DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 3, 2024
Electronic Journal,
reg. № ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru

Discrete Dynamic Systems

Calculation of the Invariant Decomposition of the Tangent Space

Osipenko G.S.[1,*], Andreeva I.A.[2,**]

[1]Sevastopol Branch of Lomonosov Moscow State University, Crimea, Russia
[2]Peter the Great St.Petersburg Polytechnic University, St. Petersburg, Russia

[*]george.osipenko@mail.ru
[**]irandr@inbox.ru

Abstract. A discrete dynamic system generated by a diffeomorphism is considered. The task of constructing an invariant decomposition of the tangent space is set. This task is reduced to calculating the components of a chain recurrent set on a projective bundle. A computer-oriented tool for such a calculation is a symbolic image of a dynamic system, which is an oriented graph. The components of the chain recurrent set are determined by calculating the components of the strong connectivity of the symbolic image. A nontrivial example of calculating the desired tangent space decomposition is given.

Keywords: discrete dynamic system, tangent space, invariant decomposition, chain recurrent trajectories, limit sets, projective space, symbolic image, localization algorithm.

1. Introduction

Consider a discrete dynamic system

$$x_{n+1} = f(x_n) \tag{1}$$

generated by a diffeomorphism $f: M \to M$ of a compact manifold. The differential Df is a linear map, which in local coordinates is a partial differential matrix or the Jacobi matrix

$$Df(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right).$$

The differential acts from the tangent space at a point x to the tangent space at a point $f(x)$

$$Df(x): TM_x \to TM_{f(x)}.$$

The differential defines the linear expansion of a discrete system

$$x_{n+1} = f(x_n)$$
$$v_{n+1} = Df(x_n)v_n,$$

where v_n is a vector in the tangent space TM_x. Often, in practical research, the problem arises of the existence of the decomposition of the tangent space, invariant for the differential.
Example 1. If x is a fixed point, i.e. $f(x) = x$, then the following decomposition takes place: $TM_x = E_1 \oplus \ldots \oplus E_l$, where each E_k is invariant for the matrix

$$Df(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right).$$

Example 2. Hyperbolicity of an invariant set $\Lambda \subset M$ is generated by the existence of invariant sub bundles E^s и E^u of the tangent space $TM|_\Lambda$, such that

$$TM|_\Lambda = E^s \oplus E^u,$$

where the invariance of the sub bundle E^* means equality

$$Df(x)E^*(x) = E^*(f(x)).$$

In this case, the differential exponentially compresses under positive iterations on E^s and under negative iterations on E^u.
Example 3. It is known [2, 14], that if the tangent bundle splits into two sub bundles E_1 and E_2, which are exponentially separated by the system (1), then the following invariant decomposition takes place:

$$TM = E_1 \oplus E_2.$$

The given examples show, that the problem of the existence of an invariant decomposition of the tangent space is quite actual.
Problem statement: to find computer-oriented methods for constructing of an invariant decomposition of a tangent space into a direct sum

$$TM = E_1 \oplus \ldots \oplus E_l.$$

1.1. A projective space

Any non-degenerate linear mapping $A: R^d \to R^d$ can be represented as:

$$Av = |Av|\,\frac{Av}{|Av|} = r|Ae|\,\frac{Ae}{|Ae|} = r|Ae|\,A_s(e),$$

where $r = |v|$, $|v| \neq 0$, $e = {}^{v}/_{|v|} \in S^{d-1}$, $S^{d-1} -$ a unit sphere. Thus, the linear mapping $A: v \to Av$ is a composition of two mappings

$$e \to A_s(e) = \frac{Ae}{|Ae|} \quad and \quad r \to r|Ae|,$$

where the first mapping acts on the sphere S^{d-1}, while the second one acts on a positive half-line R^+. The projective space P^{d-1} can be obtained by identifying diametrically opposite points of the sphere S^{d-1}. Symmetry of the display $A_s(e)$ regarding the sign change $A_s(\pm e) = \pm A_s(e)$ allows us to define the mapping $PA(e) = Ae/|Ae|$ on a projective space P^{d-1}, identifying diametrically opposite points on the sphere. For points of the space P^{d-1} we will keep the designation e, i. e. e denotes a one-dimensional subspace and (or) a unit vector on it, which does not lead to misunderstandings.

Thus, we obtain the mapping

$$Pf(x,e) = \left(f(x), \frac{Df(x)e}{|Df(x)e|}\right)$$

on the projective bundle $P = \{(x,e): x \in M, e \in P^{d-1}(x)\}$, which generates a discrete dynamic system of the form

$$x_{n+1} = f(x_n), \tag{2}$$

$$e_{n+1} = \frac{Df(x_n)e_n}{|Df(x_n)e_n|} \tag{3}$$

on the projective bundle P. A positive number $a(x,e) = |Df(x)e|$ is a coefficient of length change in the subspace e under the action of a differential at the point x. The projective bundle P is a compact manifold with a metric [14, pp. 531-539], which can be interpreted as the sum of the distance on the manifold M and the angle between the subspaces.

1.2. Chain recurrent trajectories

A sequence $\{x_n\}$ appears to be an ε-trajectory, if

$$x_{n+1} = f(x_n) + \varepsilon_n, \qquad |\varepsilon_n| < \varepsilon.$$

Let an ε-trajectory $\{x_n\}$ be a periodic one, i. e. there is $k > 0$, such as $x_{n+k} = x_n$ for every n. Then points x_n are called the ε-periodic ones.

Definition 1. *A point x is called the chain recurrent point, if it is ε-periodic for every $\varepsilon > 0$; that mean, a periodic ε-trajectory exists passing through the point x.*

A chain recurrent set consists of all chain recurrent points and is denoted by CR. The chain recurrent set CR is invariant, closed and contains all types of return trajectories: periodic, almost-periodic, non-wandering, homoclinic, etc. Note, that if the chain recurrent point is not periodic and $dim\, M > 1$, then there is an arbitrarily small perturbation f under C^0-topology, for which this point is the periodic one [27]. We can say, that chain recurrent points generate periodic trajectories under C^0- perturbations. Therefore, in computer calculations, chain recurrent points will look like periodic ones.

Definition 2. *Two chain recurrent points we call equivalent, if they can be connected by a periodic ε-trajectory for any $\varepsilon > 0$. A chain recurrent set is divided into equivalence classes $\{\Omega_i\}$, which we call the components of a chain recurrent set.*

Let us recall, that the α-limit set of the trajectory is the limit set of points of the trajectory under the condition that $n \to -\infty$, while the ω-limit set of the trajectory is the limit set of trajectory points at $n \to +\infty$. If the trajectory is not a chain recurrent one, then its α-limit set lies in some component Ω_1 of a chain recurrent set, and its ω-limit set lies in another component Ω_2. We can say that the trajectory starts at Ω_1 and ends in Ω_2. If we fix the neighborhood U of a chain recurrent set, then there exists a number $T>0$, such that any trajectory remains outside U no longer than during a time period T.

The theoretical basis of the proposed computer-oriented method gives the following theorem.

Theorem 1 [2, p. 117; 14, p. 152-153; 26]. *Let $x_{n+1} = f(x_n)$ – be a smooth system on a compact manifold M, Ω be a component of a chain recurrent set of this system, and $\pi : TM \to M$ be a projection from a tangent space onto the manifold.*

Then

1) the chain recurrent set of the reduction of the projective mapping

$$Pf(x,e) = \left(f(x), \frac{Df(x)e}{|Df(x)e|}\right)$$

on $P|_\Omega$ has l of components $\Omega_1, \dots, \Omega_l$, $1 \le l \le \dim E$,

2) every set Ω_k defines the (continuous, having constant dimension) subspace $E_k \subset TM$ over Ω having the form

$$E_k = \left\{v \in \pi^{-1}(\Omega): v \neq 0 \;\Rightarrow \frac{v}{|v|} = e \in \Omega_k\right\},$$

3) the decomposition takes place

$$TM|_\Omega = E_1 \oplus \dots \oplus E_l,$$

which is invariant for the differential Df,

4) each chain recurrent component Ω^* on the bundle $P|_\Omega$ has the properties described in the point 2.

The decomposition described in this theorem is called *the fine Morse decomposition* of the tangent bundle. Thus, in order to find an invariant decomposition of the tangent space $TM|_\Omega = E_1 \oplus \dots \oplus E_l$, it is enough to find all the components of a chain recurrent set of the mapping

$$Pf(x,e) = \left(f(x), \frac{Df(x)e}{|Df(x)e|}\right)$$

on the projective bundle.

2. Computational methods

2.1. Maps of a projective space

A projective space P^{d-1} is a set of straight lines passing through the origin of the coordinates in R^d. It can be obtained by identifying of diametrically opposite points of the unit sphere S^{d-1}. The following coordinates can be entered in the projective space. Let $(x_1, x_2, \dots, x_d)$ be the coordinates in R^d. A straight line passing through the origin and a non-zero vector $v = (x_1, x_2, \dots, x_d)$ look like

$$L = \{t(x_1, x_2, \dots, x_d), \qquad t \in R\}.$$

Let us find the coordinate x_k, which implements $\max\{|x_1|, |x_2|, \dots, |x_d|\}$ for the vector $v \neq 0$. Then $x_k \neq 0$, and the point $t(x_1, x_2, \dots, x_d)$ lies on a straight line L when $t = \frac{1}{x_k}$. This point has coordinates $(\alpha_1, \dots, \alpha_k, \dots, \alpha_d)$, where $\alpha_i = \frac{x_i}{x_k}$, $\alpha_k = 1$, and $|\alpha_i| \leq 1$ for $\neq k$. A set of numbers $(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k-1}, \dots, \alpha_d)$ defines the coordinates of a one-dimensional subspace $L \subset R^d$ in the projective space. At the same time, the vector $v = (\alpha_1, \dots, \alpha_{k-1}, 1, \alpha_{k-1}, \dots, \alpha_d)$ is situated in the subspace $L \subset R^d$, and coordinates $(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k-1}, \dots, \alpha_d)$ belong to the *(d-1)*-dimensional cube $K_k = [-1,1]^{d-1}$. A set of coordinates $(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k-1}, \dots, \alpha_d)$ we will call a map with the number *k* of the projective space P^{d-1}. If we iterate through all the values for *k=1, …, d*, we obtain *d* maps, which cover the P^{d-1}. In order to obtain the projective space, it is necessary to identify (glue) the boundary points of the described cubes. For example, a one-dimensional subspace $L = \{t(1, \dots, 1),\ t \in R\}$ has coordinates $(1, \dots, 1)$ in all maps, therefore, all maps must be glued together at these points.
A non-degenerative mapping (a matrix) $A: R^d \to R^d$ displays a one-dimensional subspace into a one-dimensional subspace and, therefore, defines a mapping in the projective space P^{d-1}. Let's consider how this mapping is defined on the maps of projective space described above. Let the vector $v = (x_1, x_2, \dots, x_d)$, $Av = (X_1, X_2, \dots, X_d)$, and subspace $L = \{tv,\ t \in R\}$ is defined by coordinates $(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k-1}, \dots, \alpha_d)$ on the map number *k*. Under the action of a non-degenerative mapping A, the subspace L is being converted into a one-dimensional subspace $AL = \{t(X_1, X_2, \dots, X_d),\ t \in R\}$. To determine the number of the subspace map for AL we find the coordinate X_m which implements $\max\{|X_1|, |X_2|, \dots, |X_d|\} \neq 0$. The number m defines the map, where the coordinates of the subspace AL look like $(\beta_1, \dots, \beta_{m-1}, \beta_{m+1}, \dots, \beta_d)$, where $\beta_t = X_t / X_m$. Thus, the subspace L has coordinates $(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k-1}, \dots, \alpha_d)$ on the map number *k*, while its image AL has coordinates $(\beta_1, \dots, \beta_{m-1}, \beta_{m+1}, \dots, \beta_d)$ on the map number m.

2.2. A symbolic image of a dynamic system [4, 5, 6]

Let $f: M \to M$ be a homeomorphism of a compact manifold *M*, which generates a discrete dynamic system

$$x_{n+1} = f(x_n),$$

and $\rho\,(x, y) -$ a distance at *M*. A tool for localization of a chain recurrent set is the symbolic image of a dynamic system. The concept of a symbolic image of a dynamic system [20] combines symbolic dynamics [1, 18] and numerical methods [16].

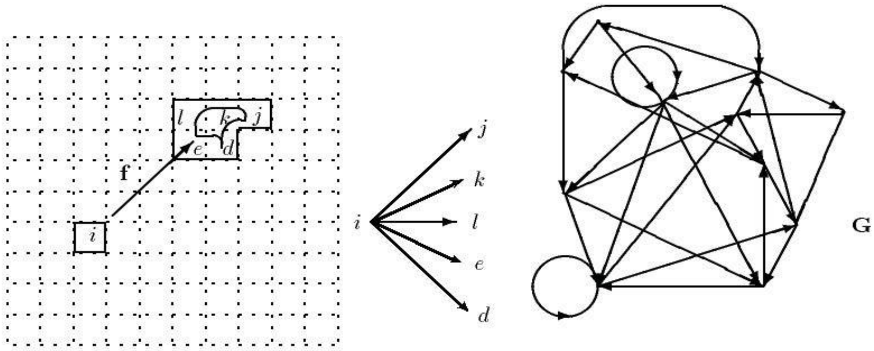

Fig. 1. Construction of a symbolic image

Let $C = \{M(1), \ldots, M(n)\}$ be a finite covering of the manifold M with closed subsets; the set $M(i)$ we will call the cell with the index i, see fig. 1. A symbolic image of a dynamic system for the covering C is an oriented graph G with vertices $\{i\}$, corresponding to cells $\{M(i)\}$. Vertices i and j connected by an oriented edge (arc) $i \to j$ if and only if

$$f\big(M(i)\big) \bigcap M(j) \neq \emptyset.$$

A symbolic image is a geometric mean for discretizing a dynamic system. The study of the symbolic image allows us to obtain useful information about the global structure of the dynamics of the system. The symbolic image depends on the covering C, the change of which changes the symbolic image.

The presence of an edge $i \to j$ guarantees the existence of such a point x inside the cell $M(i)$, whose image $f(x)$ belongs to the cell $M(j)$. In other words, the edge $i \to j$ is a trace of the display $x \to f(x)$, where $x \in M(i)$, $f(x) \in M(j)$. If an edge $i \to j$ does not exist, there are no points $x \in M(i)$, whose images $f(x) \in M(j)$. We do not impose restrictions on the covering C, but without limiting generality, we can assume that the cells $M(i)$ are polyhedral that intersect along the boundary disks. In practical calculations M is a compact area in $\mathbb{R}^d$, and the cells $M(i)$ are cubes or parallelepipeds.

E. Petrenko in his dissertation [10] considered five ways to numerically construct a symbolic image. First of all, it should be noted that the numerical construction of a symbolic image is an approximate calculation, since the image of the cell is calculated approximately. The following simple method of numerically constructing a symbolic image gives good results. Let the covering C consist of polyhedra. A certain number of points $\{x_k,\ k = 1, 2, \cdots, N\}$, which are called the *scan-points,* are evenly distributed in each cell $M(i)$. The number of scan-points N sets an accuracy of calculating of the cell image. Usually, for smooth systems, the number of scan-points $10 \leq N \leq 100$ is quite sufficient. Images $f(x_k)$ are calculated. Further, the arc $i \to j$ is considered to exist, if there is a scan-point $x_k \in M(i)$, for which $f(x_k) \in M(j)$. Therefore, each arc $i \to j$ of the symbolic image is being implemented on the manifold by a pair of points $x_i \to f(x_i)$, where $x_i \in M(i)$, $f(x_i) \in M(j)$.

Fixing for each arc $i \to j$ the described pair $x_i \to f(x_i)$, we will get a projection of a symbolic image on a manifold. At the same time, if there is a valid path on the symbolic image $\omega = (i_m,\ m \in Z)$, then each arc $i_m \to i_{m+1}$ defines the scan-point $x_m \in M(i_m)$, $f(x_m) \in M(i_{m+1})$, and the distance $\rho(f(x_m), x_{m+1})$ does not exceed the diameter of the cell $M(i_{m+1})$. Thus, the sequence of constructed points $\{x_m,\ m \in Z\}$ is an ε-trajectory, where ε is a maximum cell diameter for the covering C. In this case, we will say that the pseudo-trajectory $\{x_m,\ m \in Z\}$ is a trace of the path ω.

There is a natural multi-valued mapping $h: M \to Ver$ from the set M onto the set of vertices of the symbolic image, which is matching to a point x a set of vertices i, such as $x \in M(i)$:

$$h(x) = \{i: x \in M(i)\}.$$

It follows from the definition of a symbolic image, that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow h & & \downarrow h \\ Ver & \xrightarrow{G} & Ver \end{array}$$

is commutative in the following sense:

$$h(f(x)) \subset G(h(x)).$$

We cannot guarantee equality $h(f(x)) = G(h(x))$. However, the described inclusion is sufficient to display the trajectories of the system into acceptable paths of the symbolic image via the mapping h.

Example 4. Consider a geometric example of constructing a symbolic image. Let M be a closed area on the plane shown in the Fig. 1. The covering C consists of 100 squares, the numbering of which begins with the upper-left square, goes from left to right and from top to bottom, so that the upper left cell has the index (number) 1, while the index 10 corresponds to the upper right cell, 91 is the number of the lower left cell, and 100 is the index of the bottom right cell. Consider the cell $M(i)$ with the index *i*=63, let the image $f(M(i))$ intersect cells with numbers $\mathrm{l} = 26,\ \mathrm{k} = 27,\ \mathrm{j} = 28,\ \mathrm{l} = 36,\ \mathrm{d} = 37$, see fig. 1. As the result, we have 5 arcs radiating from the vertex *i* to vertices *j*, *k*, *l*, *e*, *d*. By doing a similar construction for each cell, we will get a symbolic image. If the image $f(M(i))$ intersects the cell $M(i)$, then the loop $i \to i$ is formed.

In this case, the following problem arises. Numbering of cells $M(i)$ is arbitrary. If the scan-point image $f(x_i)$ is calculated, the task appears of calculating of the number for that cell $M(j)$, which considers $f(x_i)$. In other words, we need to know, how to use coordinates of a point to find out the number of the cell in which this point lies. Consider the example shown in Fig. 1. Let the area be a rectangle $M = [x_0, x_1] \times [y_0, y_1]$, the covering consists of squares with a side h, the numbering of which starts from the upper left square, goes from left to right and from top to bottom. If a point on the plane has coordinates (x, y), then the number *j*: $(\mathrm{x}, \mathrm{y}) \in \mathrm{M}(\mathrm{j})$ calculated by the formula

$$j = \left[\frac{|y_1 - y|}{h}\right] s + \left[\frac{|x_0 - x|}{h}\right] + 1,$$

where $[\cdot]$ – the integer part of the number, s – the number of cells in one row, so $s = \frac{x_1 - x_0}{h}$.

2.3. Parameters of the symbolic image

Let

$$diam\, M(i) = max\ (\rho\ (x, y): x, y \epsilon\, M(i))$$

be the diameter of a cell $M(i)$, and $d = diam(C)$ be the largest one among the cell diameters. The number *d* we call the diameter of the covering C. Let *q* be the largest diameter of the cell images

$$q = \max\{diam\, f(M(i))\}.$$

Since there is a modulus of continuity $\theta(\rho)$ for the display *f*, then an estimate takes place

$$q \le \theta\,(d).$$

Let's define the number r as follows. By construction, all cells are closed. If the cell and the image do not intersect: $M(k) \cap f(M(i)) = \emptyset$, then the distance between them

$$r_{ik} = \rho\,\left(f(M(i)), M(k)\right) = \min\{\rho\,(x,y): x \in f(M(i)), y \in M(k)\}$$

is positive. Since the number of pairs (i,k) described above is finite, then the value

$$r = \min\{\, r_{ik}\colon M(k) \cap f(M(i)) = \emptyset\}$$

is the positive one. Thus, r is the smallest distance between the images $f(M(i))$ and the cells $M(k)$, that do not intersect.

Let G be a symbolic image relative to the covering C, and d be the diameter of the covering C.

Definition 3. *The vertex of a symbolic image is called a recurrent vertex, if a periodic path passes through it. The set of recurrent vertices is denoted by RV. Two recurrent vertices i and j are called equivalent if there is a periodic path passing through i and j.*

Thus, the set of recurrent vertices RV is divided into equivalence classes $\{H_k\}$. In graph theory, classes of equivalent recurrent vertices H_k are called components of strong connectivity. There are algorithms for constructing components of strong connectivity based on traversing the graph in depth [3, 12]. Robert Tarjan's algorithm is the most popular [28, 29]. This algorithm (written in codes) can be found on the Internet.

2.4. The neighborhood of a chain recurrent set

We will apply the process of sub-splitting of coverings and build a sequence of symbolic images. Let's consider the main step of the subdividing process.

Let $C = \{M(i)\}$ – be a covering and G – be a symbolic image for C. Suppose that the new NC covering is a subdivision of the C covering. This means that each cell $M(i)$ is subdivided into cells $m(i,k),\ \ k = 1, 2, \dots$, i.e.

$$\bigcup_k m(i,k) = M(i).$$

Let's denote by NG a symbolic image for the covering $NC = \{m(i,k)\}$. The vertices of NG are denoted as (ik). Such a construction generates a mapping s from NG onto G, which displays the vertices (i,k) onto the vertex i. Since the intersection of the image $f(m(i,k))$ and $m(j,l)$:

$$f(m(i,k)) \bigcap m(j,l) \ne \emptyset$$

for small cells, it guarantees a similar intersection for large cells $f(M(i))$ and $M(j)$:

$$f(M(i)) \bigcap M(j) \ne \emptyset.$$

This means that the mapping s transforms the arc $(i,k) \to (j,l)$ into the ark $i \to j$. Hence, s maps the oriented graph NG to the oriented graph G. Since the mapping s is a mapping of oriented graphs, each valid path on NG is transformed by the mapping s to some valid path on G. In particular, the image of a periodic path is a periodic path, and the image of a recurrent vertex is a recurrent vertex. Moreover, the image of the class NH of equivalent recurrent vertices (on NG) lies in the class H of equivalent recurrent vertices on G.
Multi-valued mapping of $h: M \to V$ maps to point x all vertices i such that $x \in M(i)$.

Proposition [8]. *The h mapping has the following properties.*
1) If x is a chain recurrent point and $i \in h(x)$, then i is a recurrent vertex.
2) If x_1 and x_2 lie in the same component of a chain recurrent set Ω, $i_1 \in h(x_1)$ and $i_2 \in h(x_2)$, then i_1 and i_2 are equivalent recurrent vertices.
3) If Ω is a component of a chain recurrent set, then there is a single class $H(\Omega)$ of equivalent recurrent vertices such that $h(\Omega) \subset H(\Omega)$.
Let Ω be a component of a chain recurrent set. It follows from statement 7 that the image of $h(\Omega)$ lies in a certain class of equivalent recurrent vertices, which we will denote as $H(\Omega)$; in this case, the component Ω lies in $\{\bigcup M(i) : i \in H(\Omega)\}$. Note that the class $H(\Omega)$ is uniquely determined by Ω.
Let's denote as $P(d)$ the union of cells $M(i)$, for which the vertex i is a recurrent vertex, i.e.

$$P(d) = \left\{\bigcup M(i) : i - a\, recurrent\ vertex\right\}. \tag{4}$$

Theorem 2 [6, 20].
1) The set $P(d)$ is a closed neighborhood of a chain recurrent set CR

$$CR \subset P(d). \tag{5}$$

2) For any neighborhood V of a chain recurrent set CR, there is d>0 such that

$$CR \subset P(d) \subset V. \tag{6}$$

3) The chain recurrent set CR coincides with the intersection of the sets $P(d)$ for all positive d:

$$CR = \bigcap_{d>0} P(d).$$

3. The localization process

Suppose that $C = C_0$ is the initial covering, and $C_1, C_2, \cdots, -$ were obtained by the described above subdividing process, the sets $P_0, P_1, P_2, \cdots, -$ were constructed according to (4).
Theorem 3 [8]. *The sequence of sets $P_0, P_1, P_2, \cdots, -$ has the following properties.*
1. P_k nested inside each other:

$$P_0 \supset P_1 \supset P_2 \supset \dots \supset CR.$$

2. If the diameters of the coverings $d_k \to 0$, then the limit

$$\lim_{k\to\infty} P_k = \bigcap_k P_k = CR$$

is a chain recurrent set.

3.1. The localization algorithm

Based on the results obtained, we construct the following algorithm for localization of a chain recurrent set.
1) Select the initial covering C for the compact M. We find the symbolic image G of the mapping f. Note that the cells of the initial covering can have an arbitrary diameter d_0.
2) We select the recurrent vertices $\{i\}$ on the graph G. Using them, we find a closed neighborhood $P = \{\bigcup\ M(i): i\ -\ a\ recurrent\ vertice\}$ of a chain recurrent set CR.
3) Cells $\{M(j)\}$, corresponding to non-recurrent vertices $\{j\}$, are being removed from consideration.
4) Cells $M(i)$, corresponding to recurrent vertices $\{i\}$, are being divided into small cells with diameter several times smaller than before. Thus, we determine the new covering.
5) We build a symbolic image G for the new covering.
6) Move on to the point 2.
Repeating the process of sequential grinding of the covering, according to theorem 3, we obtain a sequence of nested neighborhoods $P_0, P_1, P_2, \cdots,$ of the chain recurrent set CR, and a sequence of the largest diameters $d_0, d_1, d_2, \cdots,$ of cells corresponding to the recurrent vertices of the symbolic image for the covering C_k. According to the construction, $d_{k+1} \le \lambda\, d_k,\ 0 < \lambda < 1$, and, therefore, the covering diameter converges to zero. Thus, the described algorithm gives a monotonically decreasing sequence of neighborhoods converging to a chain recurrent set. The proposed localization method works equally well with both stable and unstable chain recurrent trajectories, since it is based on the principle of returnability of a pseudo-trajectory.

3.2. The localization of a chain recurrent set of the Van der Pol differential equation

Example 5. The Van der Pol oscillator serves as a model of a triode tube generator in the case of a nonlinear lamp characteristic [30]. The equation has the form

$$\ddot{x} - 1.5\dot{x}(1 - x^2) + x = 0.$$

Turning to a two-dimensional system, we obtain an autonomous system of differential equations

$$\dot{x} = y,$$
$$\dot{y} = 1.5\dot{y}(1 - x^2) - x.$$

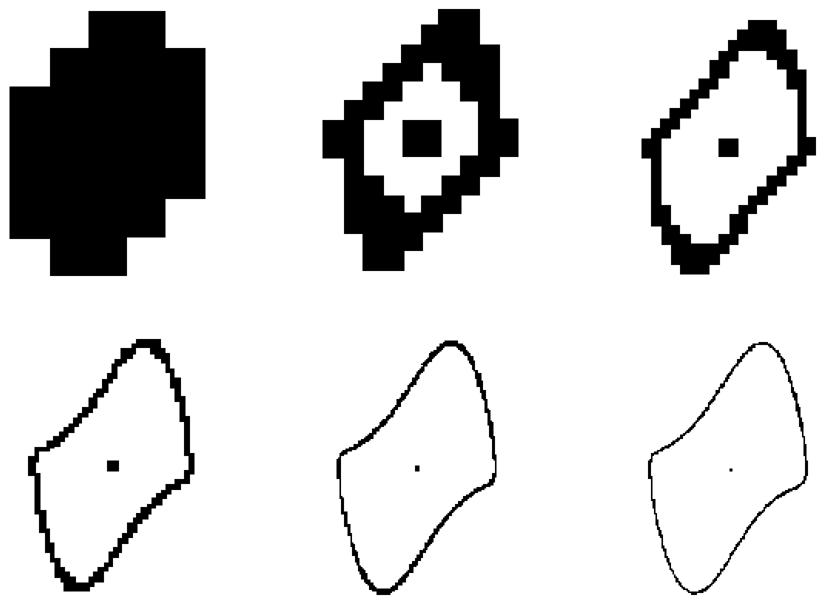

Fig. 2. Localization of the chain recurrent set of the Van der Pol equation

To construct a discrete system, we define $\Phi(x, y)$ as the mapping of the shift per unit of time along the trajectories of this system by the numerical Runge-Kutta method. We consider the discrete dynamic system

$$(x_{n+1}, y_{n+1}) = \Phi(x_n, y_n).$$

Using the method described above, we find the chain recurrent set of the resulting system. It is known, that this Van der Pol equation has an unstable equilibrium state (0,0) and a stable periodic trajectory. There are no other chain recurrent trajectories. The figure shows the sequential construction of the sets $P_k,\ k = 1, 2, 3, 4, 5, 6$. It is clear that P_6 is a fairly small neighborhood of a chain recurrent set.

3.3. The localization of a chain recurrent set in a projective space

Let's consider an example of the localization of a chain recurrent set of a dynamic system generated by the action of the matrix

$$A = \begin{pmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 1 & 2 & 5 \end{pmatrix} \tag{7}$$

in the projective space P^2; this action is denoted by $A^*: P^2 \to P^2$. Let $\pi: R^3 \to P^2$ be a projection from Euclidean space R^3 on a projective space P^2. The projective space P^2 on the maps described above is given by three squares $[-1,1]^2$, glued along the boundary. To construct a symbolic image of this system, we set the initial covering C_0, which coincides with three given squares covering the projective space P^2. We will divide each square into 4 equal parts with further sub-divisions. Applying the localization algorithm described above, we obtain a chain recurrent set of a dynamic system. The structure of a chain recurrent set depends on the eigenvalues of the matrix A, which has one real number and a pair of complex conjugate eigenvalues.

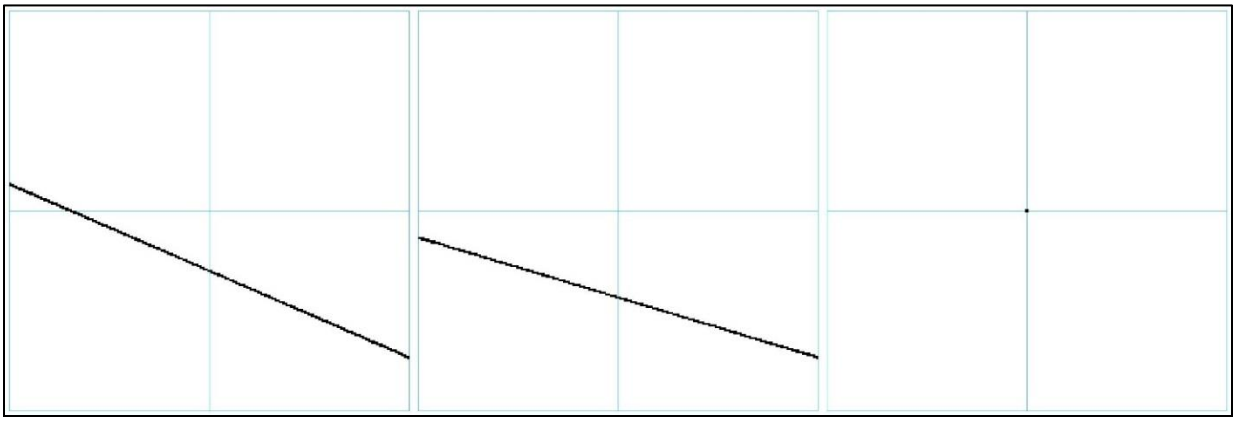

Fig. 3. The chain recurrent set of the action of the mapping $A^*: P^2 \to P^2$

For example, if the vector v is an eigenvector for the real number λ, then the eigenspace $L = \{tv,\ t \in R\}$ is a fixed point for the mapping $A^*: P^2 \to P^2$, therefore, $A^*(\pi L) = \pi L$, and πL is a chain recurrent point for A^*. M.V. Poryvai (4th year student of the Sevastopol branch of Moscow State University, 2022) implemented the described research. The resulting chain recurrent set is shown in the Fig. 3. It consists of a fixed point *(0,0)* in the third square– *Oxy* when *z = 1* and two segments in the first square – *Oyz* when *x = 1*, and the second square – *Oxz* when *y = 1*. The maps are glued together at the boundary points. We show that the segments of the first and second squares are glued into a circle O_c. Indeed, the left end of the first segment (when *x = 1*) has coordinates $(-1, \alpha)$, which corresponds to the vector $v_1 = (1, -1, \alpha)$ in R^3. The left end of the second segment (when *y = 1*) has coordinates $(-1, -\alpha)$, which corresponds to the vector $v_2 = (-1, 1, -\alpha)$ in R^3. Since $v_1 = -v_2$, then these vectors lie in the same subspace in R^3 and, therefore, the left end of the first segment is identified with the left end of the second segment in P^2. Similarly, the right end of the first segment (when *x = 1*) and the right end of the second segment (when *y = 1*) correspond to the vector $(1, 1, -\beta)$ in R^3 and have to be identified in P^2. The resulting circle in P^2 corresponds to the proper two-dimensional subspace Π (of the matrix *A*) for a pair of complex conjugate eigenvalues. The matrix *A* on *Π* rotates by a fixed angle, so the entire circle O_c is filled with chain recurrent points.

3.4. The localization of a chain recurrent set in a projective bundle over a two-dimensional manifold

As the example, we consider a discrete system generated by the Ikeda mapping

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} R + \alpha(x_n cos(\tau_n) - y_n sin(\tau_n)) \\ \beta(x_n sin(\tau_n) + y_n cos(\tau_n)) \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad (8)$$

where

$$\tau_n = 0.4 - \frac{6}{1 + {x_n}^2 + {y_n}^2},\ R = 2,\ \alpha = -0.9,\ \beta = 0.9.$$

This system was studied in the book [20]. System (8) in the domain $M = [-2,4]x[-3,2]$ has a chain recurrent attractor *A* shown in the Fig. 4.

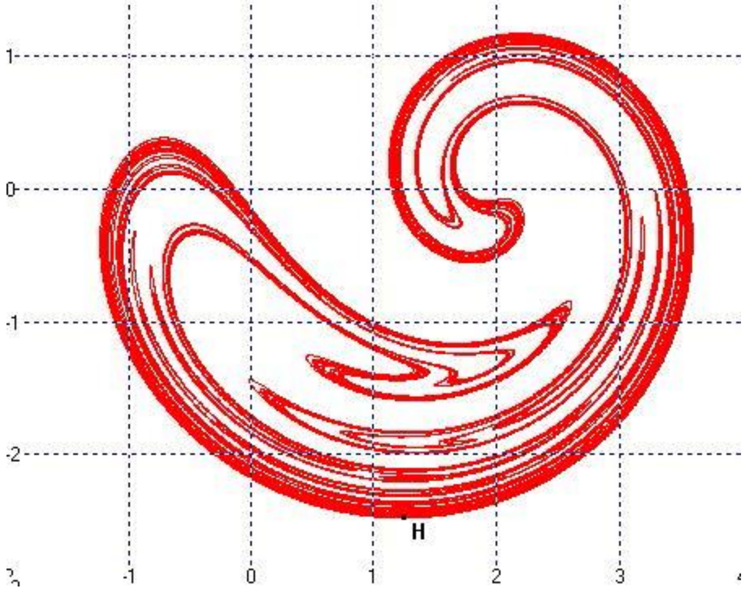

Fig. 4. The chain recurrent set of the system (8) in the domain M

The motionless point H $(1.3815, -2.4746)$ is hyperbolic and the attractor A is the closure of an unstable manifold of this point. The differential of the system does not have an invariant (nontrivial) decomposition over the attractor A, since the stable and unstable manifolds of the point H have a homoclinic tangent point.
However, if we limit the domain of study to $[-1.1, 3.4]$x$[-1.5, 1.8] = M_0$, then the hyperbolic point H will not enter this region and the chain recurrent set (in the domain M_0) splits into two components Ω_1 and Ω_2. The component Ω_1 is a 6-periodic trajectory, but Ω_2 is a Cantor set and has a nontrivial fractal structure. To localize the chain recurrent set of the differential action in the projective bundle, the Jacobi matrix is calculated

$$D(x,y) = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix}.$$

A projective bundle over the plane R^2 is a composition R^2x P^1. Denote as $v = (v_1, v_2)$ a vector in tangent space. Maps in the projective space P^1 are defined by two segments $v_2 \in [-1, 1]$ *when* $v_1 = 1$ and $v_1 \in [-1, 1]$ *when* $v_2 = 1$. A discrete dynamic system on a tangent bundle is given as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} R + \alpha(x_n cos(\tau_n) - y_n sin(\tau_n)) \\ \beta(x_n sin(\tau_n) + y_n cos(\tau_n)) \end{pmatrix},$$

$$\begin{pmatrix} v_{1_{n+1}} \\ v_{2_{n+1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial x}(x_n, y_n) & \frac{\partial X}{\partial y}(x_n, y_n) \\ \frac{\partial Y}{\partial x}(x_n, y_n) & \frac{\partial Y}{\partial y}(x_n, y_n) \end{pmatrix} \begin{pmatrix} v_{1_n} \\ v_{2_n} \end{pmatrix}.$$

Let the point e of the projective space define a straight line $E \subset R^2$, and e has a coordinate $v_1 = \alpha \in [-1, 1]$ on the map $v_2 = 1$. This means that the vector $v = (\alpha, 1)$ lies in the subspace E. Let the image $D(x,y)v = (V_1, V_2)$, $|V_2| \geq |V_1|$. Then the vector $\left({V_1}/{V_2}, 1\right)$ lies in the image $D(x,y)E$ and has a coordinate $\beta = {V_1}/{V_2} \in [-1, 1]$ on the map $v_2 = 1$.

By applying the localization of the chain recurrent set of the differential action in the projective bundle, four components are obtained: Ω_{11}, Ω_{12}, Ω_{21} and Ω_{22}. At the same time, the components Ω_{11}, Ω_{12} are projected on $\Omega_1 \subset R^2$, components Ω_{21} and Ω_{22} are projected on $\Omega_2 \subset R^2$. According to the Theorem 3, the tangent bundle over Ω_1 is decomposed into a direct sum

$$TR^2|_{\Omega_1} = E_{11} \bigoplus E_{12},$$

where the bundle E_{11} is generated by a component Ω_{11}:

$$E_{11} = \left\{v \in \pi^{-1}(\Omega_1): v \neq 0 \ \Rightarrow \frac{v}{|v|} = e \in \Omega_{11}\right\},$$

and the bundle E_{12} is generated by a component Ω_{12}:

$$E_{12} = \left\{v \in \pi^{-1}(\Omega_1): v \neq 0 \ \Rightarrow \frac{v}{|v|} = e \in \Omega_{12}\right\}.$$

To confirm the results obtained, Lyapunov exponents of the periodic trajectory Ω_1 were calculated, which showed that Ω_1 is a hyperbolic periodic trajectory. The similar decomposition also takes place over the component Ω_2

$$TR^2|_{\Omega_2} = E_{21} \oplus E_{22}.$$

To verify these data, three periodic trajectories P_2, P_4, P_6 with the periods 2, 4, 6 were found in the component Ω_2, and they also turned out to be hyperbolic ones.

4. References

[1] Alekseev, V.M. Symbolic dynamics. Eleventh Mathematical School, ed. Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR, Kiev, 1976 (in Russian).

[2] Bronstein, I.U. Nonautonomous dynamical systems. Stiinza, Chisinau, 1984.

[3] Cormen, Thomas H., Leiserson, Charles I., Rivest, Ronald L., Stein, Clifford. Algorithms: Construction and analysis. Ed. Williams, Moscow, 2011 (in Russian). Translated from: Cormen, Thomas H., Leiserson, Charles I., Rivest, Ronald L., Stein, Clifford. Introduction to Algorithms. MIT Press, Cambridge, 2002.

[4] Osipenko, G.S. On the symbolic image of a dynamic system. Boundary value problems. Collection of Works, Perm. 1983, 101-105 (in Russian).

[5] Osipenko, G.S., Ampilova, N.B. Introduction to symbolic analysis of dynamic systems. St. Petersburg University Publishing House, 2005 (in Russian).

[6] Osipenko, G.S. Computer-oriented methods of dynamic systems. Moscow, ed. INFRA-M, 2024 (in Russian).

[7] Osipenko, G.S. Lyapunov exponents and invariant measures on a projective bundle. Mathematical Notes, vol. 101, № 4, 549-561, 2017.

[8] Osipenko, G.S. The spectrum of the averaging of a function over pseudo-trajectories of a dynamical system. Mat. Sb., Vol. 209, № 8, pp. 114 – 137, 2018.

[9] Osipenko, G. S. Encoding of trajectories and invariant measures. Mat. Sb., Vol. 211, № 7, 151-176, 2020.

[10] Petrenko, E. I. Computer research of dynamic systems based on the symbolic image method. PhD thesis. St. Petersburg University, 2009 (in Russian).

[11] Pilyugin, S. Y. Introduction to rough systems of differential equations. LSU, L., 1988 (in Russian).

[12] Sedgwick R. Fundamental algorithms in C^{++}. Algorithms on graphs. LLC DiaSoftUP, St. Petersburg, 2002.

[13] Avrutin, V., Levi, P., Schanz, V., Fundinger, D., and Osipenko, G. Investigation of dynamical systems using symbolic images: efficient implementation and applications. International J. of Bifurcation and Chaos. v. 16, №. 12, 2006, 3451-3496.

[14] Colonius, F., and Kliemann, W. The Dynamics of Control. Burkhauser, 2000.

[15] Fundinger, D., Lindström, T., Osipenko, G. S. On the appearance of multiple attractors in discrete food-chains. Applied Mathematics and Computation, 184(2), 429-444, 2007.

[16] Hsu, C. S. Cell-to-Cell Mapping. Springer-Verlag, N.Y., 1987.

[17] Ikeda, K. Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system. Opt. Comm. V. 30, 257-261, 1979.

[18] Douglas, Lind, Brian, Marcus. An introduction to symbolic dynamics and coding. Cambridge University Press, 1995.

[19] Osipenko, George. Computer-oriented tests for hyperbolicity and structural stability of dynamical system. Differential Equations and Control Processes, № 3, 2023. Electronic Journal, http://di_journal.spbu.ru/ https://doi.org/10.21638/11701/spbu35.2023.302

[20] Osipenko, George. Dynamical systems, Graphs, and Algorithms. Lectures Notes in Mathematics, 1889, Springer, Berlin, 2007.

[21] Osipenko, G. S. Localization of the chain recurrent set by symbolic dynamics methods. Proceedings of Dynamics Systems and Applications. V. 1. Dynamic Publishers Inc., 227 – 282, 1994.

[22] Osipenko, G. S., Romanovsky, J. V., Ampilova, N. B., and Petrenko, E. I. Computation of the Morse Spectrum. Journal of Mathematical Sciences, v. 120:2, 1155 – 1166, 2004.

[23] Osipenko, G. Center manifold. Encyclopedia of Complexity and Systems Science. Robert A. Meyers (Ed.) N.Y., Berlin: Springer, 936-951, 2009.

[24] Osipenko, George. Symbolic images and invariant measures of dynamical systems. Ergodic Theory and Dynamical Systems, v. 30, 1217 – 1237, 2010.

[25] Sacker, R. and Sell, G. Existence of dichotomies and invariant splitting for linear differential systems I-III. J. Diff. Eq., v. 15, №. 3, 429-458 (1974); v. 22, №. 2, 476-522, 1976.

[26] Selgrade, J. Isolated invariant sets for flows on vector bundles. Trans. Amer. Math. Soc., v.203, 359-390, 1975.

[27] Shub, M. Stabilité globale de systémes denamiques. Astérisque, v. 56, 1-21, 1978.

[28] Tarjan, R. Depth-first search and linear graph algorithms. SIAM Journal on Computing, 1, 146-160, 1972.

[29] Tarjan, R. Algorithm design. Communications of the ACM, v. 30 №. 3, 204-212, 1987.

[30] Van der Pol, B. On relaxation-oscillations. The London, Edinburgh and Dublin Phil. Mag. J. of Sci., 2(7), 978—992, 1927.