_1(\tau)d\tau\right)^2+\left(\int\limits_t^T \mathbf{1}_{\{\tau>s\}}\phi_j(\tau)\Phi_2(\tau)d\tau\right)^2\right)=$$

$$=\frac{1}{2}\left(\int\limits_t^s \Phi_1^2(\tau)d\tau+\int\limits_s^T \Phi_2^2(\tau)d\tau\right)\leq\frac{1}{2}\left(\|\Phi_1\|^2_{L_2([t,T])}+\|\Phi_2\|^2_{L_2([t,T])}\right)=C<\infty. \tag{71}$$

This means that the estimate (71) can be applied to the series

$$\sum_{j_l=p+1}^{\infty}\left|\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l\right|.$$

Using the above result, Lebesgue's Dominated Convergence Theorem and (66)–(68), (70), we have

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=-\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\left.\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right)=$$

$$=-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}. \tag{72}$$

The equality (72) implies (64), (65).

Step 2. Further, let us prove that

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau, \tag{73}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\psi_2(\tau)\in L_2([t,T])$.

Let us list some useful facts that we will need further in this section.

Proposition 1 ([59], Theorem 8.1). *Let* $\mathbb{K} : L_2([t,T]) \to L_2([t,T])$ *be an integral operator defined by*

$$(\mathbb{K}f)(\tau) = \int_t^T K(\tau,s)f(s)ds,$$

where $K(\tau,s)$ *is a continuous function on* $[t,T]\times[t,T]$. *If, in addition,* $\mathbb{K}$ *is a trace class operator then*

$$tr\mathbb{K} = \int_t^T K(s,s)ds, \tag{74}$$

where trace $tr\mathbb{K}$ *is defined as a series of singular values* $s_j(\mathbb{K})$ *of* $\mathbb{K}$.

Proposition 2 ([59], P. 71). *Let*

$$(\mathbb{K}f)(\tau) = \int_t^T K(\tau,s)f(s)ds,$$

the kernel $K(\tau,s)$ *is continuous on* $[t,T]\times[t,T]$ *and satisfies the condition*

$$|K(\tau,s_2) - K(\tau,s_1)| \le C\,|s_2 - s_1|^{\alpha}, \tag{75}$$

where $0<\alpha\le 1$. *If, in addition,* $\mathbb{K}$ *is a Hermitian operator and* $\alpha > 1/2$, *then* $\mathbb{K}$ *is a trace class operator.*

Suppose that $\mathbb{A}: H \to H$ is a linear bounded operator. Recall [58] that $\mathbb{A}$ has a finite matrix trace if for any orthonormal basis $\{\phi_j(x)\}_{j=0}^{\infty}$ of the space H the series

$$\sum_{j=0}^{\infty} \langle \mathbb{A}\phi_j, \phi_j\rangle_H \tag{76}$$

converges, where $\langle\cdot,\cdot\rangle_H$ is a scalar product in H. Note that the series (76) converges absolutely since its sum does not depend on the permutation of the terms of the series (76) (any permutation of basis functions $\phi_j(x)$ forms a basis in H) [58].

Proposition 3 ([59], Theorem 5.6). *Let $\mathbb{K} : H \to H$ be a trace class operator. Then*

$$tr\mathbb{A} = \sum_{j=0}^{\infty} \langle \mathbb{A}\phi_j, \phi_j \rangle_H \tag{77}$$

for any orthonormal basis $\{\phi_j(x)\}_{j=0}^{\infty}$ of H.

Consider an integral operator $\mathbb{K}' : L_2([t,T]) \to L_2([t,T])$ defined by the equality

$$\left(\mathbb{K}'f\right)(\tau) = \int\limits_t^T K'(\tau, s) f(s) ds,$$

where the continuous kernel $K'(\tau, s)$ has the form (see [14], Sect. 2.1.2)

$$K'(t_1, t_2) = \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases} \quad (t_1, t_2 \in [t, T]),$$

where $\psi_1(\tau), \psi_2(\tau)$ are continuously differentiable functions on $[t, T]$.

Note that (see [14], Sect. 2.1.2)

$$|K'(t_2, s_2) - K'(t_1, s_1)| \leq L\left(|t_2 - t_1| + |s_2 - s_1|\right), \tag{78}$$

where $L < \infty$ and (t_1, s_1), $(t_2, s_2) \in [t, T]^2$.

Let us substitute $t_1 = t_2 = \tau$ into (78)

$$|K'(\tau, s_2) - K'(\tau, s_1)| \leq L|s_2 - s_1|. \tag{79}$$

Thus, the condition (75) is fulfilled ($\alpha = 1$). Further, using Fubini's Theorem, we have

$$\langle \mathbb{K}'x, y \rangle_{L_2([t,T])} = \int\limits_t^T \psi_2(t_2) y(t_2) \int\limits_t^{t_2} \psi_1(t_1) x(t_1) dt_1 dt_2 +$$

$$+ \int\limits_t^T \psi_1(t_2) y(t_2) \int\limits_{t_2}^T \psi_2(t_1) x(t_1) dt_1 dt_2 = \int\limits_t^T \psi_1(t_1) x(t_1) \int\limits_{t_1}^T \psi_2(t_2) y(t_2) dt_2 dt_1 +$$

$$+\int\limits_t^T \psi_2(t_1)x(t_1)\int\limits_t^{t_2}\psi_1(t_2)y(t_2)dt_2dt_1 = \left\langle \mathbb{K}'y, x\right\rangle_{L_2([t,T])}. \qquad (80)$$

The conditions of Proposition 2 are fulfilled. Then, $\mathbb{K}'$ is a trace class operator. Since the kernel $K'(t_1,t_2)$ is continuous, then by Propositions 1 and 3 (see (74) and (77)) we obtain

$$\sum_{j_1=0}^{\infty}\left\langle \mathbb{K}'\phi_{j_1},\phi_{j_1}\right\rangle_{L_2([t,T])} = \int\limits_t^T K'(s,s)ds = \int\limits_t^T \psi_1(s)\psi_2(s)ds. \qquad (81)$$

Combining (80) and (81), we get

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1(t_2)\phi_{j_1}(t_2)\int\limits_{t_2}^{T}\psi_2(t_1)\phi_{j_1}(t_1)dt_1dt_2\right) = \int\limits_t^T \psi_1(s)\psi_2(s)ds, \qquad (82)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$.

Let us substitute $\psi_2(\tau)=(\tau-t)^l$ and $\psi_1(\tau)=(\tau-t)^m$ $(l,m=0,1,2,\ldots)$ into (82)

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^T (t_2-t)^l\phi_{j_1}(t_2)\int\limits_t^{t_2}(t_1-t)^m\phi_{j_1}(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T (t_2-t)^m\phi_{j_1}(t_2)\int\limits_{t_2}^{T}(t_1-t)^l\phi_{j_1}(t_1)dt_1dt_2\right) = \int\limits_t^T (\tau-t)^l(\tau-t)^m d\tau, \qquad (83)$$

where $l,m=0,1,2,\ldots$

The equality (83) was obtained in [57] using other arguments. In addition, the formula (83) was used in [57] to obtain (73).

Consider this approach [57] in more detail. Since the equality (83) is valid for monomials with respect to $\tau-t$ $(\tau\in[t,T])$, it will obviously also be valid for Legendre polynomials that form a CONS in the space $L_2([t,T])$ and finite linear combinations of Legendre polynomials.

Let $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ and $\psi_1^{(p)}(\tau),\psi_2^{(q)}(\tau)$ be approximations of the functions $\psi_1(\tau),\psi_2(\tau)$, respectively, which are partial sums of the corresponding Fourier–Legendre series. Then we have (see (83))

$$\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1^{(p)}(t_1)\phi_j(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1^{(p)}(t_2)\phi_j(t_2)\int\limits_{t_2}^{T}\psi_2^{(q)}(t_1)\phi_j(t_1)dt_1dt_2\right)=\int\limits_t^T\psi_1^{(p)}(\tau)\psi_2^{(q)}(\tau)d\tau. \qquad (84)$$

Let us fix q in (84). The right-hand side of (84) for a fixed q defines (as a scalar product in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\psi_2^{(q)}$. The integral operator (which corresponds to the matrix trace on the left-hand side of (84)) is a trace class operator (see [57]). The matrix trace of the mentioned operator (on the left-hand side of (84)) is also a linear bounded (and therefore continuous) functional (in the space of trace class operators [58], [59]) which can be extended to the space $L_2([t,T])$ by continuity [60].

Let us implement the passage to the limit $\lim\limits_{p\to\infty}$ in (84)

$$\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_{t_2}^{T}\psi_2^{(q)}(t_1)\phi_j(t_1)dt_1dt_2\right)=\int\limits_t^T\psi_1(\tau)\psi_2^{(q)}(\tau)d\tau, \qquad (85)$$

where $q\in\mathbb{N}$. Recall that $\psi_2^{(q)}(\tau)$ is a partial sum of the Fourier–Legendre series of any function $\psi_2(\tau)\in L_2([t,T])$, i.e. the equality (85) holds on a dense subset in $L_2([t,T])$. The right-hand side of (85) defines (as a scalar product in

$L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function ψ_1. On the left-hand side of (85) (by virtue of the equality (85)) there is a linear continuous functional on a dense subset in $L_2([t,T])$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T])$ (see [61], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (85)

$$\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_{t_2}^{T}\psi_2(t_1)\phi_j(t_1)dt_1dt_2\right)=\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau. \tag{86}$$

Applying Fubini's Theorem to the left-hand side of (86), we obtain (73).

Step 3. Under the conditions of Theorem 13 we prove that

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=\frac{1}{2}C_{j_k\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)}-\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1} \tag{87}$$

or

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=\frac{1}{2}C_{j_k\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)}. \tag{88}$$

Denote

$$C_{j_{l-2}\ldots j_1}(t_{l-1})=\int\limits_t^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}.$$

Using Fubini's Theorem and (73), we obtain

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=\sum_{j_l=0}^{\infty}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times$$

$$\times\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})C_{j_{l-2}\ldots j_1}(t_{l-1})dt_{l-1}dt_ldt_{l+1}\ldots dt_k=$$

$$
= \sum_{j_l=0}^{\infty} \int\limits_t^T \psi_l(t_l)\phi_{j_l}(t_l) \int\limits_t^{t_l} \psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})C_{j_{l-2}\ldots j_1}(t_{l-1})dt_{l-1}\times
$$

$$
\times \int\limits_{t_l}^T \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\ldots \int\limits_{t_{k-1}}^T \psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_l =
$$

$$
= \frac{1}{2}\sum_{j_l=0}^{\infty} \int\limits_t^T \psi_l(t_l)\psi_{l-1}(t_l)C_{j_{l-2}\ldots j_1}(t_l)\times
$$

$$
\times \int\limits_{t_l}^T \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\ldots \int\limits_{t_{k-1}}^T \psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_l =
$$

$$
= \frac{1}{2}\sum_{j_l=0}^{\infty} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)C_{j_{l-2}\ldots j_1}(t_l)dt_l dt_{l+1}\ldots dt_k = \frac{1}{2}C_{j_k\ldots j_1}\Bigg|_{(j_l j_l)\curvearrowright(\cdot)}.
$$

The equalities (88) and (87) are proved.

Step 4. Applying (65) and (87) repeatedly, we get (see [15] (Part I of this work) for details)

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}= \tag{89}$$

$$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}+$$

$$+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} \tag{90}$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i,\ i=1,2,\ldots,r,\ r=1,2,\ldots,[k/2]\,,\ (s_r,\ldots,s_1)\in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (37) and $\mathrm{A}_{k,r}$ is defined by (38), and

$$R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl((-1)^r\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r}S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}.$$

The transition from (89) to (90) is explained in the proof of Theorem 12 (see the derivation of (58)).

By condition (63) of Theorem 13 we have (also see the property (44) of the multiple Wiener stochastic integral)

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\right)^2\right\}\le$$

$$\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2+\right.$$

$$+\sum_{l_1=1}^{r}\left(S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$+\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$\ldots$$

$$\left.+\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right)=0, \qquad (91)$$

where constant K does not depend on p.

Using (90) and (91), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}\quad \text{w. p. 1}, \qquad (92)$$

where notations are the same sa in (90).

Applying (54) for the case $p_1=\ldots=p_k=p$ as well as (61), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=$$

$$=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}=\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \qquad (93)$$

w. p. 1, where notations are the same as in Theorem 8 and $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (48). Theorem 13 is proved.

3.2 Generalization of Theorems 11–13 to the Case When the Conditions $\phi_0(x) = 1/\sqrt{T-t}$ and $\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2, 3, \ldots, k)$ are Omitted

In this section, we will consider the following generalization of Theorem 11.

Theorem 14 [14], [50]. *Assume that the CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$ *are such that*

$$\lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl.\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}} C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}\Biggr)^2 = 0 \qquad (94)$$

for all $r = 1, 2, \ldots, [k/2]$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (48) *the following expansion:*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (29), l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_{\tau}^{(0)} = \tau$.

Proof. To prove Theorem 14, we need to prove that under the conditions of Theorem 14 the following equality:

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \qquad (95)$$

holds w. p. 1, where $g_2 = g_1 + 1, \ldots, g_{2r} = g_{2r-1} + 1$, $g_{2i-1} \stackrel{\mathsf{def}}{=} s_i$, $i = 1, 2, \ldots, r$, $r = 1, 2, \ldots, [k/2]$, $(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (37) and $\mathrm{A}_{k,r}$ is defined by (38); also we put $p_1 = \ldots = p_k = p$ in (95) to simplify the notation; another notations in (95) are the same as in Sect. 2.3, 2.5, 3.1.

Using the Itô formula, we obtain w. p. 1

$$\int_t^T \psi_k(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) \int_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\ldots$$

$$\ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} dt_{l-1} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int_t^T \psi_k(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) dt_{l-1} \right) \int_t^{t_{l+1}} \psi_{l-2}(t_{l-2})\ldots$$

$$\ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} -$$

$$-\int_t^T \psi_k(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \left(\int_t^{t_{l-2}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) \times$$

$$\times \int_t^{t_{l-2}} \psi_{l-3}(t_{l-3})\ldots \int_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{l-3}}^{(i_{l-3})}d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (96)$$

where $l \geq 3$. Note that the formula (96) will change in an obvious way for the case $t_{l+1} = T$. We will also assume that the transformation (96) is not carried out for $l = 2$ since the integral

$$\int_t^{t_3} \psi_2(t_1)\psi_1(t_1)dt_1$$

is an innermost integral on the left-hand side of (96) for this case.

It is important to note that the transformation (96) fully complies with the classical rules for replacing the order of integration (Fubini's Theorem) if we replace all differentials of the form $d\mathbf{w}_{t_j}^{(i_j)}$ with dt_j in (96).

Indeed, formally changing the order of integration on the left-hand side of (96) according to the classical rules, we have

$$\int_t^T \psi_k(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) \int_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\ldots \qquad (97)$$

$$\ldots \int_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}dt_{l-1}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int_t^T \psi_k(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int_t^{t_{l+1}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots \int_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2})d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}\times\right.$$

$$\left.\times \int_{t_{l-2}}^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2}) d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} \times \right.$$

$$\left. \times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}} \right) \psi_l(t_{l-1}) \psi_{l-1}(t_{l-1}) dt_{l-1} \right) d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1}) \psi_{l-1}(t_{l-1}) dt_{l-1} \right) \int\limits_t^{t_{l+1}} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots$$

$$\ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2}) d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} -$$

$$- \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2}) \times$$

$$\times \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1}) \psi_{l-1}(t_{l-1}) dt_{l-1} \right) d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1}) \psi_{l-1}(t_{l-1}) dt_{l-1} \right) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} -$$

$$- \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1}) \psi_{l-1}(t_{l-1}) dt_{l-1} \right) \times$$

$$\times \int\limits_t^{t_{l-2}} \psi_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-3}}^{(i_{l-3})} d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)}. \qquad (98)$$

Comparing the right-hand sides of (96) and (98) we come to the conclusion that we got the same result.

The strict mathematical meaning of the transformations leading to (98) is explained in Chapter 3 [14], at least for the case when $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous functions on the interval $[t,T]$.

Note that under the conditions of Theorem 14, the derivation of the formulas (96) and (98) will remain valid if in (96) and (98) we replace all differentials of the form $d\mathbf{w}_{t_j}^{(i_j)}$ with dt_j (this follows from Fubini's Theorem).

Temporarily denote $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$. Let us carry out the transformation (96) for the iterated Itô stochastic integral $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ iteratively for $s_1,\ldots,s_r$. After this, apply Theorem 10 (see (47)) to each of the obtained iterated Itô stochastic integrals. As a result, we obtain w. p. 1

$$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}=\prod_{q=1}^{r}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}}\times$$

$$\times\sum_{d=1}^{2^r}\left(\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}-\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)=$$

$$=\prod_{q=1}^{r}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_{s_1-1},j_{s_1+2},\ldots,j_{s_r-1},j_{s_r+2},\ldots,j_k=0}^{p}\ \sum_{d=1}^{2^r}\Biggl(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}-$$

$$-\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}\Biggr)\times$$

$$\times J'[\phi_{j_1}\ldots\phi_{j_{s_1-1}}\phi_{j_{s_1+2}}\ldots\phi_{j_{s_r-1}}\phi_{j_{s_r+2}}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)},\qquad(99)$$

where some terms in the sum

$$\sum_{d=1}^{2^r}$$

can be identically equal to zero due to the remark to (96).

Taking into account that the integrals $\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and the Fourier coefficients $\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$ are formed on the basis of the same kernels (the same applies to the integrals $\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and the Fourier coefficients $\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$), as well as a remark about the relationship of the transformation (96) based on the Itô formula and on the basis of classical rules for replacing the order of integration (see the derivation of (98)), we obtain using Fubini's theorem (applying the inverse transformation from (98) to (97) in which all differentials of the form $d\mathbf{w}_{t_j}^{(i_j)}$ are replaced with dt_j)

$$\sum_{d=1}^{2^r}\left(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}-\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}\right)=$$

$$=C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}, \qquad (100)$$

where $g_2=g_1+1,\ldots,g_{2r}=g_{2r-1}+1$. Combining (99) and (100), we get w. p. 1

$$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})},$$

where we use the notations from Sect. 2.3, 2.5, 3.1. The quality (95) is proved for the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$. Thus, the condition $\phi_0(x)=1/\sqrt{T-t}$ in Theorems 11–13 can be omitted.

Let us separately explain why the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ ($l=2,3,\ldots,k$) in Theorems 11, 13 can also be omitted. Recall that this condition appeared due to the direct application of Theorem 10 to the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ defined by (37) (see the transition from (57) to (58)).

It is easy to see that the kernels $\hat{K}_d(t_1,\ldots,t_{k-2r})$ and $\bar{K}_d(t_1,\ldots,t_{k-2r})$ of the iterated Itô stochastic integrals $\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and $\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ have the same structure as (4) but with new wight functions $\hat{\psi}_1(\tau),\ldots,\hat{\psi}_{k-2r}(\tau)$ and $\bar{\psi}_1(\tau),\ldots,\bar{\psi}_{k-2r}(\tau)$, some of which possibly coincide with $\psi_1(\tau),\ldots,\psi_k(\tau)$ (see (96)). Moreover, the conditions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$ guarantees that $\hat{K}_d(t_1,\ldots,t_{k-2r}),\bar{K}_d(t_1,\ldots,t_{k-2r})\in L_2([t,T])$ (see (96)). This means that the formula (99) is true if $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$. Furthermore, the formula (100) holds under the conditions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$.

Since the condition $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ implies the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$, then the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$ can be omitted in the above reasoning.

Thus, the equalities (99) and (100) are satisfied under the condition $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ can be omitted in Theorems 11, 13. Theorem 14 is proved.

3.3 Another Definition of the Stratonovich Stochastic Integral

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a complete probability space and let $w(t,\omega)\stackrel{\mathsf{def}}{=}w_t:[0,T]\times\Omega\to\mathbb{R}$ be the standard Wiener process defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on $(\Omega,\mathrm{F},\mathsf{P})$ and connected with the Wiener process w_t in such a way that

1. $\mathrm{F}_s\subset\mathrm{F}_t\subset\mathrm{F}$ for $s<t$.

2. The Wiener process w_t is F_t-measurable for all $t\in[0,T]$.

3. The process $w_{t+\Delta}-w_t$ for all $t\ge 0,$ $\Delta>0$ is independent with the events of σ-algebra F_t.

Let $\xi(\tau,\omega)\stackrel{\sf def}{=}\xi_\tau:[0,T]\times\Omega\to\mathbb{R}$ be some random process, which is measurable with respect to the pair of variables (τ,ω) and satisfies to the following condition:

$$\int\limits_t^T|\xi_\tau|d\tau<\infty\quad \hbox{w. p. 1}\quad (t\ge 0).$$

Let $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$ such that the condition (6) is fulfilled. Consider the definition of the Stratonovich stochastic integral, which differs from the definition given in Sect. 2.2.

The mean-square limit (if it exists)

$$\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr {\rm l.i.m.}\cr\noalign{\vskip1pt} ${}_{N\to \infty}$\cr}} }\sum_{j=0}^{N-1}\frac{1}{\tau_{j+1}-\tau_j}\int\limits_{\tau_j}^{\tau_{j+1}}\xi_s ds\ \left(w_{\tau_{j+1}}-w_{\tau_j}\right)\stackrel{\sf def}{=}\int\limits_t^T\xi_\tau\circ dw_\tau \tag{101}$$

is called [63], [64] the Stratonovich stochastic integral of the process $\xi_\tau,$ $\tau\in[t,T].$ We also denote by

$$\int\limits_t^{\tau}\xi_s\circ dw_s$$

the Stratonovich stochastic integral like (101) (if it exists) of $\xi_s{\bf 1}_{\{s\in[t,\tau]\}}$ for $\tau\in[t,T],$ $t\ge 0.$

It is known [64] (Lemma A.2) that the following iterated Stratonovich stochastic integral:

$$J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)}=\int\limits_t^{\tau}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d{\bf w}_{t_1}^{(i_1)}\ldots\circ d{\bf w}_{t_k}^{(i_k)} \tag{102}$$

exists for the case $i_1=\ldots=i_k\ne 0,$ where $\tau\in[t,T],$ $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T]),$ $i_1,\ldots,i_k=0,1,\ldots,m,$ ${\bf w}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes defined as above in this section and ${\bf w}_{\tau}^{(0)}=\tau.$

Note that in [65] (2021) an analogue of Theorem 8 (1997) is proved for the integral $J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)}$ $(i_1=\ldots=i_k\neq 0,\ \psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T]))$.

3.4 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2. The Case of an Arbitrary CONS in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$

Consider the special case $k=2$ of Theorems 12, 14 in more detail. In this case, the conditions (62), (94) $(p_1=p_2=p)$ takes the following form (compare with (73)):

$$\sum_{j=0}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau, \qquad (103)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ (Theorem 14) or $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$ (Theorem 12).

Thus, from Theorem 12 (the case $k=2$) we obtain the following theorem (recall that the conditions $\phi_0(x)=1/\sqrt{T-t}$ can be omitted in Theorem 12).

Theorem 15 [14], [49]-[51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^{*}[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion:

$$J^{*}[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 1, 2.

The condition of continuity of the functions $\psi_1(\tau), \psi_2(\tau)$ is related to the definition (16) of the Stratonovich stochastic integral that we use.

Theorem 15 can be generalized to the case $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T]$ (see below) if instead of the definition (16) we use another definition of the Stratonovich stochastic integral (see the definition (101)).

From Proposition 3.1 [65] for the case $k = 2$ we obtain

$$\int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)} = \int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)} +$$

$$+\frac{1}{2} \int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1 \qquad (104)$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$, $i = 1, \ldots, m$,

$$\int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)}$$

is defined by (101), (102) and

$$\int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)}$$

is the iterated Itô stochastic integral of the form (2) ($k = 2$, $i_1 = i_2 = i$).

On the other hand, it is not difficult to show that

$$\int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(j)} = \int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(j)} \qquad (105)$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$, $i \neq j$ $(i, j = 1, \ldots, m)$, another notations are the same as in (104).

Combining (104) and (105), we get (see (48))

$$\int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i_1)} \circ d\mathbf{w}_{t_2}^{(i_2)} = \int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} +$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 \stackrel{\text{def}}{=} \bar{J}^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} \tag{106}
$$

w. p. 1, where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $i_1,i_2=1,\ldots,m$.

Summing up the above arguments, we obtain from Theorem 14 ($k=2$) the following generalization of Theorem 15 to the case $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

Theorem 16 [14], [49]-[51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral*

$$
J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\circ d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m)
$$

the following expansion:

$$
J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{107}
$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 1, 2 *and* $J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ *is defined by* (102).

3.5 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case of an Arbitrary CONS in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$

In this section, we will prove the following theorem.

Theorem 17 [14], [49], [50]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = {\int\limits_t^{*}}^{T}{\int\limits_t^{*}}^{t_3}{\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m)
$$

the following expansion:

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \qquad (108)$$

that converges in the mean-square sense is valid, where

$$C_{j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. First, note that under the conditions of Theorem 17 the equality

$$\bar{J}^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*T} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}$$

is true w. p. 1 (see Theorem 8), where $\bar{J}^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ is defined by (48).

According to Theorem 14, we come to the conclusion that Theorem 17 will be proved if we prove the following equalities (see (94)):

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2} C_{j_3 j_1 j_1}\Bigg|_{(j_1 j_1)\curvearrowright(\cdot)} \right)^2 = 0, \qquad (109)$$

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_3 j_3 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} \right)^2 = 0, \qquad (110)$$

$$\lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_2 j_1} \right)^2 = 0. \qquad (111)$$

Let us prove (109). Using Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\frac{1}{2} C_{j_3 j_1 j_1}\Bigg|_{(j_1 j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3 j_1 j_1} \right)^2 =$$

$$
= \lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\int_t^T \phi_{j_3}(\tau) \left(\frac{1}{2} \int_t^\tau ds - \sum_{j_1=0}^{p} \int_t^\tau \phi_{j_1}(s) \int_t^s \phi_{j_1}(\theta) d\theta ds \right) d\tau \right)^2 =
$$

$$
= \lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\int_t^T \phi_{j_3}(\tau) \left(\frac{1}{2} \int_t^\tau ds - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^\tau \phi_{j_1}(s) ds \right)^2 \right) d\tau \right)^2 \le
$$

$$
\le \lim_{p\to\infty} \sum_{j_3=0}^{\infty} \left(\int_t^T \phi_{j_3}(\tau) \left(\frac{1}{2}(\tau - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^\tau \phi_{j_1}(s) ds \right)^2 \right) d\tau \right)^2 =
$$

$$
= \lim_{p\to\infty} \int_t^T \left(\frac{1}{2}(\tau - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^\tau \phi_{j_1}(s) ds \right)^2 \right)^2 d\tau. \qquad (112)
$$

Applying the Parseval equality, we have

$$
\sum_{j_1=0}^{\infty} \frac{1}{2} \left(\int_t^\tau \phi_{j_1}(s) ds \right)^2 = \sum_{j_1=0}^{\infty} \frac{1}{2} \left(\int_t^T \mathbf{1}_{\{s<\tau\}} \phi_{j_1}(s) ds \right)^2 =
$$

$$
= \frac{1}{2} \int_t^T \left(\mathbf{1}_{\{s<\tau\}} \right)^2 ds = \frac{1}{2}(\tau - t). \qquad (113)
$$

Moreover,

$$
\left| \frac{1}{2}(\tau - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^\tau \phi_{j_1}(s) ds \right)^2 \right| \le \frac{1}{2}(\tau - t) \le \frac{1}{2}(T - t) < \infty. \qquad (114)
$$

Using (113), (114) and applying Lebesgue's Dominated Convergence Theorem in (112), we obtain the equality (109).

Let us prove (110). Using Fubini's Theorem and Parseval's equality, we obtain

$$
\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\frac{1}{2} C_{j_3 j_3 j_1} \Big|_{(j_3 j_3) \curvearrowright (\cdot)} - \sum_{j_3=0}^{p} C_{j_3 j_3 j_1} \right)^2 =
$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(\theta)\int\limits_t^{\theta}\phi_{j_3}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau d\theta\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_1}(s)(T-s)ds-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_1}(s)\int\limits_s^T\phi_{j_3}(\tau)\int\limits_{\tau}^T\phi_{j_3}(\theta)d\theta d\tau ds\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s)\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)ds\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(s)\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)ds\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)^2ds. \qquad (115)$$

Using the Parseval equality, we get

$$\sum_{j_3=0}^{\infty}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2=\sum_{j_3=0}^{\infty}\frac{1}{2}\left(\int\limits_t^T\mathbf{1}_{\{s<\tau\}}\phi_{j_3}(\tau)d\tau\right)^2=$$

$$=\frac{1}{2}\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2d\tau=\frac{1}{2}(T-s). \qquad (116)$$

Moreover,

$$\left|\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right|\le\frac{1}{2}(T-s)\le\frac{1}{2}(T-t)<\infty. \qquad (117)$$

Combining (115)–(117) and using the same reasoning as in the proof of (109), we obtain

$$\lim_{p\to\infty}\int\limits_t^T\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)^2ds=0.$$

The equality (110) is proved.

Let us prove (111). Applying Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\Biggl(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\Biggr)^2=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\Biggl(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(\theta)\int\limits_t^{\theta}\phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau d\theta\Biggr)^2=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\Biggl(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta d\tau\Biggr)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_2=0}^{\infty}\Biggl(\int\limits_t^T\phi_{j_2}(\tau)\sum_{j_1=0}^{p}\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta d\tau\Biggr)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\Biggl(\sum_{j_1=0}^{p}\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta\Biggr)^2d\tau. \qquad (118)$$

Using (71), we obtain

$$\left|\sum_{j_1=0}^{p}\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta\right|\le\sum_{j_1=0}^{p}\left|\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta\right|\le$$

$$\le\sum_{j_1=0}^{\infty}\left|\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta\right|\le\frac{1}{2}(T-t)<\infty. \qquad (119)$$

Applying the generalized Parseval equality, we get

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta=\sum_{j_1=0}^{\infty}\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_{j_1}(s)ds\int\limits_t^{T}\mathbf{1}_{\{s>\tau\}}\phi_{j_1}(s)ds=$$

$$=\int\limits_t^T\mathbf{1}_{\{s<\tau\}}\mathbf{1}_{\{s>\tau\}}ds=0. \qquad (120)$$

Taking into account (119), (120) and using Lebesgue's Dominated Convergence Theorem in (118), we obtain the equality (111). Theorem 17 is proved.

3.6 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case of an Arbitrary CONS in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$

In this section, we will prove the following theorem.

Theorem 18 [14], [49], [50]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t}^{(i_1 i_2 i_3 i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}
$$

the following expansion:

$$
J^*[\psi^{(4)}]_{T,t}^{(i_1 i_2 i_3 i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}
$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3,i_4=0,1,\ldots,m$,

$$
C_{j_4 j_3 j_2 j_1}=\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \qquad (121)
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_{\tau}^{(0)}=\tau$.

Proof. First, note that under the conditions of Theorem 18 the equality

$$
\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1 i_2 i_3 i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}
$$

is valid w. p. 1 (see Theorem 8), where $\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1 i_2 i_3 i_4)}$ is defined by (48).

It is easy to see that Theorem 18 will be proved if we prove the following equalities (see Theorem 14 and (94)):

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4 j_3 j_1 j_1}-\frac{1}{2}C_{j_4 j_3 j_1 j_1}\Bigg|_{(j_1 j_1)\curvearrowright(\cdot)}\right)^2=0, \tag{122}$$

$$\lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4 j_1 j_2 j_1}\right)^2=0, \tag{123}$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1 j_3 j_2 j_1}\right)^2=0, \tag{124}$$

$$\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_4 j_2 j_2 j_1}-\frac{1}{2}C_{j_4 j_2 j_2 j_1}\Bigg|_{(j_2 j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{125}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2 j_3 j_2 j_1}\right)^2=0, \tag{126}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3 j_3 j_2 j_1}-\frac{1}{2}C_{j_3 j_3 j_2 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{127}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3 j_3 j_1 j_1}=\frac{1}{4}C_{j_3 j_3 j_1 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)(j_1 j_1)\curvearrowright(\cdot)}=\frac{1}{8}(T-t)^2, \tag{128}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1 j_3 j_3 j_1}=0, \tag{129}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1 j_2 j_1}=0. \tag{130}$$

Let us prove the equalities (122)–(127). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (122)–(127):

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4 j_3 j_1 j_1}-\frac{1}{2}C_{j_4 j_3 j_1 j_1}\Bigg|_{(j_1 j_1)\curvearrowright(\cdot)}\right)^2=$$

$$
= \sum_{j_3,j_4=0}^{p} \left(\frac{1}{2} \int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3)(t_3 - t) dt_3 dt_4 - \right.
$$

$$
\left. - \sum_{j_1=0}^{p} \int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_3,j_4=0}^{p} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2}(t_3 - t) - \right.\right.
$$

$$
\left.\left. - \sum_{j_1=0}^{p} \int_t^{t_3} \phi_{j_1}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \right) dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_3,j_4=0}^{p} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2}(t_3 - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^{t_3} \phi_{j_1}(s) ds \right)^2 \right) dt_3 dt_4 \right)^2 \le
$$

$$
\le \sum_{j_3,j_4=0}^{\infty} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2}(t_3 - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^{t_3} \phi_{j_1}(s) ds \right)^2 \right) dt_3 dt_4 \right)^2 =
$$

$$
= \int_{[t,T]^2} \mathbf{1}_{\{t_3 < t_4\}} \left(\frac{1}{2}(t_3 - t) - \sum_{j_1=0}^{p} \frac{1}{2} \left(\int_t^{t_3} \phi_{j_1}(s) ds \right)^2 \right)^2 dt_3 dt_4, \qquad (131)
$$

$$
\sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_4 j_1 j_2 j_1} \right)^2 =
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=0}^{p} \int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_1}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=0}^{p} \int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_2}^{t_4} \phi_{j_1}(t_3) dt_3 dt_2 dt_4 \right)^2 =
$$

$$= \sum_{j_2,j_4=0}^{p} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_2}^{t_4} \phi_{j_1}(t_3) dt_3 dt_2 dt_4 \right)^2 \le$$

$$\le \sum_{j_2,j_4=0}^{\infty} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_2}^{t_4} \phi_{j_1}(t_3) dt_3 dt_2 dt_4 \right)^2 =$$

$$= \int_{[t,T]^2} \mathbf{1}_{\{t_2<t_4\}} \left(\sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_2}^{t_4} \phi_{j_1}(t_3) dt_3 \right)^2 dt_2 dt_4, \qquad (132)$$

$$\sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_3 j_2 j_1} \right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=0}^{p} \int_t^T \phi_{j_1}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=0}^{p} \int_t^T \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_3}^{T} \phi_{j_1}(t_4) dt_4 dt_2 dt_3 \right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p} \left(\int_t^T \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_3}^{T} \phi_{j_1}(t_4) dt_4 dt_2 dt_3 \right)^2 \le$$

$$\le \sum_{j_2,j_3=0}^{\infty} \left(\int_t^T \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_3}^{T} \phi_{j_1}(t_4) dt_4 dt_2 dt_3 \right)^2 =$$

$$= \int_{[t,T]^2} \mathbf{1}_{\{t_2<t_3\}} \left(\sum_{j_1=0}^{p} \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \int_{t_3}^{T} \phi_{j_1}(t_4) dt_4 \right)^2 dt_2 dt_3, \qquad (133)$$

$$\sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=0}^{p} C_{j_4 j_2 j_2 j_1} - \frac{1}{2} C_{j_4 j_2 j_2 j_1} \Big|_{(j_2 j_2) \curvearrowright (\cdot)} \right)^2 =$$

$$
=\sum_{j_1,j_4=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4-\right.
$$

$$
\left.-\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_4=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}dt_2dt_1dt_4-\right.
$$

$$
\left.-\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\right)^2=
$$

$$
=\sum_{j_1,j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{t_4-t_1}{2}-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)dt_1dt_4\right)^2\le
$$

$$
\le\sum_{j_1,j_4=0}^{\infty}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{t_4-t_1}{2}-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)dt_1dt_4\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_4\}}\left(\frac{1}{2}(t_4-t_1)-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)^2dt_1dt_4, \tag{134}
$$

$$
\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4dt_3\right)^2=
$$

$$
= \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_4) dt_4 dt_1 dt_3 \right)^2 =
$$

$$
= \sum_{j_1,j_3=0}^{p} \left(\int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_4) dt_4 dt_1 dt_3 \right)^2 \le
$$

$$
\le \sum_{j_1,j_3=0}^{\infty} \left(\int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_4) dt_4 dt_1 dt_3 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_3\}} \left(\sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_4) dt_4 \right)^2 dt_1 dt_3, \qquad (135)
$$

$$
\sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_2 j_1} - \frac{1}{2} C_{j_3 j_3 j_2 j_1} \Big|_{(j_3 j_3) \curvearrowright (\cdot)} \right)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \left(\frac{1}{2} \int\limits_t^T \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 - \right.
$$

$$
\left. - \sum_{j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \left(\frac{1}{2} \int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} dt_3 dt_2 dt_1 - \right.
$$

$$
\left. - \sum_{j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \right)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \left(\int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2) \left(\frac{T-t_2}{2} - \sum_{j_3=0}^{p} \frac{1}{2} \left(\int\limits_{t_2}^{T} \phi_{j_3}(s) ds \right)^2 \right) dt_2 dt_1 \right)^2 \le
$$

$$\leq \sum_{j_1,j_2=0}^{\infty}\left(\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\left(\frac{T-t_2}{2}-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^T \phi_{j_3}(s)ds\right)^2\right)dt_2dt_1\right)^2 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2\}}\left(\frac{1}{2}(T-t_2)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^T \phi_{j_3}(s)ds\right)^2\right)^2 dt_2dt_1. \qquad (136)$$

Using Parseval's equality, generalized Parseval's equality and Lebesgue's Dominated Convergence Theorem, as well as applying the same reasoning as in the proof of Theorem 17, we obtain that the right-hand sides of (131)–(136) tend to zero when $p\to\infty$. The equalities (122)–(127) are proved.

Let us prove the equalities (128)–(130). First, let us show that

$$C_{j_4j_3j_2j_1}+C_{j_1j_2j_3j_4}=C_{j_4}C_{j_3j_2j_1}-C_{j_3j_4}C_{j_2j_1}+C_{j_2j_3j_4}C_{j_1}, \qquad (137)$$

where $C_{j_4j_3j_2j_1}$ has the form (121).

Using Fubini's Theorem, we have

$$C_{j_4j_3j_2j_1}=\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$=\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{T} \phi_{j_3}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 -$$

$$-\int\limits_t^T \phi_{j_4}(t_4)\int\limits_{t_4}^{T} \phi_{j_3}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$=C_{j_4}C_{j_3j_2j_1}-\int\limits_t^T \phi_{j_4}(t_4)\int\limits_{t_4}^{T} \phi_{j_3}(t_3)\int\limits_t^{T} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 +$$

$$+\int\limits_t^T \phi_{j_4}(t_4)\int\limits_{t_4}^{T} \phi_{j_3}(t_3)\int\limits_{t_3}^{T} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$= C_{j_4}C_{j_3j_2j_1} - C_{j_3j_4}C_{j_2j_1} + \int\limits_t^T \phi_{j_4}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_2}(t_2)\int\limits_t^T \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 -$$

$$-\int\limits_t^T \phi_{j_4}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$= C_{j_4}C_{j_3j_2j_1} - C_{j_3j_4}C_{j_2j_1} + C_{j_2j_3j_4}C_{j_1} - C_{j_1j_2j_3j_4}. \tag{138}$$

From (138) we get (137). Recall that in [16] (Part II of this work) we obtained an analogue of (137) for the case $k = 6$.

It is easy to see that we can consider the following generalization of (137) for the case $k = 2r$ $(r = 2, 3, 4, \ldots)$:

$$C_{j_kj_{k-1}\ldots j_1}^{\psi_k\psi_{k-1}\ldots\psi_1} + C_{j_1j_2\ldots j_k}^{\psi_1\psi_2\ldots\psi_k} = C_{j_k}^{\psi_k}\cdot C_{j_{k-1}j_{k-2}\ldots j_1}^{\psi_{k-1}\psi_{k-2}\ldots\psi_1} - C_{j_{k-1}j_k}^{\psi_{k-1}\psi_k}\cdot C_{j_{k-2}j_{k-3}\ldots j_1}^{\psi_{k-2}\psi_{k-3}\ldots\psi_1} +$$

$$+ C_{j_{k-2}j_{k-1}j_k}^{\psi_{k-2}\psi_{k-1}\psi_k}\cdot C_{j_{k-3}j_{k-4}\ldots j_1}^{\psi_{k-3}\psi_{k-4}\ldots\psi_1} - \ldots - C_{j_3j_4\ldots j_k}^{\psi_3\psi_4\ldots\psi_k}\cdot C_{j_2j_1}^{\psi_2\psi_1} + C_{j_2j_3\ldots j_k}^{\psi_2\psi_3\ldots\psi_k}\cdot C_{j_1}^{\psi_1}, \tag{139}$$

where

$$C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{140}$$

and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Further, we will write $C_{j_k\ldots j_1}$ instead of $C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1}$ if this does not cause misunderstandings.

In principle, using (139), we can calculate any expressions of the form

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}, \tag{141}$$

where $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ are as in (18) and the following symmetry condition:

$$\psi_1(\tau) = \psi_k(\tau), \quad \psi_2(\tau) = \psi_{k-1}(\tau), \quad \ldots, \quad \psi_r(\tau) = \psi_{r+1}(\tau) \tag{142}$$

is fulfilled for $k = 2r$, $r = 2, 3, 4, \ldots$

Obviously, the case $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$ is possible since it is a special case of the symmetry condition (142). This case is important because it covers the mean-square approximation of iterated Stratonovich stochastic integrals from the classical Taylor–Stratonovich expansions [2]-[7].

Let us prove (128). Substitute $j_4 = j_3$, $j_2 = j_1$ into (137)

$$C_{j_3j_3j_1j_1} + C_{j_1j_1j_3j_3} = C_{j_3}C_{j_3j_1j_1} - C_{j_3j_3}C_{j_1j_1} + C_{j_1j_3j_3}C_{j_1}. \tag{143}$$

From (143) we obtain

$$\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_3j_1j_1} + C_{j_1j_1j_3j_3}\right) = \sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \sum_{j_1,j_3=0}^{p} C_{j_3j_3}C_{j_1j_1} +$$

$$+\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3}C_{j_1}.$$

Then

$$2\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = 2\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \left(\sum_{j_1=0}^{p} C_{j_1j_1}\right)^2. \tag{144}$$

Using (144) and Fubini's Theorem, we get

$$\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = \sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \frac{1}{2}\left(\sum_{j_1=0}^{p} C_{j_1j_1}\right)^2 =$$

$$= \sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \frac{1}{2}\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(C_{j_1}\right)^2\right)^2 = \sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \frac{1}{8}\left(\sum_{j_1=0}^{p}\left(C_{j_1}\right)^2\right)^2. \tag{145}$$

Applying Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(C_{j_1}\right)^2 = \int\limits_t^T 1^2 d\tau = T - t. \tag{146}$$

Combining (145) and (146), we get

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = \lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} - \frac{(T-t)^2}{8}. \tag{147}$$

Further, we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} =$$

$$=\frac{1}{2}\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \tag{148}$$

Using the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} = \lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T \phi_{j_3}(\tau)d\tau\int\limits_t^T \phi_{j_3}(\tau)\int\limits_t^{\tau} d\theta d\tau =$$

$$=\int\limits_t^T 1\cdot\int\limits_t^{\tau} d\theta d\tau = \frac{(T-t)^2}{2}. \tag{149}$$

From (148) and (149) we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1} =$$

$$=\frac{(T-t)^2}{4} - \lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \tag{150}$$

Combining (147) and (150), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = \frac{(T-t)^2}{8} - \lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \tag{151}$$

Due to Cauchy–Bunyakovsky's inequality and (109), (146), we get

$$\lim_{p\to\infty}\left(\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3j_1j_1}\right)\right)^2 \le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(C_{j_3}\right)^2\ \sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p} C_{j_3j_1j_1}\right)^2 \le$$

$$\leq \lim_{p\to\infty}\sum_{j_3=0}^{\infty}(C_{j_3})^2\ \sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=(T-t)\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=0. \tag{152}$$

From (152) and (151) we obtain (128).

Let us prove (129). Substitute $j_4=j_1$, $j_2=j_3$ into (137)

$$C_{j_1j_3j_3j_1}+C_{j_1j_3j_3j_1}=C_{j_1}C_{j_3j_3j_1}-C_{j_3j_1}C_{j_3j_1}+C_{j_3j_3j_1}C_{j_1}. \tag{153}$$

Using (153), we get

$$2\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=2\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2. \tag{154}$$

Applying (154), we obtain

$$\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}-\frac{1}{2}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2. \tag{155}$$

Parseval's equality gives

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2=\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\phi_{j_1}(t_1)\phi_{j_3}(t_2)dt_1dt_2\right)^2=$$

$$=\int\limits_{[t,T]^2}\left(\mathbf{1}_{\{t_1<t_2\}}\right)^2dt_1dt_2=\frac{(T-t)^2}{2}. \tag{156}$$

Combining (155) and (156), we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}-\frac{(T-t)^2}{4}. \tag{157}$$

Further, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}=$$

$$
= \frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1} C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{158}
$$

Applying Fubini's Theorem and the generalized Parseval equality, we have

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1} C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} = \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(\tau)d\tau \int\limits_t^T\int\limits_t^{t_2}\phi_{j_1}(\tau)d\tau dt_2 =
$$

$$
= \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(\tau)d\tau \int\limits_t^T \phi_{j_1}(\tau)\int\limits_\tau^T dt_2 d\tau = \int\limits_t^T 1\cdot\int\limits_\tau^T d\theta d\tau = \frac{(T-t)^2}{2}. \tag{159}
$$

From (158) and (159) we get

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1} C_{j_3j_3j_1} =
$$

$$
= \frac{(T-t)^2}{4} - \lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{160}
$$

Combining (157) and (160), we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = -\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{161}
$$

Due to Cauchy–Bunyakovsky's inequality and (110), (146), we get

$$
\lim_{p\to\infty}\left(\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right)\right)^2 \le
$$

$$
\le \lim_{p\to\infty}\sum_{j_1=0}^{p}\left(C_{j_1}\right)^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right)^2 \le
$$

$$
\le \lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(C_{j_1}\right)^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right)^2 =
$$

$$= (T-t)\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2 = 0. \qquad (162)$$

From (161) and (162) we obtain (129).

Let us prove (130). Substitute $j_3 = j_1$, $j_4 = j_2$ into (137)

$$C_{j_2j_1j_2j_1} + C_{j_1j_2j_1j_2} = C_{j_2}C_{j_1j_2j_1} - C_{j_1j_2}C_{j_2j_1} + C_{j_2j_1j_2}C_{j_1}. \qquad (163)$$

Then

$$\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1j_2j_1} + C_{j_1j_2j_1j_2}\right) = \sum_{j_1,j_2=0}^{p}\left(C_{j_2}C_{j_1j_2j_1} + C_{j_2j_1j_2}C_{j_1}\right) -$$

$$- \sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1}. \qquad (164)$$

Using (164), we have

$$\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1} = \sum_{j_1,j_2=0}^{p}C_{j_2}C_{j_1j_2j_1} - \frac{1}{2}\sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1}. \qquad (165)$$

Fubini's Theorem and the generalized Parseval equality give

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1} =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_1)dt_1dt_2\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2 =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_1\}}\mathbf{1}_{\{t_1<t_2\}}dt_1dt_2 = 0. \qquad (166)$$

The equalities (165) and (166) imply the relation

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1} = \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2}C_{j_1j_2j_1}. \qquad (167)$$

Further, we have (see the derivation of (152))

$$\lim_{p\to\infty}\left(\sum_{j_2=0}^{p}C_{j_2}\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2\le\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(C_{j_2}\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(C_{j_2}\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=(T-t)\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=0, \tag{168}$$

where (168) follows from (111).

Using (167) and (168), we obtain (130). The equalities (122)–(130) are proved. Theorem 18 is proved.

3.7 Generalization of the Results from Section 3.5 to the Case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\in L_2([t,T])$

In this section, we will prove the following two theorems.

Theorem 19 [14], [50], [51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\in L_2([t,T])$. *Then, for the sum* $\bar{J}^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}$ $(i_1,i_2,i_3=0,1,\ldots,m)$ *of iterated Itô stochastic integrals defined by* (48) *the following expansion*:

$$\bar{J}^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$), $\mathbf{w}_\tau^{(0)}=\tau$.

Theorem 20 [14], [50], [51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuous functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion:

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m$; *another notations are the same as in Theorem* 19.

Note that Theorem 20 is a simple consequence of Theorem 19 and Theorem 8 ($k=3$). Let us prove Theorem 19.

Proof. First, let us note some facts that follow from Monotone Convergence Theorem ([60], Theorem 3.5.1). Suppose that $\{g_j(x)\}_{j=0}^{\infty}$ is an arbitrary sequence of real-valued measurable functions such that the series

$$\sum_{j=0}^{\infty} g_j(x) \tag{169}$$

converges absolutely almost everywhere on X (with respect to Lebesgue's measure) to some function $f(x)$. From Monotone Convergence Theorem, in particular, it follows the following equality (see [60], Theorem 3.5.2):

$$\int\limits_X \sum_{j=0}^{\infty} g_j(x) dx = \sum_{j=0}^{\infty} \int\limits_X g_j(x) dx. \tag{170}$$

It is easy to see that under the above conditions the following equality:

$$\lim_{p\to\infty} \int\limits_X \left(\sum_{j=0}^{p} g_j(x)\right)^2 dx = \int\limits_X \left(\sum_{j=0}^{\infty} g_j(x)\right)^2 dx \tag{171}$$

is true (further, we will use the equality (171)). Indeed, we have $g_j(x) = g_j^+(x) - g_j^-(x)$, $|g_j(x)| = g_j^+(x) + g_j^-(x)$, where $g_j^+(x) = \max\{g_j(x), 0\} \geq 0$, $g_j^-(x) = -\min\{g_j(x), 0\} \geq 0$. Moreover,

$$\sum_{j=0}^{\infty} g_j(x) = \sum_{j=0}^{\infty} g_j^+(x) - \sum_{j=0}^{\infty} g_j^-(x),$$

$$\sum_{j=0}^{\infty} |g_j(x)| = \sum_{j=0}^{\infty} g_j^+(x) + \sum_{j=0}^{\infty} g_j^-(x). \tag{172}$$

Since the series (169) converges absolutely, then by virtue of the equality (172) the series (with non-negative terms) on the right-hand side of (172) converge (to some functions $f_1(x)$ and $f_2(x)$, respectively). Further, using Monotone Convergence Theorem, we obtain

$$\lim_{p\to\infty} \int\limits_X \left(\sum_{j=0}^{p} g_j(x)\right)^2 dx = \lim_{p\to\infty} \int\limits_X \left(\sum_{j=0}^{p} g_j^+(x) - \sum_{j=0}^{p} g_j^-(x)\right)^2 dx =$$

$$= \lim_{p\to\infty} \left(\int\limits_X \left(\sum_{j=0}^{p} g_j^+(x)\right)^2 dx - 2\int\limits_X \left(\sum_{j=0}^{p} g_j^+(x) \sum_{j=0}^{p} g_j^-(x)\right) dx +\right.$$

$$\left. + \int\limits_X \left(\sum_{j=0}^{p} g_j^-(x)\right)^2 dx \right) =$$

$$= \int\limits_X \lim_{p\to\infty} \left(\sum_{j=0}^{p} g_j^+(x)\right)^2 dx - 2\int\limits_X \lim_{p\to\infty} \sum_{j=0}^{p} g_j^+(x) \sum_{j=0}^{p} g_j^-(x) dx +$$

$$+ \int\limits_X \lim_{p\to\infty} \left(\sum_{j=0}^{p} g_j^-(x)\right)^2 dx =$$

$$= \int\limits_X (f_1(x))^2 dx - 2\int\limits_X f_1(x) f_2(x) dx + \int\limits_X (f_2(x))^2 dx =$$

$$= \int\limits_X (f_1(x) - f_2(x))^2 dx = \int\limits_X \left(\sum_{j=0}^{\infty} g_j(x)\right)^2 dx.$$

According to Theorem 14, we come to the conclusion that Theorem 19 will be proved if we prove the following equalities (see (94)):

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=0, \tag{173}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2=0, \tag{174}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=0. \tag{175}$$

Let us prove (173). Using Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)ds\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)ds\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\psi_3^2(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.$$

$$\left.-\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2ds= \tag{176}$$

$$= \int\limits_t^T \psi_3^2(s) \lim_{p\to\infty} \left(\frac{1}{2}\int\limits_t^s \psi_2(\tau)\psi_1(\tau)d\tau - \right.$$

$$\left. -\sum_{j_1=0}^{p} \int\limits_t^s \psi_2(\tau)\phi_{j_1}(\tau) \int\limits_t^\tau \psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2 ds = 0, \qquad (177)$$

where (177) follows from (73) and the transition from (176) to (177) is based on (171) and the absolute convergence of series on the left-hand side of (73) (the sum of this series does not depend on the order of terms since the sum is equal to the integral on the right-hand side of (73) for any basis $\{\phi_j(x)\}_{j=0}^{\infty}$ (we mean the order of numbering of the functions $\phi_j(x)$)). The equality (173) is proved.

Let us prove (174). Using Fubini's Theorem and Parseval's equality, we obtain

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\frac{1}{2} C_{j_3 j_3 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3 j_3 j_1}\right)^2 =$$

$$= \lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\frac{1}{2}\int\limits_t^T \psi_3(\tau)\psi_2(\tau) \int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds d\tau - \right.$$

$$\left. -\sum_{j_3=0}^{p} \int\limits_t^T \psi_3(\theta)\phi_{j_3}(\theta) \int\limits_t^\theta \psi_2(\tau)\phi_{j_3}(\tau) \int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds d\tau d\theta\right)^2 =$$

$$= \lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\frac{1}{2}\int\limits_t^T \psi_1(s)\phi_{j_1}(s) \int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau ds - \right.$$

$$\left. -\sum_{j_3=0}^{p} \int\limits_t^T \psi_1(s)\phi_{j_1}(s) \int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau) \int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds\right)^2 =$$

$$= \lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\int\limits_t^T \psi_1(s)\phi_{j_1}(s) \left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau - \right.\right.$$

$$\left.\left. -\sum_{j_3=0}^{p} \int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau) \int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right) ds\right)^2 \le$$

$$\leq \lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_1(s)\phi_{j_1}(s)\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)ds\right)^2 =$$

$$= \lim_{p\to\infty}\int\limits_t^T \psi_1^2(s)\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.$$

$$\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)^2 ds = \tag{178}$$

$$= \int\limits_t^T \psi_1^2(s)\lim_{p\to\infty}\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.$$

$$\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)^2 ds = 0, \tag{179}$$

where (179) follows from (73) and the transition from (178) to (179) is based on (171) and the absolute convergence of series on the left-hand side of (73). The equality (174) is proved.

Let us prove (175). Applying Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_3(\theta)\phi_{j_1}(\theta)\int\limits_t^\theta \psi_2(\tau)\phi_{j_2}(\tau)\int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds d\tau d\theta\right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_2(\tau)\phi_{j_2}(\tau)\int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds\int\limits_\tau^T \psi_3(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2 \leq$$

$$\leq \lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(\int\limits_t^T \psi_2(\tau)\phi_{j_2}(\tau)\sum_{j_1=0}^{p}\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2 =$$

$$= \lim_{p\to\infty}\int\limits_t^T \psi_2^2(\tau)\left(\sum_{j_1=0}^{p}\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_1}(\theta)d\theta\right)^2 d\tau = \qquad (180)$$

$$= \int\limits_t^T \psi_2^2(\tau)\lim_{p\to\infty}\left(\sum_{j_1=0}^{p}\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_1}(\theta)d\theta\right)^2 d\tau = 0, \qquad (181)$$

where (181) follows from the equality

$$\sum_{j_1=0}^{\infty}\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_1}(\theta)d\theta =$$

$$= \int\limits_t^T \psi_1(s)\mathbf{1}_{\{s<\tau\}}\psi_3(s)\mathbf{1}_{\{s>\tau\}}ds = 0 \qquad (182)$$

(the relation (182) follows from the generalized Parseval equality) and the transition from (180) to (181) is based on (171) and the absolute convergence of series on the left-hand side of (182) (see the derivation of (71)). The equality (175) is proved. Theorem 19 is proved.

3.8 Generalization of the Results from Section 3.6 to the Case $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$

Let us develop the approach discussed in the previous section.

Theorem 21 [14], [50], [51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$. *Then, for the sum* $\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}$ $(i_1,\ldots,i_4=0,1,\ldots,m)$ *of iterated Itô stochastic integrals defined by* (48) *the following expansion*:

$$\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where

$$C_{j_4\ldots j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_4 \qquad (183)$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_\tau^{(0)} = \tau$.

Theorem 22 [14], [50], [51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_4(\tau)$ are continuous functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \int\limits_t^{*T} \psi_4(t_4)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}$$

the following expansion:

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_4 = 0,1,\ldots,m$; another notations are the same as in Theorem 21.

Note that Theorem 22 is a simple consequence of Theorem 21 and Theorem 8 ($k=4$). Let us prove Theorem 21.

Proof. It is easy to see that Theorem 21 will be proved if we prove that (see Theorem 14 and (94))

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_4j_3j_1j_1} - \frac{1}{2}C_{j_4j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2 = 0, \qquad (184)$$

$$\lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_4j_1j_2j_1}\right)^2 = 0, \qquad (185)$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_3j_2j_1}\right)^2=0, \tag{186}$$

$$\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_4j_2j_2j_1}-\frac{1}{2}C_{j_4j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{187}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}\right)^2=0, \tag{188}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_2j_1}-\frac{1}{2}C_{j_3j_3j_2j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{189}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}=\frac{1}{4}\int\limits_t^T\psi_4(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1dt_3, \tag{190}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=0, \tag{191}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=0, \tag{192}$$

where $C_{j_4\ldots j_1}$ has the form (183), $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in the space $L_2([t,T])$, and $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

To prove (184)–(189) we modify the proof of (122)–(127). More precisely, the proof of (184)–(189) is carried out by analogy with the proof of (122)–(127) using the equality (171) instead of Lebesgue's Dominated Convergence Theorem (see the proof of Theorem 19 for details) and adjusted for the fact that in the proof of (122)–(127) the functions $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$ are replaced by $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$. Thus, the equalities (184)–(189) are proved.

In [57] an efficient method is proposed for proving equalities similar to (190)–(192). In particular, the equality (190) is proved in [57]. The above method [57] is based on the equality of the matrix and integral traces of trace class operators ([62], Theorem 3.1). In the next section, the equalities (190)–(192)

are proved using the generalized Parseval equality and (73). At that, we use some ideas from [57]. Theorem 21 is proved.

3.9 On the Calculation of Matrix Traces of Volterra–Type Integral Operators

It is easy to see that the function (4) for even $k = 2r$ $(r \in \mathbb{N})$ forms a family of integral operators $\mathbb{K} : L_2([t,T]^r) \to L_2([t,T]^r)$ (with the kernel (4)) of the form

$$(\mathbb{K}f)(t_{g_1},\ldots,t_{g_r}) = \int\limits_{[t,T]^r} K(t_1,\ldots,t_k) f(t_{g_{r+1}},\ldots,t_{g_k}) dt_{g_{r+1}}\ldots dt_{g_k}, \qquad (193)$$

where $\{g_1,\ldots,g_k\} = \{1,\ldots,k\}$, the kernel $K(t_1,\ldots,t_k)$ is defined by (4), i.e.

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ & \\ 0, & \text{otherwise} \end{cases}, \qquad (194)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $t_1,\ldots,t_k \in [t,T]$ $(k \geq 2)$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

For example,

$$(\mathbb{K}f)(t_2) = \int\limits_t^T K(t_1,t_2) f(t_1) dt_1 = \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) f(t_1) dt_1, \qquad (195)$$

$$(\mathbb{K}f)(t_2,t_3) = \int\limits_{[t,T]^2} K(t_1,\ldots,t_4) f(t_1,t_4) dt_1 dt_4 =$$

$$= \psi_2(t_2)\psi_3(t_3)\mathbf{1}_{\{t_2<t_3\}} \int\limits_t^{t_2} \psi_1(t_1) \int\limits_{t_3}^T \psi_4(t_4) f(t_1,t_4) dt_4 dt_1, \qquad (196)$$

where $K(t_1,\ldots,t_4)$ is defined by (194).

The simplest representative of the family (193) has the form

$$(\mathbb{V}f)(x) = \int\limits_0^x f(\tau)d\tau \qquad (197)$$

and is called the Volterra integral operator, where $\mathbb{V} : L_2([0,1]) \to L_2([0,1])$, $f(\tau) \in L_2([0,1])$. The kernel of the Volterra integral operator is determined by the relation: $K(\tau, x) = \mathbf{1}_{\{\tau<x\}}$ $(\tau, x \in [0,1])$. It is well known that the Volterra integral operator (197) is not a trace class operator since its singular values are equal to [62]

$$s_j(\mathbb{V}) = \frac{2}{\pi(2j+1)}.$$

On the other hand, it is known [62] that for trace class operators the equality of matrix and integral traces holds (recall that the matrix trace of a linear bounded operator is defined by (76)). It turns out that for the Volterra integral operator (197) (although it is not a trace class operator), the equality of matrix and integral traces is also true [62].

Thus, one cannot count on the fact that operators of the more general form (193) (from the same family of operators as the Volterra integral operator (197)) are operators of the trace class. Nevertheless, the proof of the equalities of matrix and integral traces for Volterra–type integral operators (193) (which is obviously a problem) provides a way to calculate the matrix traces of these operators.

Why do we talk so much in this section about matrix traces of operators from the family (193)? The point is that matrix traces of operators of the form (193) are of great importance for obtaining of expansions of iterated Stratonovich stochastic integrals.

Let us consider some illustrative examples. We have

$$\sum_{j_1=0}^{\infty} \langle \mathbb{K}\phi_{j_1}, \phi_{j_1} \rangle_{L_2([t,T])} =$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=\sum_{j_1=0}^{\infty}C_{j_1j_1}, \tag{198}$$

$$\sum_{j_1,j_2=0}^{\infty}\left\langle\mathbb{K}\Psi_{j_1j_2},\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_1)\int\limits_t^{t_2}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$
$$\times dt_1dt_2dt_3dt_4=$$

$$=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_2j_1j_1}, \tag{199}$$

where $\{\Psi_{j_1j_2}(x,y)\}_{j_1,j_2=0}^{\infty}=\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$, $(\mathbb{K}f)(t_2)$ in (198) is defined by (195), and $(\mathbb{K}f)(t_2,t_3)$ in (199) has the form (196).

The expressions on the right-hand sides of (198) and (199) were considered earlier in this article (see (73), (190)).

Let us prove the equalities (190)–(192) using a method based on generalized Parseval's equality and (73). At that, we will use some ideas from [57].

First we prove (190). Using (73), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$
$$\times dt_1dt_2dt_3dt_4=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_4\times$$
$$\times\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\frac{1}{4}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)dt_4\int\limits_t^T \psi_2(t_2)\psi_1(t_2)dt_2=$$

$$=\frac{1}{4}\int\limits_{[t,T]^2} \psi_4(t_4)\psi_3(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4, \tag{200}$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Suppose that $\psi_2(\tau)$ and $\psi_3(\tau)$ are polynomials of finite degrees. For example, $\psi_2(\tau)$ and $\psi_3(\tau)$ can be Legendre polynomials that form a CONS in $L_2([t,T])$. Denote

$$s_q(t_2,t_3)=\sum_{l_1,l_2=0}^{q} C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3), \tag{201}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3)=\bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau)\in L_2([t,T]))$, i.e.

$$C_{l_2l_1}=\int\limits_t^T \bar{\psi}_3(t_3)\bar{\phi}_{l_2}(t_3)\int\limits_t^{t_3} \bar{\psi}_2(t_2)\bar{\phi}_{l_1}(t_2)dt_2dt_3.$$

Further, we have $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^2)}=0$. From (200) we obtain (the sum on the right-hand side of (201) is finite)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_3<t_4\}}\psi_4(t_4)\phi_{j_2}(t_4)s_q(t_2,t_3)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{4}\int\limits_{[t,T]^2} \psi_4(t_4)s_q(t_2,t_4)\psi_1(t_2)dt_2dt_4. \tag{202}$$

Note that the equality (202) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$, i.e. the equality holds on a dense subset in $L_2([t,T]^2)$. The right-hand side of (202) defines (as a scalar product of $s_q(t_2,t_4)$ and $\psi_4(t_4)\psi_1(t_2)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\psi_4(t_4)\psi_1(t_2)$. On the left-hand side of (202) (by virtue of the equality (202)) there is a linear continuous functional on a dense subset in $L_2([t,T]^2)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^2)$ (see [61], Theorem I.7, P. 9). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in the equality (202) (at that we suppose that s_q is defined by (201))

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{4}\int\limits_t^T\psi_4(t_4)\bar{\psi}_3(t_4)\int\limits_t^{t_4}\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4, \tag{203}$$

where $\psi_1(\tau),\bar{\psi}_2(\tau),\bar{\psi}_3(\tau),\psi_4(\tau)\in L_2([t,T])$.

Rewrite the equality (203) in the form

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1j_1}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{4}\int\limits_t^T\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4, \tag{204}$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (204) converges absolutely since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (190) is proved.

Let us prove (192). Using the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{T}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4 =$$

$$= \sum_{j_1,j_2=0}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_4 \times$$

$$\times \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 =$$

$$= \sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_4(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_4)dt_3dt_4 \times$$

$$\times \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}}\psi_1(t_3)\psi_2(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_4)dt_3dt_4 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_2(t_4)\psi_4(t_4)\psi_1(t_3)dt_3dt_4 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_2\}}\psi_3(t_3)\psi_2(t_2)\psi_4(t_2)\psi_1(t_3)dt_3dt_2, \tag{205}$$

where $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau), \psi_4(\tau) \in L_2([t,T])$.

Suppose that $\psi_2(\tau)$ and $\psi_3(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2,t_3) = \sum_{l_1,l_2=0}^{q} C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3), \tag{206}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3) = \bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau), \bar{\psi}_3(\tau) \in L_2([t,T]))$, i.e.

$$C_{l_2l_1} = \int\limits_t^T \bar{\psi}_3(t_3)\bar{\phi}_{l_2}(t_3) \int\limits_t^{t_3} \bar{\psi}_2(t_2)\bar{\phi}_{l_1}(t_2)dt_2dt_3$$

and

$$\lim_{q\to\infty} \|s_q - g\|^2_{L_2([t,T]^2)} = 0.$$

From (205) we obtain (the sum on the right-hand side of (206) is finite)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_3<t_4\}}\psi_4(t_4)s_q(t_2,t_3)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_2\}}s_q(t_2,t_3)\psi_1(t_3)\psi_4(t_2)dt_3dt_2. \qquad (207)$$

The right-hand side of (207) defines (as a scalar product of $s_q(t_2,t_3)$ and $\mathbf{1}_{\{t_3<t_2\}}\psi_1(t_3)\psi_4(t_2)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\mathbf{1}_{\{t_3<t_2\}}\psi_1(t_3)\psi_4(t_2)$. On the left-hand side of (207) there is also a linear continuous functional in $L_2([t,T]^2)$ (see note below the formula (202)).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (207)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2>t_3\}}\mathbf{1}_{\{t_2<t_3\}}\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_3)\psi_4(t_2)dt_3dt_2=0. \qquad (208)$$

Rewrite the equality (208) in the form

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0, \qquad (209)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (209) converges absolutely since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (192) is proved.

Let us prove (191). Using Fubini's Theorem and generalized Parseval's equality, we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4 =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1}^{\psi_4}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1} = \frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} -$$

$$-\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} - \sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right) =$$

$$= \frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_4(s)\phi_{j_1}(s)ds\int\limits_t^T\psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)\psi_1(s)dsd\tau -$$

$$-\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} - \sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right) =$$

$$= \frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_4(s)\phi_{j_1}(s)ds\int\limits_t^T\phi_{j_1}(s)\psi_1(s)\int\limits_s^{T}\psi_3(\tau)\psi_2(\tau)d\tau ds -$$

$$-\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} - \sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right) =$$

$$= \frac{1}{2}\int\limits_t^T\psi_4(s)\psi_1(s)\int\limits_s^T\psi_3(\tau)\psi_2(\tau)d\tau ds -$$

$$-\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} - \sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right), \qquad (210)$$

where $C_{j_1}^{\psi_4}$ and $C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}$ are defined by (140).

Due to Cauchy–Bunyakovsky's inequality, Parseval's equality and (174), we get

$$\lim_{p\to\infty}\left(\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(C_{j_1}^{\psi_4}\right)^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(C_{j_1}^{\psi_4}\right)^2\ \sum_{j_2=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2=$$

$$=\int_t^T\psi_4^2(s)ds\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2=0. \qquad (211)$$

Combining (210) and (211), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int_t^T\psi_3(t_3)\phi_{j_2}(t_3)\int_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int_t^T\psi_4(s)\psi_1(s)\int_s^T\psi_3(\tau)\psi_2(\tau)d\tau ds=$$

$$=\frac{1}{2}\int\limits_{[t,T]^2}\psi_3(t_3)\psi_4(t_4)\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3, \qquad (212)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Suppose that $\psi_3(\tau)$ and $\psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4), \qquad (213)$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4)=\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_3<t_4\}}$ ($\bar{\psi}_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T])$), i.e.

$$C_{l_2l_1}=\int_t^T\bar{\psi}_4(t_4)\bar{\phi}_{l_2}(t_4)\int_t^{t_4}\bar{\psi}_3(t_3)\bar{\phi}_{l_1}(t_3)dt_3dt_4$$

and

$$\lim_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^2)}=0.$$

From (212) we obtain (the sum on the right-hand side of (213) is finite)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_1}(t_4)\phi_{j_2}(t_3)s_q(t_3,t_4)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int\limits_{[t,T]^2}s_q(t_3,t_4)\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3. \qquad (214)$$

The right-hand side of (214) defines (as a scalar product of $s_q(t_3,t_4)$ and $\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)$. On the left-hand side of (214) there is also a linear continuous functional in $L_2([t,T]^2)$ (see note below the formula (202)).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (214)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\bar\psi_4(t_4)\phi_{j_1}(t_4)\bar\psi_3(t_3)\phi_{j_2}(t_3)\psi_2(t_2)\phi_{j_2}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int\limits_{[t,T]^2}\bar\psi_3(t_3)\bar\psi_4(t_4)\mathbf{1}_{\{t_3<t_4\}}\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3=0. \quad (215)$$

Rewrite the equality (215) in the form

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0, \qquad (216)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (216) converges absolutely since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (191) is proved. The equalities (190)–(192) are proved.

4 Conclusion

Recall that this article is Part III of the work devored to a new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of a multidimensional Wiener process ([15] and [16] are Parts I and II of this work, respectively). The results of the work make it possible to construct efficient procedures for the mean-square approximation of iterated Stratonovich stochastic integrals that appear in strong methods with orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Itô SDEs with multidimensional non-commutative noise (approach based on the Taylor–Stratonovich expansion). The above procedures based on multiple Fourier–Legendre series have been successfully implemented as part of the software package in the Python programming language in [66].

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