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Nonlinear partial differential equations
Group analysis of differential equations

Solution of the Carleman system using group analysis

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Abstract. In this paper, we consider the discrete kinetic Carleman system. The Carleman system is the Boltzmann kinetic equation, and for this model momentum and energy are not conserved. Using group analysis methods, we obtain a solution representing the density of gas particles in a certain area. This limitation is due to the non-negativity of solution. Similarly, it is possible to find exact solutions for other kinetic models.

Keywords: Carleman system, group analysis, invariant solution

1 Introduction

We consider the one-dimensional Carleman system [9, 15]:

$$\begin{aligned} &\partial_t u + \partial_x u = \frac{1}{\varepsilon}(v^2 - u^2), \quad x \in \mathbb{R},\ t > 0, \\ &\partial_t v - \partial_x v = -\frac{1}{\varepsilon}(v^2 - u^2). \end{aligned} \tag{1}$$

Here $u = u(x,t), v = v(x,t)$ are the densities of two groups of particles with velocities $c = 1, -1$, ε is the Knudsen parameter from the kinetic theory of gases. This system describes a monatomic rarefied gas consisting of two groups of particles. The Carleman system is a non-integrable system, i.e. the Painlevé test is not applicable. The interaction is as follows. The Carleman system describes particles of two groups, namely, the first group of particles moves at a unit speed along the axis Ox, and the second group moves at a unit speed in the opposite direction. It is worth noting that only particles within one group interact, that is, only with themselves, changing the direction of motion.

The method of group analysis is a well-known method for finding solutions, in particular invariant solutions of equations of mathematical physics. This method is described in detail in [1, 2, 3, 5]. A general description of the Boltzmann equation is described in the article [4]. In this works [6, 7], Ilyin O. V. obtained an optimal system of one-dimensional subalgebras and classes of invariant solutions for the stationary kinetic Broadwell model and the one-dimensional Boltzmann integro-differential equation for Maxwellian particles with inelastic collisions. In [8] the results of group analysis of the multidimensional Boltzmann and Vlasov equations are presented. Also, solutions of kinetic systems using Painlevé series were found in [9, 10, 11, 15, 16]. Asymptotic stability for Boltzmann models as well as the numerical study are presented in [14, 17, 18, 19]. The analysis described in this paper provides solutions of the Carleman system that have not yet been found in the literature.

2 Method of group analysis

We consider a system of two second order partial differential equations

$$F_1(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2}$$

$$F_2(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, ...) = 0, \tag{3}$$

where $u = u(x,t), v = v(x,t)$ are unknown functions. According to the group analysis methods, we will look for the prolonged operator in the form

$$\underset{1}{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u} + \chi\frac{\partial}{\partial v} + \zeta_1\frac{\partial}{\partial u_x} + \zeta_2\frac{\partial}{\partial u_t} + \chi_1\frac{\partial}{\partial v_x} + \chi_2\frac{\partial}{\partial v_t},$$

where $\xi = \xi(x,t,u,v)$, $\eta = \xi(x,t,u,v)$, $\zeta = \zeta(x,t,u,v)$, $\chi = \chi(x,t,u,v)$. Here

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u} + \chi\frac{\partial}{\partial v}. \tag{4}$$

is called the infinitesimal operator of the group. A universal invariant of the group and the operator (4) is a function $I(x,t,u,v)$

$$XI = \xi\frac{\partial I}{\partial x} + \eta\frac{\partial I}{\partial t} + \zeta\frac{\partial I}{\partial u} + \chi\frac{\partial I}{\partial v} = 0. \tag{5}$$

The coordinates of the first prolongation are given by

$$\zeta_1 = D_x(\zeta) - u_x D_x(\xi) - u_t D_x(\eta).$$

$$\zeta_2 = D_t(\zeta) - u_x D_t(\xi) - u_t D_t(\eta)$$

and

$$\chi_1 = D_x(\chi) - v_x D_x(\xi) - v_t D_x(\eta),$$
$$\chi_2 = D_t(\chi) - v_x D_t(\xi) - v_t D_t(\eta),$$

where D_x, D_t are the operators of total differentiation with respect to x and t:

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + v_{xx} \frac{\partial}{\partial v_x} + v_{xt} \frac{\partial}{\partial v_t} + \cdots,$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + v_{xt} \frac{\partial}{\partial v_x} + v_{tt} \frac{\partial}{\partial v_t} + \cdots.$$

After calculations, we have

$$\zeta_1 = \zeta_x + \zeta_u u_x + \zeta_v v_x - u_x(\xi_x + \xi_u u_x + \xi_v v_x) - u_t(\eta_x + \eta_u u_x + \eta_v v_x), \qquad (6)$$

$$\zeta_2 = \zeta_t + \zeta_u u_t + \zeta_v v_t - u_x(\xi_t + \xi_u u_t + \xi_v v_t) - u_t(\eta_x + \eta_u u_t + \eta_v v_t) \qquad (7)$$

and

$$\chi_1 = \chi_x + u_x \chi_u + v_x \chi_v - v_x(\xi_x + u_x \xi_u + v_x \xi_v) - v_t(\eta_x + u_x \eta_u + v_x \eta_v), \qquad (8)$$

$$\chi_2 = \xi_t + \xi_u u_t + \xi_v v_t - v_x(\xi_t + \xi_u u_t + \xi_v v_t) - v_t(\eta_t + \eta_u u_t + \eta_v v_t). \qquad (9)$$

We require that

$$\underset{1}{X} F_1|_{F_1=F_2=0} = 0, \underset{1}{X} F_2|_{F_1=F_2=0} = 0. \qquad (10)$$

Relations (10) is called the invariance conditions.

We illustrate the above procedure for the Carleman system.

3 Application of the method

Let us insert

$$F_1 = u_t + u_x - \frac{1}{\varepsilon}(v^2 - u^2),$$

$$F_2 = v_t - v_x + \frac{1}{\varepsilon}(v^2 - u^2),$$

into the invariance conditions (10)

$$\underset{1}{X} F_1|_{F_1=0,F_2=0} = (\zeta_2 + \zeta_1 - \frac{1}{\varepsilon}2v\chi + \frac{1}{\varepsilon}2u\zeta)|_{F_1=0,F_2=0}, \tag{11}$$

$$\underset{1}{X} F_2|_{F_1=0,F_2=0} = (\chi_2 - \chi_1 + \frac{1}{\varepsilon}2v\chi - \frac{1}{\varepsilon}2u\zeta)|_{F_1=0,F_2=0}. \tag{12}$$

Now, replacing u_t, v_t by $-u_x + \frac{1}{\varepsilon}(v^2 - u^2), v_x - \frac{1}{\varepsilon}(v^2 - u^2)$ and taking into account the expressions (6)-(7) and (8)-(9) for the coordinates of the first prolongation from the first equation (11), we have:

$$u_x : -\xi_t - \frac{1}{\varepsilon}(v^2 - u^2)\xi_u + \xi_v \frac{1}{\varepsilon}(v^2 - u^2) + \eta_t + \eta_u \frac{1}{\varepsilon}(v^2 - u^2) -$$

$$-\frac{1}{\varepsilon}\eta_v(v^2 - u^2) - \xi_x + \eta_x = 0,$$

$$u_x^2 : \xi_u - \eta_u - \xi_u + \eta_u = 0,$$

$$v_x : \zeta_v - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v + \zeta_v - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v = 0,$$

$$u_x v_x : -\xi_v + \eta_v - \xi_v + \eta_v = 0,$$

$$1 : \zeta_t + \zeta_u \frac{1}{\varepsilon}(v^2 - u^2) - \zeta_v \frac{1}{\varepsilon}(v^2 - u^2) - \frac{1}{\varepsilon}(v^2 - u^2)\eta_t -$$

$$-\frac{1}{\varepsilon}(v^2 - u^2)\eta_u \frac{1}{\varepsilon}(v^2 - u^2) + \frac{1}{\varepsilon}(v^2 - u^2)\eta_v \frac{1}{\varepsilon}(v^2 - u^2) + \zeta_x -$$

$$-\frac{1}{\varepsilon}(v^2 - u^2)\eta_x - \frac{2}{\varepsilon}\chi v + \frac{2}{\varepsilon}u\zeta = 0.$$

From the second equation (12):

$$u_x : \chi_u - \frac{1}{\varepsilon}(v^2 - u^2)\eta_u - \chi_u - \frac{1}{\varepsilon}(v^2 - u^2)\eta_u = 0,$$

$$u_x v_x : \xi_u + \eta_u + \xi_u + \eta - u = 0,$$

$$v_x : \chi_v - \xi_t - \frac{1}{\varepsilon}(v^2 - u^2)\xi_u + \xi_v \frac{1}{\varepsilon}(v^2 - u^2) - \eta_t -$$

$$-\eta_u \frac{1}{\varepsilon}(v^2 - u^2) + \eta_v \frac{1}{\varepsilon}(v^2 - u^2) + \frac{1}{\varepsilon}(v^2 - u^2)\eta_v - \chi_v + \xi_x +$$

$$+\eta_x - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v = 0,$$

$$v_x^2 : -\xi_v - \eta_v + \xi_v + \eta_v = 0,$$
$$1 : \chi_t + \chi_u \frac{1}{\varepsilon}(v^2 - u^2) - \chi_v \frac{1}{\varepsilon}(v^2 - u^2) + \frac{1}{\varepsilon}(v^2 - u^2)\eta_t +$$
$$+\frac{1}{\varepsilon}(v^2 - u^2)\eta_u \frac{1}{\varepsilon}(v^2 - u^2) - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v \frac{1}{\varepsilon}(v^2 - u^2) - \chi_x -$$
$$-\frac{1}{\varepsilon}(v^2 - u^2)\eta_x + \frac{2}{\varepsilon}\chi v - \frac{2}{\varepsilon}\zeta u = 0.$$

Rewrite the system in a more compact form

$$\eta_v = \xi_v,$$
$$2\frac{1}{\varepsilon}(v^2 - u^2)\eta_u - \xi_t + \eta_t - \eta_x - \xi_x = 0,$$
$$\zeta_v - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v = 0,$$
$$\frac{1}{\varepsilon}(v^2 - u^2)(\zeta_u - \zeta_v - \eta_t - \eta_u \frac{1}{\varepsilon}(v^2 - u^2) + \eta_v \frac{1}{\varepsilon}(v^2 - u^2) - \eta_x) +$$
$$+\zeta_x + \zeta_t - \frac{2}{\varepsilon}\chi v + \frac{2}{\varepsilon}u\zeta = 0$$

and

$$\eta_u = -\xi_u,$$
$$\chi_u + \frac{1}{\varepsilon}(v^2 - u^2)\eta_u = 0,$$
$$2\frac{1}{\varepsilon}(v^2 - u^2)\eta_v - \eta_t - \xi_t + \eta_x + \xi_x = 0,$$
$$\frac{1}{\varepsilon}(v^2 - u^2)(\chi_u - \chi_v + \eta_t + \frac{1}{\varepsilon}(v^2 - u^2)\eta_u - \frac{1}{\varepsilon}(v^2 - u^2)\eta_v - \eta_x) +$$
$$+\chi_t - \chi_x + \frac{2}{\varepsilon}\chi v - \frac{2}{\varepsilon}\zeta u = 0.$$

Integrating the system, we get

$$\eta(t) = \alpha t + \beta, \xi(x) = \alpha x + \gamma, \zeta(u) = -\alpha u, \chi(v) = -\alpha v,$$

where α, β, γ are arbitrary constants. The corresponding characteristic system of ordinary differential equations for (5)

$$\frac{dt}{\eta} = \frac{dx}{\xi} = \frac{du}{\zeta} = \frac{dv}{\chi}$$

or

$$\frac{dt}{\alpha t + \beta} = \frac{dx}{\alpha x + \gamma} = \frac{du}{-\alpha u} = \frac{dv}{-\alpha v}.$$

We obtain

$$\omega = \frac{\alpha x + \gamma}{\alpha t + \beta},$$

where α, γ, β is an integration constants. Invariant solution are sought in the form

$$u = \frac{\Phi(\omega)}{\alpha(\alpha t + \beta)}, v = \frac{\Psi(\omega)}{\alpha(\alpha t + \beta)},$$

where Φ, Ψ unknown functions of similarity variable that must be defined.

$$\frac{-(\alpha t + \beta)\Phi + (\alpha t - \alpha x + \beta - \gamma)\Phi'}{(\alpha t + \beta)^3} = \frac{\Psi^2 - \Phi^2}{\alpha^2(\alpha t + \beta)^2 \varepsilon}, \tag{13}$$

$$\frac{-(\alpha t + \beta)\Psi - (\alpha t + \alpha x + \beta + \gamma)\Psi'}{(\alpha t + \beta)^3} = \frac{\Phi^2 - \Psi^2}{\alpha^2(\alpha t + \beta)^2 \varepsilon} \tag{14}$$

or we rewrite

$$\begin{aligned} \alpha^2\varepsilon(\Phi(1-\omega))' &= -\Phi^2 + \Psi^2, \\ \alpha^2\varepsilon(\Psi(1+\omega))' &= -\Phi^2 + \Psi^2. \end{aligned} \tag{15}$$

$$\Phi(\omega) = \frac{1+\omega}{1-\omega}\Psi(\omega) + (1-\omega)C,$$

where C is an arbitrary constant of integration.

For finding function Ψ, we have the Riccati equation

$$\Psi' = \frac{-4\omega}{\varepsilon\alpha^2(1-\omega)^2(1+\omega)}\Psi^2 + \Big(\frac{-2C - \varepsilon\alpha^2 + 2(\varepsilon\alpha^2 - C)\omega - \varepsilon\alpha^2\omega^2}{\varepsilon\alpha^2(1-\omega)^2(1+\omega)}\Big)\Psi - \\ - \frac{C^2}{\varepsilon\alpha^2(1-\omega)^2(1+\omega)}.$$

We can write the particular solution at $C = 0$

$$\begin{aligned} u &= \frac{-\varepsilon\alpha(\varepsilon\alpha(t+x) + \beta + \gamma)}{-2(\varepsilon\alpha t + \beta)^2 + C_1((\varepsilon\alpha t + \beta)^2 - (\varepsilon\alpha t + \gamma)^2)}, \\ v &= \frac{-\varepsilon\alpha(\varepsilon\alpha(t-x) + \beta - \gamma)}{-2(\varepsilon\alpha t + \beta)^2 + C_1((\varepsilon\alpha t + \beta)^2 - (\varepsilon\alpha t + \gamma)^2)}, \end{aligned} \tag{16}$$

where C_1 is an arbitrary constant of integration. The non-negativity of the solution can be chosen by choosing a constant C_1 in the region of the plane x, t.

4 Conclusion

In this article, the solution was found using group analysis methods. We determined three different infinitesimal groups of transformations leaving the invariant condition. A well-known and very important special class of invariant solutions was represented by self similar solution which was constructed on the basis of invariants of extension groups. Future work will present solutions to the remaining kinetic systems.

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