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Permanence analysis for continuous and discrete-time generalized Lotka–Volterra models with delay and switching of parameters

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Abstract. The paper is addressed to the permanence problem for a generalized Lotka–Volterra system modeling interaction of species in a biological community. The impact of a constant delay and switching of parameters on the dynamics of the system is taken into account. Our analysis is based on the Lyapunov direct method. An original construction of a Lyapunov–Krasovskii functional is proposed. The conditions for the existence of such a functional are formulated in terms of feasibility of special systems of linear algebraic inequalities. It is proven that, under these conditions, the investigated system is permanent for any constant positive delay and any admissible switching signal. In addition, a discrete-time counterpart of the considered model is studied for which the permanence analysis is fulfilled, as well. A comparison of the obtained permanence conditions with known ones is provided. It is shown that the constraints on the system parameters derived in this paper are less conservative.

Keywords: population dynamics, ultimate boundedness, permanence, delay, switching, Lyapunov–Krasovskii functional.

1 Introduction

Continuous and discrete-time generalized Lotka–Volterra systems are widely used for modeling species interactions in biological communities [1, 2, 3, 4]. An actual problem that arises when analyzing the dynamics of such systems is to find the conditions ensuring sustainable coexistence of species. A sustainable biological system should possess the following two important properties: ultimate boundedness and persistence (see [1]). The first one means the existence of a compact domain in the state space of the system such that each motion enters into it in a finite time and remains in this domain thereafter. The property of persistence means that in the process of evolution species do not die out, and, moreover, no matter how small their initial quantities are, starting from a certain time instant, the quantities of species will exceed some fixed positive values. If a system admits both of these properties, then it is called permanent [1].

Permanence conditions are well investigated for biological models described by ordinary differential or difference equations with constant parameters (see, e.g., [1, 2, 3, 5, 6, 7]) and the references therein). However, it is worth noticing that numerous realistic models should take into account such factors as delay and switching of operation modes [3, 4, 8, 9]. For instance, the presence of a delay may be due to the dependence of the growth rates of some species on the prehistory of others, different rates of reproduction and death processes for different age groups, etc. On the other hand, switching can be a result of the impact of such natural and artificial factors as droughts, rainy seasons, deforestation, fires, radiation, that cause sudden changes in the internal connections in the biological community and the characteristics of the habitat of populations.

The presence of delays and switching significantly complicates the permanence analysis. Some sufficient conditions of permanence for time-delay Lotka–Volterra type systems were obtained in [10, 11, 12, 13], whereas, in [14, 15, 16, 17], permanence of biological models with switching of parameters was studied. However, there are few results on permanence for systems containing both switching and delays (see, e.g., [14, 18, 19, 20]).

In particular, in [20], certain classes of continuous and discrete-time generalized Lotka–Volterra systems with delays and switching were investigated. Using special constructions of Lyapunov–Krasovskii functionals, conditions were derived providing that the considered systems are permanent for any constant delay and any admissible switching law. These conditions were formulated in terms of feasibility of auxiliary linear inequalities.

In the present paper the same classes of systems are studied. New constructions of Lyapunov–Krasovskii functionals will be proposed. Our objective is to prove that, with the aid of these functionals, less conservative permanence conditions can be obtained than those presented in [20].

2 Statement of the problem

In what follows, $\mathbb{R}$ denotes the field of real numbers and $\mathbb{R}^n$ is the vector space of n-tuples of real numbers with the Euclidean norm $\|\cdot\|$. Let $\mathbb{R}^n_+$ be the nonnegative cone of $\mathbb{R}^n$: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\operatorname{int}\mathbb{R}^n_+$ be the interior of $\mathbb{R}^n_+$. A matrix is said to be nonnegative if all its entries are nonnegative. A square matrix is called Metzler if all its off-diagonal entries are nonnegative. For a given number $\tau > 0$, $C([-\tau,0],\operatorname{int}\mathbb{R}^n_+)$ is the space of continuous functions $\varphi(\xi) : [-\tau,0] \mapsto \operatorname{int}\mathbb{R}^n_+$ with the uniform norm $\|\varphi\|_\tau = \max_{\xi\in[-\tau,0]}\|\varphi(\xi)\|$.

Let the switched system

$$\dot{x}_i(t) = x_i(t)\left(\sum_{j=1}^n a_{ij}^{(\sigma)} f_j(x_j(t)) + \sum_{j=1}^n b_{ij}^{(\sigma)} f_j(x_j(t-\tau)) + c_i^{(\sigma)}\right), \quad i = 1,\dots,n, \tag{1}$$

and the associated family of subsystems

$$\dot{x}_i(t) = x_i(t)\left(\sum_{j=1}^n a_{ij}^{(s)} f_j(x_j(t)) + \sum_{j=1}^n b_{ij}^{(s)} f_j(x_j(t-\tau)) + c_i^{(s)}\right), \tag{2}$$

$$i = 1,\dots,n, \quad s = 1,\dots,N,$$

be given. Here $t \geq 0$, $x_i(t) \in \mathbb{R}$, the functions $f_j(x_j)$ are defined for $x_j \in [0,+\infty)$, $a_{ij}^{(s)}$, $b_{ij}^{(s)}$, $c_i^{(s)}$ are constant coefficients, τ is a constant positive delay, $\sigma = \sigma(t)$ is a piece-wise constant function defining the switching law, $\sigma(t) : [0,+\infty) \mapsto \{1,\dots,N\}$. Let the function $\sigma(t)$ admit only finitely many discontinuities on every bounded interval. Such switching laws will be called admissible.

The system (1) is a generalized Lotka–Volterra model with delay and switching of parameters describing the interaction of species in a biological community [1, 2, 3]. In this model, $x_i(t)$ is the population density of the i-th species, the coefficients $c_i^{(\sigma)}$ characterise the intrinsic growth rate of the i-th population, the terms $a_{ii}^{(\sigma)} x_i(t) f_i(x_i(t))$ and $b_{ii}^{(\sigma)} x_i(t) f_i(x_i(t-\tau))$ reflect the self-interaction in the i-th population, whereas the terms $a_{ij}^{(\sigma)} x_i(t) f_j(x_j(t))$ and $b_{ij}^{(\sigma)} x_i(t) f_j(x_j(t-\tau))$ for $j \neq i$ describe the influence of population j on population i.

Taking into account the biological meaning of the model, we will consider the system (1) in $\operatorname{int}\mathbb{R}^n_+$. Let initial functions $\varphi(\xi)$ for solutions $x(t) = x(t, t_0, \varphi)$ of (1) belong to the space $C([-\tau, 0], \operatorname{int}\mathbb{R}^n_+)$ and, for a solution $x(t)$, x_t be the restriction of the solution to the segment $[t - \tau, t]$, i.e., $x_t : \xi \mapsto x(t + \xi)$ for $\xi \in [-\tau, 0]$.

According to the standard assumptions (see [2, 3]), we consider the case where functions $f_i(x_i)$ possess the following properties:

1) $f_i(x_i)$ are continuous and locally Lipschitz for $x_i \geq 0$;

2) $f_i(x_i) \geq 0$ for $x_i \geq 0$, and $f_i(0) = 0$;

3) $f_i(x_i) \to +\infty$ as $x_i \to +\infty$.

Remark 1. From these properties it follows that $\operatorname{int}\mathbb{R}^n_+$ is an invariant set for (1) [2].

Definition 1 [1]. The system (1) is called permanent if there exist numbers δ_1 and δ_2 satisfying the inequalities $0 < \delta_1 < \delta_2$ and, for any solution $x(t, t_0, \varphi) = (x_1(t, t_0, \varphi), \ldots, x_n(t, t_0, \varphi))^\top$ with $t_0 \geq 0$, $\varphi(\xi) \in C([-\tau, 0], \operatorname{int}\mathbb{R}^n_+)$, one can choose $T \geq t_0$ such that $\delta_1 \leq x_i(t, t_0, \varphi) \leq \delta_2$, $i = 1, \ldots, n$, for $t \geq T$.

Our objective is to determine the conditions ensuring permanence of (1) for any constant positive delay and any admissible switching law. In addition, we will investigate the permanence problem for a discrete-time counterpart of the model (1).

3 Permanence conditions for the continuous model

Consider the case where the following restrictions are imposed on the coefficients in the system (1).

Assumption 1. Let $c_i^{(s)} > 0$, $i = 1, \ldots, n$, the matrices $A^{(s)} = \{a_{ij}^{(s)}\}_{i,j=1}^n$ be Metzler and the matrices $B = \{b_{ij}^{(s)}\}_{i,j=1}^n$ be nonnegative, $s = 1, \ldots, N$.

Remark 2. Asumption 1 means that, for each species, birth rate is greater than mortality rate, and between any two species in the community there are interactions of the following types: "symbiosis", "compensalism" or "neutralism" [1].

Under Assumption 1, conditions of permanence for (1) were obtained in [20] on the basis of the Lyapunov direct method. A common Lyapunov–Krasovskii functional for the family (2) was constructed in the form

$$V(x_t) = \sum_{i=1}^{n} \lambda_i \ln x_i(t) + \sum_{i=1}^{n} \mu_i \int_{t-\tau}^{t} f_i(x_i(\xi))d\xi$$

$$+\omega \sum_{i=1}^{n} \int_{t-\tau}^{t} (\xi - t + \tau) f_i(x_i(\xi))d\xi,$$

where λ_i, μ_i, ω are positive coefficients. With the aid of this functional, it was proven that, if there exists a vector $\theta > 0$ such that

$$\left(A^{(s)} + B^{(r)}\right)^{\top} \theta < 0, \quad s, r = 1, \ldots, N, \tag{3}$$

then the system (1) is permanent for any positive delay and any admissible switching law. Here and in what follows vector inequalities are understood componentwise.

In this paper, we will propose another construction of Lyapunov–Krasovskii functional. To do this, we will use the approach that was first suggested in [21] for the stability analysis of delay-free switched linear systems and was later extended in [22, 23] to some classes of linear and nonlinear time-delay switched systems. We will show that, using the new functional, less conservative permanence conditions can be derived than those found in [20].

Theorem 1 *Let Assumption 1 be fulfilled. If there exist numbers ν, h_1, h_2 and vectors $\eta > 0$, $\zeta > 0$ such that $\nu > 0$, $\nu h_1 + h_2 < 0$,*

$$(A^{(s)} + B^{(s)})\eta \le h_1 \eta, \quad s = 1, \ldots, N, \tag{4}$$

$$(A^{(s)} + B^{(r)})^{\top} \zeta \le h_2 \zeta, \quad s, r = 1, \ldots, N, \tag{5}$$

then the system (1) is permanent for any constant positive delay and any admissible switching law.

Proof. Choose a Lyapunov–Krasovskii functional candidate in the form

$$\widetilde{V}(x_t) = \sum_{i=1}^{n} \frac{\zeta_i}{\eta_i^{\nu}} \int_{1}^{x_i(t)} \frac{f_i^{\nu}(u)}{u} du + \sum_{i=1}^{n} \rho_i \int_{t-\tau}^{t} f_i^{\nu+1}(x_i(\xi))d\xi$$

$$+\omega \sum_{i=1}^{n} \int_{t-\tau}^{t} (\xi - t + \tau) f_i^{\nu+1}(x_i(\xi))d\xi, \tag{6}$$

where ω, ρ_i are positive coefficients, η_i, ζ_i are components of the vectors η, ζ, respectively.

Differentiating the functional (6) along the solutions of (1), we obtain

$$\dot{\tilde{V}} = \sum_{i=1}^{n} \frac{\zeta_i}{\eta_i^{\nu}} f_i^{\nu}(x_i(t)) \left(\sum_{j=1}^{n} a_{ij}^{(\sigma)} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}^{(\sigma)} f_j(x_j(t-\tau)) + c_i^{(\sigma)} \right)$$

$$+ \sum_{i=1}^{n} \rho_i \left(f_i^{\nu+1}(x_i(t)) - f_i^{\nu+1}(x_i(t-\tau)) \right)$$

$$+ \omega\tau \sum_{i=1}^{n} f_i^{\nu+1}(x_i(t)) - \omega \sum_{i=1}^{n} \int_{t-\tau}^{t} f_i^{\nu+1}(x_i(\xi)) d\xi.$$

Let $z_i(t) = f_i(x_i(t))/\eta_i$, $i = 1, \ldots, n$. Then

$$\dot{\tilde{V}} = \sum_{i=1}^{n} \zeta_i z_i^{\nu}(t) \left(\sum_{j=1}^{n} a_{ij}^{(\sigma)} \eta_j z_j(t) + \sum_{j=1}^{n} b_{ij}^{(\sigma)} \eta_j z_j(t-\tau) + c_i^{(\sigma)} \right)$$

$$+ \sum_{i=1}^{n} \rho_i \eta_i^{\nu+1} \left(z_i^{\nu+1}(t) - z_i^{\nu+1}(t-\tau) \right)$$

$$+ \omega\tau \sum_{i=1}^{n} \eta_i^{\nu+1} z_i^{\nu+1}(t) - \omega \sum_{i=1}^{n} \eta_i^{\nu+1} \int_{t-\tau}^{t} z_i^{\nu+1}(\xi) d\xi.$$

Applying the Young inequality, we arrive at the estimates

$$\dot{\tilde{V}} \leq \frac{\nu}{\nu+1} \sum_{i=1}^{n} \zeta_i z_i^{\nu+1}(t) \sum_{j=1}^{n} \left(a_{ij}^{(\sigma)} + b_{ij}^{(\sigma)} \right) \eta_j$$

$$+ \sum_{i=1}^{n} \eta_i z_i^{\nu+1}(t) \left(\rho_i \eta_i^{\nu} + \omega\tau \eta_i^{\nu} + \frac{1}{\nu+1} \sum_{j=1}^{n} a_{ji}^{(\sigma)} \zeta_j \right)$$

$$+ \sum_{i=1}^{n} \eta_i z_i^{\nu+1}(t-\tau) \left(\frac{1}{\nu+1} \sum_{j=1}^{n} b_{ji}^{(\sigma)} \zeta_j - \rho_i \eta_i^{\nu} \right)$$

$$- \omega \sum_{i=1}^{n} \eta_i^{\nu+1} \int_{t-\tau}^{t} z_i^{\nu+1}(\xi) d\xi + D \sum_{i=1}^{n} z_i^{\nu}(t)$$

$$\leq \frac{\nu}{\nu+1} h_1 \sum_{i=1}^{n} \eta_i \zeta_i z_i^{\nu+1}(t) - \omega \sum_{i=1}^{n} \eta_i^{\nu+1} \int_{t-\tau}^{t} z_i^{\nu+1}(\xi) d\xi$$

$$+\sum_{i=1}^{n}\eta_i z_i^{\nu+1}(t)\left(\rho_i\eta_i^{\nu}+\omega\tau\eta_i^{\nu}+\frac{1}{\nu+1}\sum_{j=1}^{n}a_{ji}^{(\sigma)}\zeta_j\right)$$

$$+\sum_{i=1}^{n}\eta_i z_i^{\nu+1}(t-\tau)\left(\frac{1}{\nu+1}\sum_{j=1}^{n}b_{ji}^{(\sigma)}\zeta_j-\rho_i\eta_i^{\nu}\right)$$

$$+\frac{nD^{\nu+1}}{(\nu+1)\varepsilon^{\nu+1}}+\frac{\nu}{\nu+1}\varepsilon^{\frac{\nu+1}{\nu}}\sum_{i=1}^{n}z_i^{\nu+1}(t),$$

where D is a positive constant and ε is a positive tuning parameter.

Let

$$\rho_i=\frac{1}{(\nu+1)\eta_i^{\nu}}\max_{s=1,\dots,N}\sum_{j=1}^{n}b_{ji}^{(s)}\zeta_j+\delta,\quad i=1,\dots,n. \tag{7}$$

Here $\delta=\text{const}>0$. Then

$$\dot{\tilde{V}}\le\frac{\nu}{\nu+1}h_1\sum_{i=1}^{n}\eta_i\zeta_i z_i^{\nu+1}(t)-\omega\sum_{i=1}^{n}\eta_i^{\nu+1}\int_{t-\tau}^{t}z_i^{\nu+1}(\xi)d\xi$$

$$+\frac{1}{\nu+1}\sum_{i=1}^{n}\eta_i z_i^{\nu+1}(t)\left(\sum_{j=1}^{n}a_{ji}^{(\sigma)}\zeta_j+\max_{s=1,\dots,N}\sum_{j=1}^{n}b_{ji}^{(s)}\zeta_j\right)$$

$$+\sum_{i=1}^{n}\eta_i^{\nu+1}z_i^{\nu+1}(t)\left(\delta+\omega\tau\right)-\delta\sum_{i=1}^{n}\eta_i^{\nu+1}z_i^{\nu+1}(t-\tau)$$

$$+\frac{nD^{\nu+1}}{(\nu+1)\varepsilon^{\nu+1}}+\frac{\nu}{\nu+1}\varepsilon^{\frac{\nu+1}{\nu}}\sum_{i=1}^{n}z_i^{\nu+1}(t)$$

$$\le\frac{1}{\nu+1}(\nu h_1+h_2)\sum_{i=1}^{n}\eta_i\zeta_i z_i^{\nu+1}(t)-\omega\sum_{i=1}^{n}\eta_i^{\nu+1}\int_{t-\tau}^{t}z_i^{\nu+1}(\xi)d\xi$$

$$+\sum_{i=1}^{n}\eta_i^{\nu+1}z_i^{\nu+1}(t)\left(\delta+\omega\tau\right)-\delta\sum_{i=1}^{n}\eta_i^{\nu+1}z_i^{\nu+1}(t-\tau)$$

$$+\frac{nD^{\nu+1}}{(\nu+1)\varepsilon^{\nu+1}}+\frac{\nu}{\nu+1}\varepsilon^{\frac{\nu+1}{\nu}}\sum_{i=1}^{n}z_i^{\nu+1}(t).$$

Hence, if values of $\omega,\varepsilon,\delta$ are sufficiently small, then

$$\dot{\tilde{V}}\le-\alpha_1\sum_{i=1}^{n}z_i^{\nu+1}(t)-\omega\sum_{i=1}^{n}\eta_i^{\nu+1}\int_{t-\tau}^{t}z_i^{\nu+1}(\xi)d\xi-\delta\sum_{i=1}^{n}\eta_i^{\nu+1}z_i^{\nu+1}(t-\tau)+\tilde{D}$$

$$\leq -\alpha_2 \sum_{i=1}^{n} \left(f_i^{\nu+1}(x_i(t)) + \int_{t-\tau}^{t} f_i^{\nu+1}(x_i(\xi))d\xi \right) + \tilde{D},$$

where α_1, α_2 are positive constants and $\tilde{D} = nD^{\nu+1}/((\nu+1)\varepsilon^{\nu+1})$.

Denote

$$Q = \left\{ x_t \in C([-\tau, 0], \operatorname{int} \mathbb{R}_+^n) : \sum_{i=1}^{n} \left(f_i^{\nu+1}(x_i(t)) + \int_{t-\tau}^{t} f_i^{\nu+1}(x_i(\xi))d\xi \right) > \frac{\tilde{D}}{\alpha_2} \right\}.$$

If $x_t \in Q$, then $\dot{\tilde{V}} < 0$.

From Assumption 1 it follows that

$$\dot{x}_i(t) \geq x_i(t) \left(a_{ii}^{(\sigma)} f_i(x_i(t)) + c_i^{(\sigma)} \right), \quad i = 1, \ldots, n.$$

Therefore, one can choose a number $\delta_1 > 0$ such that $\dot{x}_i(t) > 0$ for $x_i(t) \in (0, \delta_1)$, $i = 1, \ldots, n$. Let

$$G = \left\{ x_t \in C([-\tau, 0], \operatorname{int} \mathbb{R}_+^n) : \sum_{i=1}^{n} \left(f_i^{\nu+1}(x_i(t)) + \int_{t-\tau}^{t} f_i^{\nu+1}(x_i(\xi))d\xi \right) \leq \frac{2\tilde{D}}{\alpha_2}, \right.$$

$$\left. x_i(t) \geq \delta_1, \ i = 1, \ldots, n \right\}, \qquad W = \sup_{x_t \in G} \widetilde{V}(x_t),$$

$$\Omega = \left\{ x_t \in C([-\tau, 0], \operatorname{int} \mathbb{R}_+^n) : \widetilde{V}(x_t) \leq W, \ x_i(t) \geq \delta_1, \ i = 1, \ldots, n \right\}.$$

We obtain $0 < W < +\infty$, $G \subset \Omega$. Moreover, there exists $\delta_2 > 0$ such that $x_i(t) \leq \delta_2$, $i = 1, \ldots, n$, for $x_t \in \Omega$.

Consider a solution $\bar{x}(t) = x(t, t_0, \varphi)$ of (1) with $t_0 \geq 0$, $\varphi(\xi) \in C([-\tau, 0], \operatorname{int} \mathbb{R}_+^n)$. First, one can find $T_1 \geq t_0$ such that $\bar{x}_i(t) \geq \delta_1$, $i = 1, \ldots, n$, for $t \geq T_1$. Next, there exists a time instant T_2 such that $T_2 \geq T_1$ and $\bar{x}_{T_2} \in G$. Hence, $\bar{x}_t \in \Omega$ and $\bar{x}_i(t) \leq \delta_2$, $i = 1, \ldots, n$, for $t \geq T_2$. This completes the proof.

Remark 3. It is worth noticing that, under Assumption 1, the conditions of Theorem 1 are valid if and only if one of the following conditions is satisfied:

(a) there exists a vector $\eta > 0$ such that

$$(A^{(s)} + B^{(s)})\eta < 0, \quad s = 1, \ldots, N,$$

(b) there exists a vector $\zeta > 0$ such that

$$(A^{(s)} + B^{(r)})^\top \zeta < 0, \quad s, r = 1, \ldots, N.$$

Really, in the case (a) the conditions of Theorem 1 are fulfilled for sufficiently large values of ν, whereas in the case (b), the conditions of Theorem 1 are fulfilled for sufficiently small values of ν. However, in some problems, e.g., estimation of ultimate boundedness domain, estimation of convergence rate of solutions, analysis of the impact of external disturbances, etc., to derive less conservative results, it is useful to have opportunity for an appropriate choice of the parameter ν (see, for instance, [24]). The presented statement of Theorem 1 permits us to determine admissible domain for this parameter.

4 Permanence conditions for a discrete-time model

In this section, we consider the following discrete-time counterpart of the model (1):

$$y_i(k+1) = y_i(k)\exp\left(h\left(\sum_{j=1}^{n} a_{ij}^{(\sigma)} f_j(y_j(k)) + \sum_{j=1}^{n} b_{ij}^{(\sigma)} f_j(y_j(k-m)) + c_i^{(\sigma)}\right)\right), \tag{8}$$

$$i = 1, \dots, n.$$

Here $y_i(k) \in \mathbb{R}$ is the density of population i at the kth generation; functions $f_i(x_i)$ possess the same properties as for (1), $\sigma = \sigma(k)$, $k = 0, 1, \dots$, with $\sigma(k) \in \{1, \dots, N\}$ defines a switching law, $c_i^{(s)}, a_{ij}^{(s)}, b_{ij}^{(s)}$, $s = 1, \dots, N$, $i, j = 1, \dots, n$, are constant coefficients, m is a positive integer delay, h is a positive parameter.

It is worth noticing that, in numerous cases, discrete-time models are more appropriate than continuous ones [3, 5, 7]. In addition, the system (8) can be used for the numerical simulation of the model (1).

Let $y^{(k)} = (y^\top(k), y^\top(k-1), \dots, y^\top(k-m))^\top$ be the augmented state vector and $y(k, k_0, y^{(k_0)})$ be a solution of (8) with initial conditions $k_0 \geq 0$, $y^{(k_0)} > 0$.

Definition 2 [5, 7]. The system (8) is called permanent if there exist numbers δ_1 and δ_2 satisfying the inequalities $0 < \delta_1 < \delta_2$ and, for any solution $y(k, k_0, y^{(k_0)}) = (y_1(k, k_0, y^{(k_0)}), \dots, y_n(k, k_0, y^{(k_0)}))^\top$ with $k_0 \geq 0$, $y^{(k_0)} > 0$, one can choose $K \geq k_0$ such that $\delta_1 \leq y_i(k, k_0, y^{(k_0)}) \leq \delta_2$, $i = 1, \dots, n$, for $k \geq K$.

Compared with the continuous case, we will impose an additional constraint on the functions $f_i(x_i)$.

Assumption 2. The functions $g_i(z_i) = f_i(\exp(z_i))$ are globally Lipschitz for $z_i \in (-\infty, +\infty)$ with a constant $L > 0$, $i = 1, \dots, n$.

In [20], it has been proven that if Assumptions 1, 2 are satisfied, the system (3) admits a positive solution and the value of h is sufficiently small, then (8) is permanent for any positive delay and any switching law. Our objective is to show that, with the aid of a discrete-time counterpart of the functional (6), the permanence can be guaranteed under less conservative conditions imposed on the coefficients $a_{ij}^{(s)}$ and $b_{ij}^{(s)}$.

Theorem 2 *Let the conditions of Theorem 1 and Assumption 2 be fulfilled. Then there exists $h_0 > 0$ such that the system (8) is permanent for any $h \in (0, h_0)$, any constant positive delay and any switching law.*

Proof. Construct a Lyapunov–Krasovskii functional candidate as follows:

$$\widetilde{V}(y^{(k)}) = \sum_{i=1}^{n} \frac{\zeta_i}{\eta_i^{\nu}} \int_1^{y_i(k)} \frac{f_i^{\nu}(u)}{u} du + h \sum_{i=1}^{n} \sum_{l=1}^{m} \rho_i f_i^{\nu+1}(y_i(k-l))$$

$$+h \sum_{i=1}^{n} \sum_{l=1}^{m} \omega_l f_i^{\nu+1}(y_i(k-l)), \tag{9}$$

where ω_l, ρ_i are positive coefficients, η_i, ζ_i are components of the vectors η, ζ satisfying the inequalities (4), (5), respectively.

The difference of the functional (9) along the solutions of the system (8) takes the form

$$\Delta \widetilde{V} = \widetilde{V}(y^{(k+1)}) - \widetilde{V}(y^{(k)}) = \sum_{i=1}^{n} \frac{\zeta_i}{\eta_i^{\nu}} \int_{y_i(k)}^{y_i(k+1)} \frac{f_i^{\nu}(u)}{u} du$$

$$+h \sum_{i=1}^{n} \rho_i \left(f_i^{\nu+1}(y_i(k)) - f_i^{\nu+1}(y_i(k-m)) \right) + h\omega_1 \Psi(k) - h\omega_m \Psi(k-m)$$

$$+h \sum_{l=1}^{m-1} (\omega_{l+1} - \omega_l) \Psi(k-l),$$

where $\Psi(k) = \sum_{i=1}^{n} f_i^{\nu+1}(y_i(k))$.

With the aid of the mean value theorem and the Lipschitz condition, we obtain

$$\int_{y_i(k)}^{y_i(k+1)} \frac{f_i^{\nu}(u)}{u} du = \int_{\ln y_i(k)}^{\ln y_i(k+1)} f_i^{\nu}(e^{\beta}) d\beta = f_i^{\nu}(y_i(k)) \ln \frac{y_i(k+1)}{y_i(k)}$$

$$
+\left(g_i^{\nu}\left(\ln y_i(k)+\theta_{ik}\ln\frac{y_i(k+1)}{y_i(k)}\right)-g_i^{\nu}(\ln y_i(k))\right)\ln\frac{y_i(k+1)}{y_i(k)}
$$

$$
=f_i^{\nu}(y_i(k))\ln\frac{y_i(k+1)}{y_i(k)}+\nu\Bigg(f_i(y_i(k))
$$

$$
+\bar{\theta}_{ik}\left(g_i\left(\ln y_i(k)+\theta_{ik}\ln\frac{y_i(k+1)}{y_i(k)}\right)-g_i(\ln y_i(k))\right)\Bigg)^{\nu-1}\Bigg(g_i\Bigg(\ln y_i(k)
$$

$$
+\theta_{ik}\ln\frac{y_i(k+1)}{y_i(k)}\Bigg)-g_i(\ln y_i(k))\Bigg)\ln\frac{y_i(k+1)}{y_i(k)}
$$

$$
\le hf_i^{\nu}(y_i(k))\left(\sum_{j=1}^{n}a_{ij}^{(\sigma)}f_j(y_j(k))+\sum_{j=1}^{n}b_{ij}^{(\sigma)}f_j(y_j(k-m))+c_i^{(\sigma)}\right)
$$

$$
+\nu L\left(f_i(y_i(k))+L\left|\ln\frac{y_i(k+1)}{y_i(k)}\right|\right)^{\nu-1}\left(\ln\frac{y_i(k+1)}{y_i(k)}\right)^2
$$

$$
\le hf_i^{\nu}(y_i(k))\left(\sum_{j=1}^{n}a_{ij}^{(\sigma)}f_j(y_j(k))+\sum_{j=1}^{n}b_{ij}^{(\sigma)}f_j(y_j(k-m))+c_i^{(\sigma)}\right)
$$

$$
+h^2(1+h^{\nu-1})\tilde{L}_i\left(\sum_{j=1}^{n}f_j^{\nu+1}(y_j(k))+\sum_{j=1}^{n}f_j^{\nu+1}(y_j(k-m))+\tilde{c}\right),
$$

where $\tilde{c}>0$, $\theta_{ik},\bar{\theta}_{ik}\in(0,1)$, $\tilde{L}_i>0$, $i=1,\dots,n$.

Hence,

$$
\Delta\widetilde{V}\le h\sum_{i=1}^{n}\frac{\zeta_i}{\eta_i^{\nu}}f_i^{\nu}(y_i(k))\left(\sum_{j=1}^{n}a_{ij}^{(\sigma)}f_j(y_j(k))+\sum_{j=1}^{n}b_{ij}^{(\sigma)}f_j(y_j(k-m))+c_i^{(\sigma)}\right)
$$

$$
+h\sum_{i=1}^{n}\rho_i\left(f_i^{\nu+1}(y_i(k))-f_i^{\nu+1}(y_i(k-m))\right)+h\omega_1\Psi(k)-h\omega_m\Psi(k-m)
$$

$$
+h\sum_{l=1}^{m-1}(\omega_{l+1}-\omega_l)\Psi(k-l)+h^2(1+h^{\nu-1})\tilde{L}\left(\Psi(k)+\Psi(k-m)+\tilde{c}\right),
$$

where $\tilde{L}=\mathrm{const}>0$.

Let the coefficients ρ_i be defined by the formula (7). In addition, assume that $\omega_{l+1}<\omega_l$, $l=1,\dots,m-1$. Then (see the proof of Theorem 1), under sufficiently small values of δ, ω_1, h, the estimate

$$\Delta\widetilde{V} \leq -\alpha_1 \sum_{l=0}^{m} \Psi(k-l) + \alpha_2$$

holds, where α_1, α_2 are positive constants. The remaining part of the proof is a similar to that of Theorem 1.

5 Discussion

In this paper, using a special construction of Lyapunov–Krasovskii functional and its discrete-time counterpart, new permanence conditions for some continuous and discrete-time biological models with delays and switching are obtained. Let us show that these conditions are less conservative than those found in [20].

Really, if the system (3) admits a positive solution, then there exist vectors $\eta > 0$, $\zeta > 0$ and numbers h_1, h_2 such that $h_2 < 0$ and inequalities (4), (5) are satisfied. Thus, the condition $\nu h_1 + h_2 < 0$ is fulfilled for sufficiently small values of ν.

On the other hand, consider the case where $n = N = 2$,

$$A^{(1)} = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -5 & 0 \\ 0 & -10 \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}.$$

We obtain

$$A^{(1)} + B^{(2)} = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Hence, the system (3) does not admit positive solutions.

It can be easily verified that the inequalities $h_1 \geq \sqrt{5} - 3$, $h_2 \geq 2\sqrt{2}$ are necessary and sufficient conditions for the existence of positive vectors η and ζ satisfying the systems (4) and (5), respectively. Thus, if $h_1 = \sqrt{5} - 3$, $h_2 = 2\sqrt{2}$ and $\nu > 2\sqrt{2}//(3 - \sqrt{5})$, then $\nu h_1 + h_2 < 0$.

As a result, we obtain that theorems from the paper [20] are not applicable to systems of the forms (1) and (8) with the considered matrices, whereas Theorems 1 and 2 from the present work guarantee that these systems are permanent.

An interesting direction for further research is an extension of the obtained results to switched generalized Lotka–Volterra models with variable delays.

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