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A new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The case of arbitrary complete orthonormal systems in Hilbert space. II

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Abstract. The article is Part IV of the author's work devoted to a new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The above approach was proposed by the author in 2022 and is based on generalized multiple Fourier series in complete orthonormal systems of functions in Hilbert space. In the previous parts of the work, expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 6 (the case of Legendre polynomials and the trigonometric Fourier basis) and multiplicities 1 to 4 (the case of an arbitrary complete orthonormal system of functions in Hilbert space) were obtained. In this article, an expansion of iterated Stratonovich stochastic integrals of multiplicity 5 (the case of an arbitrary complete orthonormal system of functions in Hilbert space) is obtained. The mentioned expansion is generalized to the case of an arbitrary multiplicity of iterated Stratonovich stochastic integrals. The results of the article will be useful for construction of strong numerical methods

with orders 1.0, 1.5, 2.0, ... (based on the Taylor–Stratonovich expansion) for systems of Itô stochastic differential equations with non-commutative noise.
Key words: iterated Stratonovich stochastic integral, iterated Itô stochastic integral, Itô stochastic differential equation, multidimensional Wiener process, generalized multiple Fourier series, factorized Volterra–type kernel, integral operator, trace series, mean-square convergence, expansion.

Contents

1 Introduction

Let (Ω, F, P) be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-

dimensional Wiener stochastic process which is F_t-measurable for any $t \in [0,T]$ and has independent components $\mathbf{w}_t^{(i)}$ $(i=1,\ldots,m)$. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau + \sum_{j=1}^m \int\limits_0^t B_j(\mathbf{x}_\tau,\tau)d\mathbf{w}_\tau^{(j)}, \quad \mathbf{x}_0 = \mathbf{x}(0,\omega), \quad \omega \in \Omega, \quad (1)$$

where $\mathbf{x}_t$ is the n-dimensional stochastic process satisfying (1), the functions $\mathbf{a}, B_j : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1], the second integral in (1) is the Itô stochastic integral, $\mathbf{x}_0$ is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). Futhermore, we assume that $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

Consider the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (2)$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (3)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau) : [t,T] \to \mathbb{R}$, $i_1,\ldots,i_k = 0,\ 1,\ \ldots,\ m$, $\mathbf{w}_\tau^{(0)} = \tau$,

$$\int \quad \text{and} \quad \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively.

It is well known that the stochastic integrals (2) and (3) are an essential ingredient for the numerical solution of Itô SDEs using the Taylor–Itô and Taylor–Stratonovich expansions [2]-[14]. Moreover, Itô SDEs have a wide range of applications, which confirms the importance of the problem of their approximate solution [2]-[13].

Note that the so-called classical [2]-[8] $(\psi_1(\tau),\ldots,\psi_k(\tau) \equiv 1,\ i_1,\ldots,i_k = 0,1,\ldots,m)$ and unified [9]-[14] $(\psi_l(\tau) \equiv (t-\tau)^{q_l},\ q_l = 0,1,\ldots,\ l = 1,\ldots,k,$ $i_1,\ldots,i_k = 1,\ldots,m)$ Taylor–Itô and Taylor–Stratonovich expansions are known.

This article is Part IV of the work devoted to a new approach to the series expansion of iterated Stratonovich stochastic integrals (3) ([15]–[17] are Parts I–III of the above work, respectlively). Throughout this article we will use the definition of the Stratonovich stochastic integral from Sect. 2.2 [17] (see also [2]).

Other approaches to the expansion of iterated Itô and Stratonovich stochastic integrals (2) and (3) should also be noted [2]–[5], [11], [14], [18]–[51].

2 Preliminary Results

2.1 Connection Between Iterated Stratonovich and Itô Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$)

Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{q=1}^{l} \mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})} dt_{s_l+1} d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (4)$$

where $(s_l,\ldots,s_1)\in \mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l} = \bigl\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;\ s_l,\ldots,s_1=1,\ldots,k-1\bigr\}, \qquad (5)$$

$l=1,2,\ldots,[k/2]$, $i_1,\ldots,i_k=0,1,\ldots,m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals (3) and (2) of arbitrary multiplicity k ($k \in \mathbb{N}$).

Theorem 1 [51] (1997) (also see [11]-[14], [40], [50]). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *are continuous nonrandom functions at the interval* $[t, T]$. *Then, the following relation between iterated Stratonovich and Itô stochastic integrals* (3) *and* (2) *is correct*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \tag{6}$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\sum\limits_{\emptyset}$ *is supposed to be equal to zero, w. p.* 1 (*here and further*) *means with probability* 1.

Note that the condition of continuity of the functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ is related to the definition of the Stratonovich stochastic integral that we use (see [17], Sect. 2.2 for details).

2.2 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$) Under the Condition on Trace Series

In this section, we recall Theorem 14 from [17] (Part III of this work) on the expansion of iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbb{N}$) and introduce some notations.

Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{7}$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Consider the Fourier coefficient

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{8}$$

corresponding to the factorized Volterra–type kernel of the form

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{9}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system (CONS) in the space $L_2([t,T])$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_{l-1}j_{l-2}\ldots j_1}\Biggl|_{(j_lj_{l-1})\curvearrowright(\cdot)}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k, \tag{10}$$

where we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (7)).

Let

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\bar J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}, \tag{11}$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero; another notations are the same as in Theorem 1.

Theorem 2 [14], [17], [52]–[54]. *Assume that the CONS $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ are such that*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\sum_{j_{g_3}=0}^{\min\{p_{g_3},p_{g_4}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=0 \tag{12}$$

for all $r=1,2,\ldots,[k/2]$ *and for all possible* $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (7)*). Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (11) *the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (8), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $\mathbf{w}_\tau^{(0)}=\tau.$

Using Theorem 1, we obtain the following corollary of Theorem 2.

Theorem 3 [14], [17], [52]–[54]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$*. Furthermore, assume that the condition* (12) *is satisfied for* $p_1=\ldots=p_k=p$*. Then, for the iterated Stratonovich stochastic integral* (3) *of multiplicity*

k ($k \in \mathbb{N}$) *the following expansion*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where notations are the same as in Theorem 2.

2.3 Auxiliary Result That Follows From Lebesgue's Dominated Convergence Theorem

Let us note one fact that follows from Lebesgue's Dominated Convergence Theorem. Suppose that $\{g_j(x)\}_{j=0}^{\infty}$ is an arbitrary sequence of real-valued measurable functions such that

$$\left|\sum_{j=0}^{p} g_j(x)\right| \le K < \infty \quad \forall p \in \mathbb{N} \tag{13}$$

almost everywhere on X (with respect to Lebesgue's measure), where constant K does not depend on x and p, and

$$\lim_{p\to\infty} \sum_{j=0}^{p} g_j(x) \ \ \text{exists} \tag{14}$$

almost everywhere on X (with respect to Lebesgue's measure).

Then by Lebesgue's Dominated Convergence Theorem we obtain

$$\lim_{p\to\infty} \int\limits_X g^2(x) \left(\sum_{j=0}^{p} g_j(x)\right)^2 dx = \int\limits_X g^2(x) \left(\sum_{j=0}^{\infty} g_j(x)\right)^2 dx, \tag{15}$$

where $g(x) \in L_2(X)$.

The equality (15) will be used further[1].

[1]In our article [17] there was an inaccuracy. Namely, the formula (171) (see [17]) is true under the additional condition

$$\sum_{j=0}^{p} |g_j(x)| \le K < \infty,$$

where constant K does not depend on x and p, or under another sufficient conditions (13), (14) (see this article). Therefore, the conditions of Theorems 19–22 (see [17]) must be clarified. More precise formulations of these theorems can be found in [14] (see Theorems 2.52–2.55). This clarification in [14] is carried out using (13), (14) (see this article).

3 Main Results

3.1 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case of an Arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv 1$

In this section, we will prove the following theorem.

Theorem 4 [14], [52]–[54]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)}
$$

the following expansion

$$
J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5=0,1,\ldots,m,$

$$
C_{j_5\ldots j_1}=\int\limits_t^T\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. Step 1. According to Theorem 3, we conclude that Theorem 4 will be proved if we prove the following equalities (see (12) for $k=5, r=1$ and $k=5, r=2$ $(p_1=\ldots=p_5=p)$) under the conditions of Theorem 4

$$
\lim_{p\to\infty}\sum_{j_3,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_4j_3j_1j_1}-\frac{1}{2}C_{j_5j_4j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0, \qquad (16)
$$

$$\lim_{p\to\infty}\sum_{j_2,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_4j_1j_2j_1}\right)^2=0, \tag{17}$$

$$\lim_{p\to\infty}\sum_{j_2,j_3,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_1j_3j_2j_1}\right)^2=0, \tag{18}$$

$$\lim_{p\to\infty}\sum_{j_2,j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_4j_3j_2j_1}\right)^2=0, \tag{19}$$

$$\lim_{p\to\infty}\sum_{j_1,j_4,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_4j_2j_2j_1}-\frac{1}{2}C_{j_5j_4j_2j_2j_1}\Bigr|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{20}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_2j_3j_2j_1}\right)^2=0, \tag{21}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_4j_3j_2j_1}\right)^2=0, \tag{22}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_5=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_5j_3j_3j_2j_1}-\frac{1}{2}C_{j_5j_3j_3j_2j_1}\Bigr|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{23}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_4j_3j_2j_1}\right)^2=0, \tag{24}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\left(\sum_{j_4=0}^{p}C_{j_4j_4j_3j_2j_1}-\frac{1}{2}C_{j_4j_4j_3j_2j_1}\Bigr|_{(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{25}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_5j_3j_3j_1j_1}-\frac{1}{4}C_{j_5j_3j_3j_1j_1}\Bigr|_{(j_1j_1)\curvearrowright(\cdot),(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{26}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_4j_3j_1j_1}\right)^2=0, \tag{27}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_3j_1j_1}-\frac{1}{4}C_{j_4j_4j_3j_1j_1}\Bigr|_{(j_1j_1)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{28}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_5j_2j_1j_2j_1}\right)^2=0, \tag{29}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_4j_1j_2j_1}\right)^2=0, \tag{30}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_2j_1}\right)^2=0, \tag{31}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_5j_1j_2j_2j_1}\right)^2=0, \tag{32}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_2j_1}\right)^2=0, \tag{33}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_3j_1j_3j_2j_1}\right)^2=0, \tag{34}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_4j_2j_2j_1}\right)^2=0, \tag{35}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_3j_2j_1}\right)^2=0, \tag{36}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_2j_1}\right)^2=0, \tag{37}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2j_1}-\frac{1}{4}C_{j_4j_4j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{38}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1}\right)^2=0, \tag{39}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1}\right)^2=0. \tag{40}$$

Step 2. Let us prove the equalities (16)–(25). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (16)–(25)

$$\sum_{j_3,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_5 j_4 j_3 j_1 j_1}-\frac{1}{2}C_{j_5 j_4 j_3 j_1 j_1}\Biggl|_{(j_1 j_1)\curvearrowright(\cdot)}\right)^2=$$

$$=\sum_{j_3,j_4,j_5=0}^{p}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)\times\right.$$

$$\left.\times dt_3dt_4dt_5\right)^2\le$$

$$\le\sum_{j_3,j_4,j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)\times\right.$$

$$\left.\times dt_3dt_4dt_5\right)^2=$$

$$=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_3<t_4<t_5\}}\right)^2\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)^2dt_3dt_4dt_5,\qquad(41)$$

$$\sum_{j_2,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_5 j_4 j_1 j_2 j_1}\right)^2=$$

$$=\sum_{j_2,j_4,j_5=0}^{p}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\times\right.$$

$$\left.\times dt_2dt_4dt_5\right)^2\le$$

$$\le\sum_{j_2,j_4,j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\times\right.$$

$$
\left.\times dt_2dt_4dt_5\right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_2<t_4<t_5\}}\right)^2 \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \phi_{j_1}(t_3)dt_3\right)^2 dt_2dt_4dt_5, \qquad (42)
$$

$$
\sum_{j_2,j_3,j_5=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_5j_1j_3j_2j_1}\right)^2 =
$$

$$
= \sum_{j_2,j_3,j_5=0}^{p} \left(\int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_3}^{t_5} \phi_{j_1}(t_4)dt_4 \times\right.
$$

$$
\left.\times dt_2dt_3dt_5\right)^2 \le
$$

$$
\le \sum_{j_2,j_3,j_5=0}^{\infty} \left(\int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_3}^{t_5} \phi_{j_1}(t_4)dt_4 \times\right.
$$

$$
\left.\times dt_2dt_3dt_5\right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_2<t_3<t_5\}}\right)^2 \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_3}^{t_5} \phi_{j_1}(t_4)dt_4\right)^2 dt_2dt_3dt_5, \qquad (43)
$$

$$
\sum_{j_2,j_3,j_4=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1j_4j_3j_2j_1}\right)^2 =
$$

$$
= \sum_{j_2,j_3,j_4=0}^{p} \left(\int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_4}^{T} \phi_{j_1}(t_5)dt_5 \times\right.
$$

$$
\left.\times dt_2dt_3dt_4\right)^2 \le
$$

$$\leq \sum_{j_2,j_3,j_4=0}^{\infty}\Biggl(\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\times$$

$$\times dt_2dt_3dt_4\Biggr)^2 =$$

$$= \int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_2<t_3<t_4\}}\right)^2\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_2dt_3dt_4, \qquad (44)$$

$$\sum_{j_1,j_4,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_4j_2j_2j_1}-\frac{1}{2}C_{j_5j_4j_2j_2j_1}\Bigr|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2 =$$

$$= \sum_{j_1,j_4,j_5=0}^{p}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2\times$$

$$\times dt_1dt_4dt_5-\frac{1}{2}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\Biggr)^2 =$$

$$= \sum_{j_1,j_4,j_5=0}^{p}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)\times$$

$$\times dt_1dt_4dt_5\Biggr)^2 \leq$$

$$\leq \sum_{j_1,j_4,j_5=0}^{\infty}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)\times$$

$$\times dt_1dt_4dt_5\Biggr)^2 =$$

$$= \int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_1<t_4<t_5\}}\right)^2\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1dt_4dt_5, \qquad (45)$$

$$
\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_2j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_3dt_5\right)^2=
$$

$$
=\sum_{j_1,j_3,j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_1dt_3dt_5\right)^2\le
$$

$$
\le\sum_{j_1,j_3,j_5=0}^{\infty}\left(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_1dt_3dt_5\right)^2=
$$

$$
=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_1<t_3<t_5\}}\right)^2\left(\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\right)^2dt_1dt_3dt_5, \tag{46}
$$

$$
\sum_{j_1,j_3,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_4j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_1,j_3,j_4=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_2}(t_5)dt_5\times\right.
$$

$$
\left.\times dt_4\right)^2=
$$

$$= \sum_{j_1,j_3,j_4=0}^{p} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int_{t_4}^{T} \phi_{j_2}(t_5) dt_5 \times \right.$$

$$\left. \times dt_1 dt_3 dt_4 \right)^2 \le$$

$$\le \sum_{j_1,j_3,j_4=0}^{\infty} \left(\int_t^T \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int_{t_4}^{T} \phi_{j_2}(t_5) dt_5 \times \right.$$

$$\left. \times dt_1 dt_3 dt_4 \right)^2 =$$

$$= \int_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_3<t_4\}} \right)^2 \left(\sum_{j_2=0}^{p} \int_{t_1}^{t_3} \phi_{j_2}(t_2) dt_2 \int_{t_4}^{T} \phi_{j_2}(t_5) dt_5 \right)^2 dt_1 dt_3 dt_4, \qquad (47)$$

$$\sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_5 j_3 j_3 j_2 j_1} - \frac{1}{2} C_{j_5 j_3 j_3 j_2 j_1} \Big|_{(j_3 j_3) \curvearrowright (\cdot)} \right)^2 =$$

$$= \sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} \int_t^T \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_1}(t_1) \int_{t_1}^{t_5} \phi_{j_2}(t_2) \int_{t_2}^{t_5} \phi_{j_3}(t_3) \int_{t_3}^{t_5} \phi_{j_3}(t_4) dt_4 dt_3 \times \right.$$

$$\left. \times dt_2 dt_1 dt_5 - \frac{1}{2} \int_t^T \phi_{j_5}(t_5) \int_t^{t_5} \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_5 \right)^2 =$$

$$= \sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} \int_t^T \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_1}(t_1) \int_{t_1}^{t_5} \phi_{j_2}(t_2) \int_{t_2}^{t_5} \phi_{j_3}(t_3) \int_{t_3}^{t_5} \phi_{j_3}(t_4) dt_4 dt_3 \times \right.$$

$$\left. \times dt_2 dt_1 dt_5 - \frac{1}{2} \int_t^T \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_1}(t_1) \int_{t_1}^{t_5} \phi_{j_2}(t_2) \int_{t_2}^{t_5} dt_3 dt_2 dt_1 dt_5 \right)^2 =$$

$$= \sum_{j_1,j_2,j_5=0}^{p} \left(\int_t^T \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_1}(t_1) \int_{t_1}^{t_5} \phi_{j_2}(t_2) \left(\sum_{j_3=0}^{p} \frac{1}{2} \left(\int_{t_2}^{t_5} \phi_{j_3}(t_3) dt_3 \right)^2 - \frac{t_5 - t_2}{2} \right) \times \right.$$

$$
\times dt_2 dt_1 dt_5\Biggr)^2 \le
$$

$$
\le \sum_{j_1,j_2,j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\left(\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2-\frac{t_5-t_2}{2}\right)\times\right.
$$

$$
\left.\times dt_2 dt_1 dt_5\right)^2 =
$$

$$
= \int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_1<t_2<t_5\}}\right)^2\left(\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2-\frac{t_5-t_2}{2}\right)^2 dt_2 dt_1 dt_5, \qquad (48)
$$

$$
\sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_4j_3j_2j_1}\right)^2 =
$$

$$
= \sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3\times\right.
$$

$$
\left.\times dt_2 dt_1\right)^2 =
$$

$$
= \sum_{j_1,j_2,j_4=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_4}(t_4)\sum_{j_3=0}^{p}\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4\times\right.
$$

$$
\left.\times dt_2 dt_1\right)^2 \le
$$

$$
\le \sum_{j_1,j_2,j_4=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_4}(t_4)\sum_{j_3=0}^{p}\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4\times\right.
$$

$$
\left.\times dt_2 dt_1\right)^2 =
$$

$$= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_2<t_4\}}\right)^2 \left(\sum_{j_3=0}^{p} \int\limits_{t_4}^{T} \phi_{j_3}(t_5)dt_5 \int\limits_{t_2}^{t_4} \phi_{j_3}(t_3)dt_3\right)^2 dt_4dt_2dt_1, \qquad (49)$$

$$\sum_{j_1,j_2,j_3=0}^{p} \left(\sum_{j_4=0}^{p} C_{j_4j_4j_3j_2j_1} - \frac{1}{2}C_{j_4j_4j_3j_2j_1}\Big|_{(j_4j_4)\curvearrowright(\cdot)}\right)^2 =$$

$$= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(\int\limits_{t}^{T} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1dt_2 \sum_{j_4=0}^{p} \int\limits_{t_3}^{T} \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_4}(t_5)dt_5dt_4 \times$$

$$\times dt_3 - \frac{1}{2}\int\limits_{t}^{T}\int\limits_{t}^{t_4} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\Biggr)^2 =$$

$$= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(\int\limits_{t}^{T} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1) \left(\sum_{j_4=0}^{p} \frac{1}{2}\left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4\right)^2 - \frac{T-t_3}{2}\right) \times$$

$$\times dt_1dt_2dt_3\Biggr)^2 \le$$

$$\le \sum_{j_1,j_2,j_3=0}^{\infty} \Biggl(\int\limits_{t}^{T} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1) \left(\sum_{j_4=0}^{p} \frac{1}{2}\left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4\right)^2 - \frac{T-t_3}{2}\right) \times$$

$$\times dt_1dt_2dt_3\Biggr)^2 =$$

$$= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_2<t_3\}}\right)^2 \left(\sum_{j_4=0}^{p} \frac{1}{2}\left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4\right)^2 - \frac{T-t_3}{2}\right)^2 dt_1dt_2dt_3. \qquad (50)$$

Further, applying the Parseval equality and the generalized Parseval equality (Parseval's equality for two functions) as well as using the Cauchy–Bunyakovsky inequality, we have

$$\sum_{j=0}^{\infty} \left(\int\limits_{t_1}^{t_2} \phi_j(s)ds\right)^2 = \int\limits_{t}^{T} \left(\mathbf{1}_{\{t_1<s<t_2\}}\right)^2 ds = t_2 - t_1, \qquad (51)$$

$$\sum_{j=0}^{\infty} \int\limits_{t_1}^{t_2} \phi_j(s)ds \int\limits_{t_3}^{t_4} \phi_j(s)ds = \sum_{j=0}^{\infty} \int\limits_{t}^{T} \mathbf{1}_{\{t_1<s<t_2\}}\phi_j(s)ds \int\limits_{t}^{T} \mathbf{1}_{\{t_3<s<t_4\}}\phi_j(s)ds =$$

$$= \int\limits_{t}^{T} \mathbf{1}_{\{t_1<s<t_2\}}\mathbf{1}_{\{t_3<s<t_4\}}ds = 0, \qquad (52)$$

$$\left|(t_2-t_1) - \sum_{j=0}^{p} \left(\int\limits_{t_1}^{t_2} \phi_j(s)ds\right)^2\right| \le t_2 - t_1 \le T - t < \infty, \qquad (53)$$

$$\left(\sum_{j=0}^{p} \int\limits_{t_1}^{t_2} \phi_j(s)ds \int\limits_{t_3}^{t_4} \phi_j(s)ds\right)^2 \le \sum_{j=0}^{p} \left(\int\limits_{t_1}^{t_2} \phi_j(s)ds\right)^2 \sum_{j=0}^{p} \left(\int\limits_{t_3}^{t_4} \phi_j(s)ds\right)^2 \le$$

$$\le (t_2-t_1)(t_4-t_3) \le (T-t)^2 < \infty, \qquad (54)$$

where $t \le t_1 < t_2 \le t_3 < t_4 \le T$.

Using Lebesgue's Dominated Convergence Theorem and (51)–(54), we obtain that the right-hand sides of (41)–(50) tend to zero when $p \to \infty$. The equalities (16)–(25) are proved.

Step 3. Before proving the equalities (26)–(40), we show that

$$\left|\sum_{j_1,j_3=0}^{p} C_{j_3 j_3 j_1 j_1}(s,\tau)\right| \le K, \qquad (55)$$

$$\left|\sum_{j_1,j_3=0}^{p} C_{j_1 j_3 j_3 j_1}(s,\tau)\right| \le K, \qquad (56)$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_2 j_1}(s,\tau)\right| \le K, \qquad (57)$$

$$\sum_{j_2=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_2 j_1}(s,\tau)\right)^2 \le \int\limits_{\tau}^{s} \left(\sum_{j_1=0}^{p} \int\limits_{\tau}^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{s} \phi_{j_1}(t_3)dt_3\right)^2 dt_2, \qquad (58)$$

where constant K does not depend on p, s, τ; here and further in this proof

$$C_{j_k\ldots j_1}(s,\tau) = \int\limits_{\tau}^{s} \phi_{j_k}(t_k) \ldots \int\limits_{\tau}^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_k \quad (k = 1, \ldots, 4,\ t \le \tau < s \le T).$$

Further, by K, K_1, K_2 we will denote contants that can change from line to line.

By analogy with the formulas (145), (155), (165) and (128)–(130) from [17], we get

$$\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau) = \sum_{j_1,j_3=0}^{p} C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau) - \frac{1}{8}\left(\sum_{j_1=0}^{p}\bigl(C_{j_1}(s,\tau)\bigr)^2\right)^2, \quad (59)$$

$$\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau) = \sum_{j_1,j_2=0}^{p} C_{j_2}(s,\tau)C_{j_1j_2j_1}(s,\tau) - \frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_1j_2}(s,\tau)C_{j_2j_1}(s,\tau), \quad (60)$$

$$\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau) = \sum_{j_1,j_3=0}^{p} C_{j_1}(s,\tau)C_{j_3j_3j_1}(s,\tau) - \frac{1}{2}\sum_{j_1,j_3=0}^{p} \bigl(C_{j_3j_1}(s,\tau)\bigr)^2, \quad (61)$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau) = \frac{1}{8}(s-\tau)^2, \quad (62)$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau) = 0, \quad (63)$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau) = 0. \quad (64)$$

Using (59), Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$\left(\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau)\right)^2 \le 2\left(\sum_{j_1,j_3=0}^{p} C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau)\right)^2 +$$

$$+2\cdot\frac{1}{64}\left(\sum_{j_1=0}^{p}\bigl(C_{j_1}(s,\tau)\bigr)^2\right)^4 \le$$

$$\le 2\sum_{j_3=0}^{p}\left(C_{j_3}(s,\tau)\right)^2\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}(s,\tau)\right)^2 + K_1 \le$$

$$\le K_2\sum_{j_3=0}^{\infty}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}(s,\tau)\right)^2 + K_1 =$$

$$
= K_2 \sum_{j_3=0}^{\infty} \left(\int_{\tau}^{s} \phi_{j_3}(t_3) \sum_{j_1=0}^{p} \int_{\tau}^{t_3} \phi_{j_1}(t_2) \int_{\tau}^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \right)^2 + K_1 =
$$

$$
= K_2 \int_{\tau}^{s} \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int_{\tau}^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 \right)^2 dt_3 + K_1 \leq
$$

$$
\leq K_2 \int_{\tau}^{s} \left(\frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int_{\tau}^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 \right)^2 dt_3 + K_1 =
$$

$$
= K_2 \int_{\tau}^{s} \left(\frac{1}{2}(t_3 - \tau) \right)^2 dt_3 + K_1 \leq K < \infty,
$$

where constants K, K_1, K_2 do not depend on p, s, τ. The equality (55) is proved.

Let us prove (56). Using (61) and the above reasoning, we get

$$
\left(\sum_{j_1, j_3=0}^{p} C_{j_1 j_3 j_3 j_1}(s, \tau) \right)^2 \leq 2 \left(\sum_{j_1, j_3=0}^{p} C_{j_1}(s, \tau) C_{j_3 j_3 j_1}(s, \tau) \right)^2 +
$$

$$
+2 \cdot \frac{1}{4} \left(\sum_{j_1, j_3=0}^{p} \left(C_{j_3 j_1}(s, \tau) \right)^2 \right)^2 \leq
$$

$$
\leq 2 \sum_{j_1=0}^{p} \left(C_{j_1}(s, \tau) \right)^2 \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1}(s, \tau) \right)^2 + K_1 \leq
$$

$$
\leq K_2 \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1}(s, \tau) \right)^2 + K_1 =
$$

$$
= K_2 \sum_{j_1=0}^{\infty} \left(\int_{\tau}^{s} \phi_{j_1}(t_1) \sum_{j_3=0}^{p} \int_{t_1}^{s} \phi_{j_3}(t_2) \int_{t_2}^{s} \phi_{j_3}(t_3) dt_3 dt_2 dt_1 \right)^2 + K_1 =
$$

$$
= K_2 \int_{\tau}^{s} \left(\frac{1}{2} \sum_{j_3=0}^{p} \left(\int_{t_1}^{s} \phi_{j_3}(t_2) dt_2 \right)^2 \right)^2 dt_1 + K_1 \leq
$$

$$\leq K_2\int\limits_{\tau}^{s}\left(\frac{1}{2}\sum_{j_3=0}^{\infty}\left(\int\limits_{t_1}^{s}\phi_{j_3}(t_2)dt_2\right)^2\right)^2 dt_1+K_1=$$

$$=K_2\int\limits_{\tau}^{s}\left(\frac{1}{2}(s-t_1)\right)^2 dt_1+K_1\leq K<\infty,$$

where constants K, K_1, K_2 do not depend on p, s, τ. The equality (56) is proved.

Let us prove (57), (58). Applying (60), (54) and the above reasoning, we have

$$\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(s,\tau)\right)^2\leq 2\left(\sum_{j_1,j_2=0}^{p}C_{j_2}(s,\tau)C_{j_1j_2j_1}(s,\tau)\right)^2+$$

$$+2\cdot\frac{1}{4}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2}(s,\tau)C_{j_2j_1}(s,\tau)\right)^2\leq$$

$$\leq 2\sum_{j_2=0}^{p}\left(C_{j_2}(s,\tau)\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+$$

$$+\frac{1}{2}\sum_{j_1,j_2=0}^{p}\left(C_{j_1j_2}(s,\tau)\right)^2\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1}(s,\tau)\right)^2\leq$$

$$\leq K_2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+K_1\leq K_2\sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+K_1= \tag{65}$$

$$=K_2\sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{s}\phi_{j_1}(t_3)dt_3dt_2\right)^2+K_1=$$

$$=K_2\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{s}\phi_{j_1}(t_3)dt_3\right)^2 dt_2+K_1\leq \tag{66}$$

$$\leq K_2\int\limits_{\tau}^{s}\left((t_2-\tau)(s-t_2)\right)^2 dt_2+K_1\leq K<\infty,$$

where constants K, K_1, K_2 do not depend on p, s, τ. The equalities (57) and (58) (see (65), (66)) are proved.

Step 4. Let us start proving the equalities (26)–(40). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (26), (29), (32), (38)–(40)

$$\sum_{j_5=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_5j_3j_3j_1j_1}-\frac{1}{4}C_{j_5j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot),(j_3j_3)\curvearrowright(\cdot)}\right)^2=$$

$$=\sum_{j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{4}\int\limits_t^{t_5}(\tau-t)d\tau\right)dt_5\right)^2\le$$

$$\le\sum_{j_5=0}^{\infty}\left(\int\limits_t^T\phi_{j_5}(t_5)\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{4}\int\limits_t^{t_5}(\tau-t)d\tau\right)dt_5\right)^2=$$

$$=\int\limits_t^T\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{8}(t_5-t)^2\right)^2dt_5, \qquad (67)$$

$$\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_2j_1j_2j_1}\right)^2=\sum_{j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(t_5,t)dt_5\right)^2\le$$

$$\le\sum_{j_5=0}^{\infty}\left(\int\limits_t^T\phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(t_5,t)dt_5\right)^2=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(t_5,t)\right)^2dt_5, \qquad (68)$$

$$\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_1j_2j_2j_1}\right)^2=\sum_{j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}(t_5,t)dt_5\right)^2\le$$

$$\le\sum_{j_5=0}^{\infty}\left(\int\limits_t^T\phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}(t_5,t)dt_5\right)^2=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}(t_5,t)\right)^2dt_5, \qquad (69)$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2j_1}-\frac{1}{4}C_{j_4j_4j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\sum_{j_2,j_4=0}^{p}\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_4}(t_5)dt_5dt_4dt_3dt_2dt_1-\right.$$

$$\left.-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_5\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\left(\sum_{j_2,j_4=0}^{p}C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{4}\int\limits_{t_1}^T(T-t_3)dt_3\right)dt_1\right)^2\le$$

$$\le\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(t_1)\left(\sum_{j_2,j_4=0}^{p}C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{8}(T-t_1)^2\right)dt_1\right)^2=$$

$$=\int\limits_t^T\left(\sum_{j_2,j_4=0}^{p}C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{8}(T-t_1)^2\right)^2dt_1,\qquad(70)$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2j_1}\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4\times\right.$$

$$\left.\times dt_3dt_2dt_1\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2}(T,t_1)dt_1\right)^2\le$$

$$\le\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2}(T,t_1)dt_1\right)^2=$$

$$=\int\limits_t^T\left(\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2}(T,t_1)\right)^2dt_1,\qquad(71)$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1}\right)^2 =$$

$$=\sum_{j_1=0}^{p}\Biggl(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_3}(t_4)\int\limits_{t_4}^T \phi_{j_2}(t_5)dt_5dt_4\times$$

$$\times dt_3dt_2dt_1\Biggr)^2 =$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)dt_1\right)^2 \le$$

$$\le \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)dt_1\right)^2 = \int\limits_t^T\left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)\right)^2 dt_1. \tag{72}$$

Using Lebesgue's Dominated Convergence Theorem and (55)–(57), (62)–(64), we obtain that the right-hand sides of (67)–(72) tend to zero when $p\to\infty$. The equalities (26), (29), (32), (38)–(40) are proved.

Further, let us prove the equalities (28), (30), (33), (34), (36). Using Fubini's Theorem, Parseval's equality and Cauchy–Bunyakovsky's inequality, we have the following relations for the prelimit expressions on the left-hand sides of (28), (30), (33), (34), (36)

$$\sum_{j_3=0}^{p}\left(\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_3j_1j_1} - \frac{1}{4}C_{j_4j_4j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2 =$$

$$=\sum_{j_3=0}^{p}\Biggl(\int\limits_t^T \phi_{j_3}(t_3)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_3} \phi_{j_1}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^T \phi_{j_4}(t_4)\int\limits_{t_4}^T \phi_{j_4}(t_5)dt_5dt_4dt_3-$$

$$-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}dt_1dt_3dt_4\Biggr)^2 \le$$

$$\le \sum_{j_3=0}^{\infty}\left(\int\limits_t^T \phi_{j_3}(t_3)\left(\sum_{j_1,j_4=0}^{p}\frac{1}{4}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\left(\int\limits_{t_3}^T \phi_{j_4}(t_4)dt_4\right)^2 -\right.\right.$$

$$
-\frac{1}{4}(t_3-t)\int\limits_{t_3}^{T} dt_4\Biggr)\, dt_3\Biggr)^2 =
$$

$$
=\int\limits_t^T \left(\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{1}{4}(t_3-t)(T-t_3)\right)^2 dt_3, \tag{73}
$$

$$
\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_4j_1j_2j_1}\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_2}(t_5)dt_5dt_4\right)^2\le
$$

$$
\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_4)dt_4\right)^2=
$$

$$
=\int\limits_t^T\left(\sum_{j_2=0}^{p}\sum_{j_1=0}^{p}C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_4)\right)^2dt_4\le
$$

$$
\le\int\limits_t^T\sum_{j_2=0}^{p}\left(C_{j_2}(T,t_4)\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(t_4,t)\right)^2dt_4\le
$$

$$
\le\int\limits_t^T\sum_{j_2=0}^{\infty}\left(C_{j_2}(T,t_4)\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(t_4,t)\right)^2dt_4\le
$$

$$
\le K_1\int\limits_t^T\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(t_4,t)\right)^2dt_4\le \tag{74}
$$

$$
\le K_1\int\limits_t^T\int\limits_t^{t_4}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\right)^2dt_2dt_4= \tag{75}
$$

$$
= K_1 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<t_4\}} \left(\sum_{j_1=0}^{p} \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \phi_{j_1}(t_3)dt_3 \right)^2 dt_2 dt_4, \tag{76}
$$

where constant K_1 does not depend on p and the transition from (74) to (75) is based on (58);

$$
\sum_{j_3=0}^{p} \left(\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_2 j_1} \right)^2 =
$$

$$
= \sum_{j_3=0}^{p} \left(\int\limits_{t}^{T} \phi_{j_3}(t_3) \sum_{j_1,j_2=0}^{p} \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_1}(t_4) \int\limits_{t_4}^{T} \phi_{j_2}(t_5)dt_5 dt_4 dt_3 \right)^2 \le
$$

$$
\le \sum_{j_3=0}^{\infty} \left(\int\limits_{t}^{T} \phi_{j_3}(t_3) \sum_{j_1,j_2=0}^{p} \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2)dt_2 dt_1 dt_3 \right)^2 =
$$

$$
= \int\limits_{t}^{T} \left(\sum_{j_1,j_2=0}^{p} \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2)dt_2 dt_1 \right)^2 dt_3 =
$$

$$
= \int\limits_{t}^{T} \left(\sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}} \phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1 dt_2 \times \right.
$$

$$
\left. \times \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}} \phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1 dt_2 \right)^2 dt_3, \tag{77}
$$

where, using the generalized Parseval equality and the Cauchy–Bunyakovsky inequality, we obtain

$$
\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}} \phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1 dt_2 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}} \phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1 dt_2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}} \mathbf{1}_{\{t_2>t_1>t_3\}} dt_1 dt_2 = 0,
$$

$$\left(\sum_{j_1,j_2=0}^{p}\ \int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le$$

$$\le\sum_{j_1,j_2=0}^{p}\ \left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\times$$

$$\times\sum_{j_1,j_2=0}^{p}\ \left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le K_1<\infty,$$

where constant K_1 does not depend on p;

$$\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_3j_2j_1}\right)^2=$$

$$=\sum_{j_2=0}^{p}\left(\int\limits_t^T\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3dt_2\right)^2\le$$

$$\le\sum_{j_2=0}^{\infty}\left(\int\limits_t^T\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3dt_2\right)^2=$$

$$=\int\limits_t^T\left(\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3\right)^2dt_2=$$

$$=\int\limits_t^T\left(\sum_{j_1=0}^{p}C_{j_1}(t_2,t)\sum_{j_3=0}^{p}\int\limits_{t_2}^T\phi_{j_3}(t_5)\int\limits_{t_2}^{t_5}\phi_{j_1}(t_4)\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4dt_5\right)^2dt_2=$$

$$=\int\limits_t^T\left(\sum_{j_1=0}^{p}C_{j_1}(t_2,t)\sum_{j_3=0}^{p}C_{j_3j_1j_3}(T,t_2)\right)^2dt_2\le$$

$$\le\int\limits_t^T\sum_{j_1=0}^{p}\left(C_{j_1}(t_2,t)\right)^2\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_1j_3}(T,t_2)\right)^2dt_2\le$$

$$\leq K_1 \int\limits_t^T \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_1 j_3}(T, t_2) \right)^2 dt_2 \leq \tag{78}$$

$$\leq K_1 \int\limits_t^T \int\limits_{t_2}^T \left(\sum_{j_3=0}^{p} \int\limits_{t_2}^{\theta} \phi_{j_3}(t_1) dt_1 \int\limits_{\theta}^{T} \phi_{j_3}(t_3) dt_3 \right)^2 d\theta dt_2 = \tag{79}$$

$$= K_1 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<\theta\}} \left(\sum_{j_3=0}^{p} \int\limits_{t_2}^{\theta} \phi_{j_3}(t_1) dt_1 \int\limits_{\theta}^{T} \phi_{j_3}(t_3) dt_3 \right)^2 d\theta dt_2, \tag{80}$$

where constant K_1 does not depend on p and the transition from (78) to (79) is based on (58);

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1,j_2=0}^{p} C_{j_1 j_2 j_3 j_2 j_1} \right)^2 =$$

$$= \sum_{j_3=0}^{p} \left(\int\limits_t^T \phi_{j_3}(t_3) \sum_{j_1,j_2=0}^{p} \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_4) \int\limits_{t_4}^{T} \phi_{j_1}(t_5) dt_5 dt_4 dt_3 \right)^2 \leq$$

$$\leq \sum_{j_3=0}^{\infty} \left(\int\limits_t^T \phi_{j_3}(t_3) \sum_{j_1,j_2=0}^{p} \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \right)^2 =$$

$$= \int\limits_t^T \left(\sum_{j_1,j_2=0}^{p} \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_1}(t_1) dt_1 dt_2 \right)^2 dt_3 =$$

$$= \int\limits_t^T \left(\sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}} \phi_{j_2}(t_2) \phi_{j_1}(t_1) dt_1 dt_2 \times \right.$$

$$\left. \times \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1>t_2>t_3\}} \phi_{j_2}(t_2) \phi_{j_1}(t_1) dt_1 dt_2 \right)^2 dt_3, \tag{81}$$

where, using the generalized Parseval equality and the Cauchy–Bunyakovsky inequality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_1>t_2>t_3\}}dt_1dt_2=0,$$

$$\left(\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le$$

$$\le\sum_{j_1,j_2=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\times$$

$$\times\sum_{j_1,j_2=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le K_1<\infty,$$

where constant K_1 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem, we obtain that the right-hand sides of (73), (76), (77), (80), (81) tend to zero when $p\to\infty$. The equalities (28), (30), (33), (34), (36) are proved.

Step 5. Finally, let us prove the equalities (27), (31), (35), (37). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2\le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (27)

$$\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_4j_3j_1j_1}\right)^2=$$

$$=\sum_{j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2\le$$

$$\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2=$$

$$= \int\limits_t^T \left(\sum_{j_1,j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 =$$

$$= \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 \mp \frac{t_3 - t}{2} \right) dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \le$$

$$\le 2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 +$$

$$+2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \le$$

$$\le 2 \int\limits_t^T \sum_{j_3=0}^{p} \left(C_{j_3}(T, t_4) \right)^2 \times$$

$$\times \sum_{j_3=0}^{p} \left(\int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \right)^2 dt_4 + \varepsilon_p \le$$

$$\le K_1 \int\limits_t^T \sum_{j_3=0}^{p} \left(\int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \right)^2 dt_4 + \varepsilon_p \le$$

$$\le K_1 \int\limits_t^T \sum_{j_3=0}^{\infty} \left(\int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \right)^2 dt_4 + \varepsilon_p =$$

$$= K_1 \int\limits_t^T \int\limits_t^{t_4} \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right)^2 dt_3 dt_4 + \varepsilon_p =$$

$$= K_1 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3 < t_4\}} \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right)^2 dt_3 dt_4 + \varepsilon_p, \qquad (82)$$

where constant K_1 does not depend on p,

$$\varepsilon_p = 2\int\limits_t^T \left(\sum_{j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4.$$

By analogy with (52), (54) we get

$$\left(\sum_{j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 \le K_2<\infty, \tag{83}$$

$$\sum_{j_3=0}^{\infty}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5 = 0, \tag{84}$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (51), (53), (83), (84), we obtain that the right-hand side of (82) tends to zero when $p\to\infty$. The equality (27) is proved.

Let us prove the equality (31). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2\le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (31)

$$\sum_{j_2=0}^{p}\left(\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_2j_1}\right)^2 =$$

$$= \sum_{j_2=0}^{p}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_4}(t_4)\int\limits_{t_4}^{T}\phi_{j_4}(t_5)dt_5dt_4dt_3dt_2\right)^2 \le$$

$$\le \sum_{j_2=0}^{\infty}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_4}(t_4)\int\limits_{t_4}^{T}\phi_{j_4}(t_5)dt_5dt_4dt_3dt_2\right)^2 =$$

$$= \int\limits_t^T \left(\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_4}(t_4)\int\limits_{t_4}^{T}\phi_{j_4}(t_5)dt_5dt_4dt_3\right)^2 dt_2 =$$

$$= \int\limits_t^T \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{T} \phi_{j_1}(t_3) \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 \mp \frac{T-t_3}{2} \right) dt_3 \right)^2 dt_2 \le$$

$$\le 2 \int\limits_t^T \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{T} \phi_{j_1}(t_3) \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right) dt_3 \right)^2 dt_2 +$$

$$+2 \int\limits_t^T \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{T} \phi_{j_1}(t_3) \frac{T-t_3}{2} dt_3 \right)^2 dt_2 \le$$

$$\le 2 \int\limits_t^T \sum_{j_1=0}^{p} \left(C_{j_1}(t_2,t)\right)^2 \times$$

$$\times \sum_{j_1=0}^{p} \left(\int\limits_{t_2}^{T} \phi_{j_1}(t_3) \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right) dt_3 \right)^2 dt_2 + \mu_p \le$$

$$\le K_1 \int\limits_t^T \sum_{j_1=0}^{p} \left(\int\limits_{t_2}^{T} \phi_{j_1}(t_3) \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right) dt_3 \right)^2 dt_2 + \mu_p \le$$

$$\le K_1 \int\limits_t^T \sum_{j_1=0}^{\infty} \left(\int\limits_{t_2}^{T} \phi_{j_1}(t_3) \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right) dt_3 \right)^2 dt_2 + \mu_p =$$

$$= K_1 \int\limits_t^T \int\limits_{t_2}^T \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right)^2 dt_3 dt_2 + \mu_p =$$

$$= K_1 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<t_3\}} \left(\frac{1}{2} \sum_{j_4=0}^{p} \left(\int\limits_{t_3}^{T} \phi_{j_4}(t_4)dt_4 \right)^2 - \frac{T-t_3}{2} \right)^2 dt_3 dt_2 + \mu_p, \qquad (85)$$

where constant K_1 does not depend on p,

$$\mu_p = 2 \int\limits_t^T \left(\sum_{j_1=0}^{p} \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{T} \phi_{j_1}(t_3) \frac{T-t_3}{2} dt_3 \right)^2 dt_2.$$

By analogy with (52), (54) we get

$$\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3\right)^2\le K_2<\infty, \qquad (86)$$

$$\sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3=0, \qquad (87)$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (51), (53), (86), (87), we obtain that the right-hand side of (85) tends to zero when $p\to\infty$. The equality (31) is proved.

Let us prove the equality (35). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2\le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (35)

$$\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_4j_2j_2j_1}\right)^2=$$

$$=\sum_{j_4=0}^{p}\left(\int\limits_t^{T}\phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4\right)^2\le$$

$$\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^{T}\phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4\right)^2=$$

$$=\int\limits_t^{T}\left(\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4=$$

$$=\int\limits_t^{T}\left(\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4=$$

$$=\int\limits_t^{T}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2\mp\frac{t_4-t_1}{2}\right)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4\le$$

$$
\leq 2\int\limits_t^T \left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_4+
$$

$$
+2\int\limits_t^T \left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_4 \leq
$$

$$
\leq 2\int\limits_t^T \sum_{j_1=0}^{p}\left(C_{j_1}(T,t_4)\right)^2 \times
$$

$$
\times \sum_{j_1=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p \leq
$$

$$
\leq K_1\int\limits_t^T \sum_{j_1=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p \leq
$$

$$
\leq K_1\int\limits_t^T \sum_{j_1=0}^{\infty}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p =
$$

$$
= K_1\int\limits_t^T \int\limits_t^{t_4}\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1 dt_4+\rho_p =
$$

$$
= K_1\int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_4\}}\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1 dt_4+\rho_p, \qquad (88)
$$

where constant K_1 does not depend on p,

$$
\rho_p = 2\int\limits_t^T \left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_4.
$$

By analogy with (52), (54) we get $(t_4-t_1=(t_4-t)+(t-t_1))$

$$\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2\le K_2<\infty, \tag{89}$$

$$\sum_{j_1=0}^{\infty}\int\limits_{t}^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5=0, \tag{90}$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (51), (53), (89), (90), we obtain that the right-hand side of (88) tends to zero when $p\to\infty$. The equality (35) is proved.

Let us prove the equality (37). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2\le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (37)

$$\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_2j_1}\right)^2=$$

$$=\sum_{j_2=0}^{p}\left(\int\limits_{t}^{T}\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4dt_3dt_2\right)^2\le$$

$$\le\sum_{j_2=0}^{\infty}\left(\int\limits_{t}^{T}\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4dt_3dt_2\right)^2=$$

$$=\int\limits_{t}^{T}\left(\sum_{j_1,j_3=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4dt_3\right)^2dt_2=$$

$$=\int\limits_{t}^{T}\left(\sum_{j_1,j_3=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4dt_5\right)^2dt_2=$$

$$=\int\limits_{t}^{T}\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2\mp\frac{t_5-t_2}{2}\right)dt_5\right)^2dt_2\le$$

$$\leq 2\int\limits_t^T\left(\sum_{j_1=0}^p\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2dt_2+$$

$$+2\int\limits_t^T\left(\sum_{j_1=0}^p\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2dt_2\leq$$

$$\leq 2\int\limits_t^T\sum_{j_1=0}^p\left(C_{j_1}(t_2,t)\right)^2\times$$

$$\times\sum_{j_1=0}^p\left(\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2dt_2+\chi_p\leq$$

$$\leq K_1\int\limits_t^T\sum_{j_1=0}^p\left(\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2dt_2+\chi_p\leq$$

$$\leq K_1\int\limits_t^T\sum_{j_1=0}^\infty\left(\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2dt_2+\chi_p=$$

$$=K_1\int\limits_t^T\int\limits_{t_2}^T\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)^2dt_5dt_2+\chi_p=$$

$$=K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_5\}}\left(\frac{1}{2}\sum_{j_3=0}^p\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)^2dt_5dt_2+\chi_p,\qquad(91)$$

where constant K_1 does not depend on p,

$$\chi_p=2\int\limits_t^T\left(\sum_{j_1=0}^p\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2dt_2.$$

By analogy with (52), (54) we get $(t_5-t_2=(t_5-t)+(t-t_2))$

$$\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2 \le K_2<\infty, \tag{92}$$

$$\sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5=0, \tag{93}$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (51), (53), (92), (93), we obtain that the right-hand side of (91) tends to zero when $p\to\infty$. The equality (37) is proved. The equalities (16)–(40) are proved. Theorem 4 is proved.

3.2 On the Calculation of Matrix Traces of Volterra–Type Integral Operators

Consider the so-called factorized Volterra–Type kernel (9), where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. It is easy to see that the function (9) for even $k=2r$ $(r\in\mathbb{N})$ forms a family of integral operators $\mathbb{K}: L_2([t,T]^r)\to L_2([t,T]^r)$ (with the kernel (9)) of the form

$$\left(\mathbb{K}f\right)(t_{g_1},\ldots,t_{g_r})=\int\limits_{[t,T]^r}K(t_1,\ldots,t_k)f(t_{g_{r+1}},\ldots,t_{g_k})dt_{g_{r+1}}\ldots dt_{g_k}, \tag{94}$$

where $\{g_1,\ldots,g_k\}=\{1,\ldots,k\}$, $k=2r$, the kernel $K(t_1,\ldots,t_k)$ is defined by (9), $f(t_{g_{r+1}},\ldots,t_{g_k})\in L_2([t,T]^r)$. For example,

$$\left(\mathbb{K}f\right)(t_2,t_3)=\int\limits_{[t,T]^2}K(t_1,\ldots,t_4)f(t_1,t_4)dt_1dt_4=$$

$$=\psi_2(t_2)\psi_3(t_3)\mathbf{1}_{\{t_2<t_3\}}\int\limits_t^{t_2}\psi_1(t_1)\int\limits_{t_3}^{T}\psi_4(t_4)f(t_1,t_4)dt_4dt_1. \tag{95}$$

Consider the matrix trace of integral operator defined by (95)

$$\sum_{j_1,j_2=0}^{\infty}\left\langle\mathbb{K}\Psi_{j_1j_2},\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)}=$$

$$
= \sum_{j_1,j_2=0}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1 dt_2 dt_3 dt_4 =
$$

$$
= \sum_{j_1,j_2=0}^{\infty} C_{j_2 j_2 j_1 j_1}, \tag{96}
$$

where $\{\Psi_{j_1 j_2}(x,y)\}_{j_1,j_2=0}^{\infty} = \{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$.

As we have already seen, matrix traces of integral operators (94) play an important role when proving expansions for iterated Stratonovich integrals. In [14] (Sect. 2.27) it was explained why calculation the above matrix traces is a problem (see also [56]). Further, we will develop the results from [17] (Sect. 3.9).

Recall that in [56], the following relation

$$
\lim_{p\to\infty} \sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{p} C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2 j_2} = \frac{1}{2^r}\int\limits_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times
$$

$$
\times \int\limits_t^{t_{2r}} \psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots \int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}, \tag{97}
$$

is proved, where $r\in\mathbb{N}$, $C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2 j_2}$ is defined by (8), $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$, and $\psi_1(\tau),\ldots,\psi_{2r}(\tau)\in L_2([t,T])$.

Let us prove the equality (97) by another method, namely by induction. The case $r=1$ is considered in [56]. Thus, the equality (97) is true for $r=1$. Suppose that the equality (97) is true for some $r>1$. Then, using the induction hypothesis and (97) for $r=1$, we get

$$
\lim_{p\to\infty} \sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p} \int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2}) \int\limits_t^{t_{2r+2}} \psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1})\times
$$

$$
\times \int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r}) \int\limits_t^{t_{2r}} \psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1})\ldots
$$

$$
\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_2}(t_1)dt_1 dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2} =
$$

$$= \sum_{j_{2r+2}=0}^{\infty} \int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2}) \int\limits_t^{t_{2r+2}} \psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1})dt_{2r+1}dt_{2r+2}\times$$

$$\times \sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{\infty} \int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r}) \int\limits_t^{t_{2r}} \psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1})\times$$

$$\times \int\limits_t^{t_{2r-1}} \psi_{2r-2}(t_{2r-2})\phi_{j_{2r-2}}(t_{2r-2}) \int\limits_t^{t_{2r-2}} \psi_{2r-3}(t_{2r-3})\phi_{j_{2r-2}}(t_{2r-3})\ldots$$

$$\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_2}(t_1)dt_1dt_2\ldots dt_{2r-3}dt_{2r-2}dt_{2r-1}dt_{2r} =$$

$$= \frac{1}{2}\int\limits_t^T \psi_{2r+2}(t_{2r+2})\psi_{2r+1}(t_{2r+2})dt_{2r+2}\cdot\frac{1}{2^r}\int\limits_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times$$

$$\times \int\limits_t^{t_{2r}} \psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots \int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}. \qquad (98)$$

Let us rewrite the equality (98) in the form

$$\lim_{p\to\infty} \sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p} \int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2}) \int\limits_t^{t_{2r+2}} \psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1})\times$$

$$\times \int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r}) \int\limits_t^{t_{2r}} \psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1})\ldots$$

$$\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_2}(t_1)dt_1dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2} =$$

$$= \frac{1}{2^{r+1}}\int\limits_t^T \psi_{2r+2}(t_{2r+2})\psi_{2r+1}(t_{2r+2}) \int\limits_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times$$

$$\times \int\limits_t^{t_{2r}} \psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots \int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}dt_{2r+2}, \qquad (99)$$

where $\psi_1(\tau),\ldots,\psi_{2r+2}(\tau)\in L_2([t,T])$.

Suppose that $\psi_1(\tau),\psi_3(\tau),\ldots,\psi_{2r-3}(\tau),\psi_{2r}(\tau),\psi_{2r+1}(\tau)$ in (99) are Legendre polynomials of finite degrees. Denote

$$h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})=$$
$$=\psi_2(t_2)\psi_4(t_4)\ldots\psi_{2r-2}(t_{2r-2})\psi_{2r-1}(t_{2r-1})\psi_{2r+2}(t_{2r+2}),$$

$$g(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})=$$
$$=\bar{\psi}_1(t_1)\bar{\psi}_3(t_3)\ldots\bar{\psi}_{2r-3}(t_{2r-3})\bar{\psi}_{2r}(t_{2r})\bar{\psi}_{2r+1}(t_{2r+1})\mathbf{1}_{\{t_{2r}<t_{2r+1}\}}, \qquad (100)$$

$$s_q(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})=$$
$$=\sum_{l_1,\ldots,l_{r+1}=0}^{q} C_{l_{r+1}\ldots l_1}\bar{\phi}_{l_1}(t_1)\bar{\phi}_{l_2}(t_3)\ldots\bar{\phi}_{l_{r-1}}(t_{2r-3})\bar{\phi}_{l_r}(t_{2r})\bar{\phi}_{l_{r+1}}(t_{2r+1}), \qquad (101)$$

where $C_{l_{r+1}\ldots l_1}$ are Fourier–Legendre coefficients for the function (100), $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2([t,T])$, and $\bar{\psi}_1(\tau),\bar{\psi}_3(\tau)$, $\ldots,\bar{\psi}_{2r-3}(\tau),\bar{\psi}_{2r}(\tau),\bar{\psi}_{2r+1}(\tau)\in L_2([t,T])$. Then we have

$$\lim_{q\to\infty}\left\|s_q-g\right\|^2_{L_2([t,T]^{r+1})}=0.$$

From (99) we obtain (the sum on the right-hand side of (101) is finite)

$$\lim_{p\to\infty}\sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p}\ \int\limits_{[t,T]^{2r+2}}\mathbf{1}_{\{t_1<t_2<\ldots<t_{2r}\}}\mathbf{1}_{\{t_{2r+1}<t_{2r+2}\}}s_q(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})\times$$
$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})\times$$
$$\times\prod_{d=1}^{r+1}\phi_{j_{2d}}(t_{2d-1})\phi_{j_{2d}}(t_{2d})dt_1dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2}=$$
$$=\frac{1}{2^{r+1}}\int\limits_{[t,T]^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}s_q(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})\times$$
$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})dt_2dt_4\ldots dt_{2r-2}dt_{2r}dt_{2r+2}. \qquad (102)$$

The right-hand side of the equality (102) defines (as a scalar product of $s_q(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})$ and

$$\frac{1}{2^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})$$

in $L_2([t,T]^{r+1})$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^{r+1})$. The mentioned functional is given by the function

$$\frac{1}{2^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2}).$$

Note that the equality (102) will also remain true if s_q is replaced by $\bar{s}_q$ ($\bar{s}_q$ is the partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^{r+1})$), i.e. the modified equality (102) is true on a dense subset in $L_2([t,T]^{r+1})$. On the left-hand side of (102) (by virtue of the equality (102)) there is a linear continuous functional on a dense subset in $L_2([t,T]^{r+1})$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^{r+1})$ (see [57], Theorem I.7, P. 9). Thus, we have the equality of two linear continuous functionals in $L_2([t,T]^{r+1})$. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in the mentioned equality if instead of $\bar{s}_q$ we choose s_q of the form (101) (i.e. passage to the limit $\lim\limits_{q\to\infty}$ in (102))

$$\lim_{p\to\infty}\sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p}\ \int\limits_{[t,T]^{2r+2}}\mathbf{1}_{\{t_1<t_2<\ldots<t_{2r}\}}\mathbf{1}_{\{t_{2r+1}<t_{2r+2}\}}g(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})\times$$

$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})\times$$

$$\times\prod_{d=1}^{r+1}\phi_{j_{2d}}(t_{2d-1})\phi_{j_{2d}}(t_{2d})dt_1dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2}=$$

$$=\frac{1}{2^{r+1}}\int\limits_{[t,T]^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}g(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})\times$$

$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})dt_2dt_4\ldots dt_{2r-2}dt_{2r}dt_{2r+2},\qquad(103)$$

where $\bar{\psi}_1(\tau),\bar{\psi}_3(\tau),\ldots,\bar{\psi}_{2r-3}(\tau)\bar{\psi}_{2r}(\tau),\bar{\psi}_{2r+1}(\tau)\in L_2([t,T])$.

It is easy to see that the equality (103) (up to notations) is the equality (97) in which r is replaced by $r+1$. So, we proved the equality (97) by induction.

Note that the series on the left-hand side of (97) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$).

Further, let us show that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \qquad (104)$$

for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)), where $k = 2r$ ($r = 2, 3, \ldots$), $C_{j_k\ldots j_1}$ is defined by (8), another notations are the same as in (97).

The case

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=1$$

corresponds to (97). Thus, it remains to prove that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=0 \qquad (105)$$

for the case

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=0.$$

Below we consider two examples that clearly explain the algorithm for the proof of equality (105). After this we will formulate the algorithm.

First, let us prove that

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p} C_{j_3j_4j_4j_3j_1j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{t_6}\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=0, \qquad (106)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 1. Using (97) ($r=1$) and generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^T\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6= \tag{107}$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)dt_6\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)dt_3\times$$

$$\times\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)dt_4dt_5\times$$

$$\times\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_t^T\psi_6(t_6)\psi_3(t_6)dt_6\cdot\frac{1}{2}\int\limits_t^T\psi_5(t_4)\psi_4(t_4)dt_4\cdot\frac{1}{2}\int\limits_t^T\psi_2(t_2)\psi_1(t_2)dt_2. \tag{108}$$

Let us rewrite (108) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^T\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_t^T\psi_6(t_6)\psi_3(t_6)\int\limits_t^T\psi_5(t_4)\psi_4(t_4)\int\limits_t^T\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \tag{109}$$

Step 2. Suppose that $\psi_2(\tau),\psi_3(\tau),\psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2,t_3,t_4)=\sum_{l_1,l_2.l_3=0}^{q}C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3)\bar{\phi}_{l_3}(t_4), \tag{110}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2([t,T])$ and $C_{l_3 l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3,t_4)=\bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\times$ $\times\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0.$

From (109) we obtain (the sum on the right-hand side of (110) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\ \int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_4<t_5\}}s_q(t_2,t_3,t_4)\psi_6(t_6)\psi_5(t_5)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\times$$

$$\times\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}s_q(t_2,t_6,t_4)\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)dt_2dt_4dt_6. \tag{111}$$

Note that the equality (111) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (111) defines (as a scalar product of $s_q(t_2,t_6,t_4)$ and $\frac{1}{4}\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$. On the left-hand side of (111) (by virtue of the equality (111)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [57], Theorem I.7, P. 9). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (111) (at that we suppose that s_q is defined by (110))

$$\sum_{j_1,j_3,j_4=0}^{\infty}\ \int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\bar{\psi}_3(t_6)\psi_5(t_4)\bar{\psi}_4(t_4)\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \tag{112}$$

Rewrite the equality (112) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6, \qquad (113)$$

where $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 3. Suppose that $\psi_3(\tau),\psi_4(\tau),\psi_1(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4,t_1)=\sum_{l_1,l_2.l_3=0}^{q}C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4)\bar{\phi}_{l_3}(t_1), \qquad (114)$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (110) and $C_{l_3l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4,t_1)=\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\bar{\psi}_1(t_1)\mathbf{1}_{\{t_3<t_4\}}$ $(\bar{\psi}_3(\tau),\bar{\psi}_4(\tau),\bar{\psi}_1(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0$.

From (113) we obtain (the sum on the right-hand side of (114) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}s_q(t_3,t_4,t_1)\psi_6(t_6)\psi_5(t_5)\psi_2(t_2)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}s_q(t_6,t_4,t_2)\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)dt_2dt_4dt_6. \qquad (115)$$

Note that the equality (115) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (115) defines (as a scalar product of $s_q(t_6,t_4,t_2)$ and $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)$. On the left-hand side of (115) (by virtue of the equality (115)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [57], Theorem I.7, P. 9). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (115) (at that we suppose that s_q is defined by (114))

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\psi_2(t_2)\bar{\psi}_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_6(t_6)\bar{\psi}_3(t_6)\psi_5(t_4)\bar{\psi}_4(t_4)\psi_2(t_2)\bar{\psi}_1(t_2)dt_2dt_4dt_6. \qquad (116)$$

Rewrite (116) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6, \qquad (117)$$

where $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 4. Suppose that $\psi_5(\tau),\psi_6(\tau),\psi_2(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_5,t_6,t_2)=\sum_{l_1,l_2.l_3=0}^{q}C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_5)\bar{\phi}_{l_2}(t_6)\bar{\phi}_{l_3}(t_2), \qquad (118)$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (110) and $C_{l_3l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_5,t_6,t_2)=\bar{\psi}_5(t_5)\bar{\psi}_6(t_6)\bar{\psi}_2(t_2)\mathbf{1}_{\{t_5<t_6\}}$ $(\bar{\psi}_5(\tau),\bar{\psi}_6(\tau),\bar{\psi}_2(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0$.

From (117) we obtain (the sum on the right-hand side of (118) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}s_q(t_5,t_6,t_2)\psi_4(t_4)\psi_3(t_3)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$
$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$
$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}s_q(t_4,t_6,t_2)\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)dt_2dt_4dt_6. \qquad (119)$$

Note that the equality (119) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (119) defines (as a scalar product of $s_q(t_4,t_6,t_2)$ and $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)$. On the left-hand side of (119) (by virtue of the equality (119)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [57], Theorem I.7, P. 9). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (119) (at that we suppose that s_q is defined by (118))

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5<t_6\}}\bar{\psi}_6(t_6)\bar{\psi}_5(t_5)\psi_4(t_4)\psi_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$
$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$
$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\mathbf{1}_{\{t_4<t_6\}}\bar{\psi}_6(t_6)\psi_3(t_6)\bar{\psi}_5(t_4)\psi_4(t_4)\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4dt_6=0. \qquad (120)$$

It is obvious that the equality (120) (up to notations) is (106). The equality (106) is proved.

As a second example, we will prove the equality

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}=0. \tag{121}$$

In this case, we will use the same approach as in the proof of equality (106).

Step 1. Using generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4= \tag{122}$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)dt_2\times$$

$$\times\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)dt_1=$$

$$=\int\limits_t^T\psi_4(t_4)\psi_2(t_4)dt_4\int\limits_t^T\psi_3(t_3)\psi_1(t_3)dt_3. \tag{123}$$

Rewrite the equality (123) in the form

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\psi_4(t_4)\psi_2(t_4)\psi_3(t_2)\psi_1(t_2)dt_2dt_4. \tag{124}$$

Step 2. Suppose that $\psi_1(\tau),\psi_2(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_1,t_2)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_1)\bar{\phi}_{l_2}(t_2),$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (110), $C_{l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_1,t_2)=\bar{\psi}_1(t_1)\bar{\psi}_2(t_2)\mathbf{1}_{\{t_1<t_2\}}$ $(\bar{\psi}_1(\tau),\bar{\psi}_2(\tau)\in L_2([t,T]))$.

From (124) we obtain

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4} s_q(t_1,t_2)\psi_4(t_4)\psi_3(t_3)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2} s_q(t_2,t_4)\psi_4(t_4)\psi_3(t_2)dt_2dt_4. \qquad (125)$$

The left-hand and right-hand sides of (125) define linear continuous functionals in $L_2([t,T]^2)$ (see explanation earlier in this section). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (125)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\psi_4(t_4)\psi_3(t_3)\bar{\psi}_2(t_2)\bar{\psi}_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4(t_4)\bar{\psi}_2(t_4)\psi_3(t_2)\bar{\psi}_1(t_2)dt_2dt_4. \qquad (126)$$

Rewrite the equality (126) in the form

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4(t_4)\psi_2(t_4)\psi_3(t_2)\psi_1(t_2)dt_2dt_4, \qquad (127)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Step 3. Suppose that $\psi_2(\tau),\psi_3(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2,t_3)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3),$$

where $\{\bar{\phi}_j(x)\}_{j=0}^{\infty}$ as in (110), $C_{l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3)=\bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau)\in L_2([t,T]))$.

From (127) we obtain

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}s_q(t_2,t_3)\psi_4(t_4)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}s_q(t_4,t_2)\psi_4(t_4)\psi_1(t_2)dt_2dt_4. \qquad (128)$$

The left-hand and right-hand sides of (128) define linear continuous functionals in $L_2([t,T]^2)$. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (128)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\mathbf{1}_{\{t_4<t_2\}}\psi_4(t_4)\bar{\psi}_2(t_4)\bar{\psi}_3(t_2)\psi_1(t_2)dt_2dt_4=0. \qquad (129)$$

Rewrite the equality (129) in the form

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0. \qquad (130)$$

Step 4. Suppose that $\psi_3(\tau),\psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4),$$

where $\{\bar{\phi}_j(x)\}_{j=0}^{\infty}$ as in (110), $C_{l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4)=\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_3<t_4\}}$ $(\bar{\psi}_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T]))$.

From (130) we obtain

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}s_q(t_3,t_4)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0. \qquad (131)$$

The left-hand and right-hand sides of (131) define linear continuous functionals in $L_2([t,T]^2)$ (we interpret the right-hand side of (131) as the zero functional in $L_2([t,T]^2)$). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (131)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0. \qquad (132)$$

It is easy to see that the equality (132) (up to notations) is the equality (121). The equality (121) is proved.

Let us formulate the ideas used when considering the two above examples in the form of an algorithm.

Algorithm 1. Step 1. Suppose $k=2r$ $(r=2,3,\ldots)$, where r is the number of pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (7)). Let us select blocks in the multi-index $j_k\ldots j_1$ that correspond to the fulfillment of the condition

$$\prod_{l=1}^{r_d}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=1,$$

where r_d is the number of pairs (see (7)) in the block with number d.

Step 2. Let us write the Volterra–type kernel (9) in the form

$$K(t_1,\ldots,t_k)=\psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2<t_3\}}\ldots\mathbf{1}_{\{t_{k-1}<t_k\}}, \qquad (133)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$, $k\geq 4$.

Let us save multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the expression (133) that correspond to the above blocks. At that, we remove the remaining multipliers

of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ from the expression (133). As a result, we get a modified kernel $\bar{K}(t_1,\ldots,t_k)$. Let us write an analogue of the left-hand side of equality (105) for the modified kernel $\bar{K}(t_1,\ldots,t_k)$ (see (107) and (122) as examples). For definiteness, let us denote this expression by $(^{-})$.

Step 3. Using generalized Parseval's equality and (97), we represent the expression $(^{-})$ as an integral over the hypercube $[t,T]^r$ (see the right-hand sides of (109) and (124) as examples). For definiteness, let us denote the obtained equality by $(\bar{K})$ ((109) and (124) are examples of $(\bar{K})$).

Step 4. Further, transformations and passages to the limit in the equality $(\bar{K})$ are performed iteratively in such a way as to restore the removed multipliers $\mathbf{1}_{\{t_n<t_{n+1}\}}$ on the left-hand side of $(\bar{K})$ (for more details, see the proof of formulas (106), (121)). As a result, we obtain the equality (105). More precisely, we can move from left to right along a multi-index corresponding to the left-hand side of $(\bar{K})$. Let us assume that at the n-th step we need to restore the multiplier $\mathbf{1}_{\{t_n<t_{n+1}\}}$. Then the function g (see the proof of formulas (106), (121)) will be the product of $\mathbf{1}_{\{t_n<t_{n+1}\}}\psi_n(t_n)\psi_{n+1}(t_{n+1})$ and $r-2$ weight functions that are chosen so that on the right-hand side of the equality $(\bar{K})$ there is a scalar product in $L_2([t,T]^r)$ involving s_q (s_q is an approximation of g).

Using the above algorithm, we prove the equality (104) for the case $k=2r$ ($r=2,3,\ldots$). The equality (104) is proved.

Note that the series on the left-hand side of (104) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$).

Let us generalize (104) to the case $k\ge 2r$, i.e. we prove that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \qquad (134)$$

for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)), where $k \ge 2r$, $r = 1, 2, \ldots, [k/2]$, $C_{j_k \ldots j_1}$ is defined by (8), another notations are the same as in Theorem 2.

Moreover (assuming that (134) is proved), the series on the left-hand side of (134) converges absolutly (the case $k = 2r$; this case is considered above) and converges absolutly for any fixed $j_1, \ldots, j_q, \ldots, j_k$ and $q \ne g_1, g_2, \ldots, g_{2r-1}$, g_{2r} (the case $k > 2r$) since its sum does not depend on permutations of basis functions (here the basis in $L_2([t, T]^r)$ is $\{\phi_{j_1}(x_1) \ldots \phi_{j_r}(x_r)\}_{j_1, \ldots, j_r = 0}^{\infty}$).

Using Fubini's Theorem, we obtain

$$\int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots$$

$$\ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l) dt_l \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l) dt_l \right) \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l \right) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-2}dt_{l-1}dt_{l+1} \ldots dt_k, \tag{135}$$

where $2 < l < k-1$ and $h_1(\tau), \ldots, h_k(\tau) \in L_2([t,T])$.

By analogy with (135) we have for $l = k$

$$\int\limits_t^T h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}dt_l =$$

$$= \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^T h_l(t_l)dt_l dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l \right) \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1} \ldots dt_2 dt_1 -$$

$$- \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2) \ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l \right) dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l \right) \int\limits_t^T h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1} -$$

$$- \int\limits_t^T h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l \right) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots \int\limits_t^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}. \tag{136}$$

We will assume that for $l = 1$ the transformation (135) is not carried out since

$$\int\limits_t^{t_2} h_1(t_1)dt_1$$

is the innermost integral on the left-hand side of (135). The formulas (135), (136) will be used further.

Let us carry out the transformations (135), (136) for

$$C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \qquad (k>2r)$$

iteratively for $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\,g_{2r})$. As a result, we obtain

$$C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right), \qquad (137)$$

where some terms in the sum

$$\sum_{d=1}^{2^{k-2r}}$$

can be identically equal to zero due to the remark to (135), (136).

Using (137), we obtain

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)=$$

$$=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right). \qquad (138)$$

Further, consider 3 possible cases.

Case 1. The quantities

$$\hat{C}^{(d)}_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},\quad \bar{C}^{(d)}_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{139}$$

are such that

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=1 \tag{140}$$

for $d=1,2,\ldots,2^{k-2r}$ and

$$C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{141}$$

is such that the condition (140) is fulfilled for (141).

Case 2. The quantities (139) are such that the condition (140) is satisfied for $d=1,2,\ldots,2^{k-2r}$ and (141) is such that the condition

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=0 \tag{142}$$

is fulfilled for (141).

Case 3. The quantities (139) are such that the condition (142) is satisfied for $d=1,2,\ldots,2^{k-2r}$ and (141) is such that the condition (142) is fulfilled for (141).

For Case 1, applying (104) ($k=2r$) and (138), we get for any fixed $j_1,\ldots,$ $j_q,\ldots,j_k$ ($q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r}$)

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\ \bar{C}^{(d)}_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)=$$

$$
= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \times
$$

$$
\times \left(\hat{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} - \right.
$$

$$
\left. - \ \bar{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} \right) = \qquad (143)
$$

$$
= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \left(\hat{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} - \right.
$$

$$
\left. - \ \bar{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} \right), \qquad (144)
$$

where $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (7), $k > 2r$, $r = 1, 2, \ldots, [k/2]$ (it is not difficult to see that the left-hand side of (140) is a constant for the quantities (139) for all $d = 1, 2, \ldots, 2^{k-2r}$).

Using (135), (136), we obtain

$$
\sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \left(\hat{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} - \right.
$$

$$
\left. - \ \bar{C}^{(d)}_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}} \right) =
$$

$$
= \frac{1}{2^r} C_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}= \ j_{g_2},\ldots,j_{g_{2r-1}}= \ j_{g_{2r}}}. \qquad (145)
$$

Combining (144) and (145), we have for any fixed $j_1, \ldots, j_q, \ldots, j_k$ ($q \neq g_1, g_2, \ldots, g_{2r-1},\ g_{2r}$)

$$
\lim_{p \to \infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1} \Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$= \frac{1}{2^r} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}}, \tag{146}$$

where $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (7), $k > 2r$, $r = 1, 2, \ldots, [k/2]$.

From (104) ($k = 2r$) and (146) ($k > 2r$) we obtain (134) for the case $k \geq 2r$. The equality (134) is proved for Case 1.

For Case 2, applying (104) ($k = 2r$) and (138), we get (144) for any fixed $j_1, \ldots, j_q, \ldots, j_k$ ($q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}$). Further, note that

$$\hat{C}^{(d)}_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} =$$

$$= \bar{C}^{(d)}_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} \tag{147}$$

for Case 2. Combining (144) and (147), we obtain (Case 2) for any fixed $j_1, \ldots, j_q, \ldots, j_k$ ($q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}$)

$$\lim_{p \to \infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}} = 0}^{p} C_{j_k \ldots j_1} \Bigg|_{j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} = 0. \tag{148}$$

From (104) ($k = 2r$) and (148) ($k > 2r$) we obtain (148) for the case $k \geq 2r$. The equality (134) is proved for Case 2.

For Case 3, applying (104) ($k = 2r$) and (138), we get (143) for any fixed $j_1, \ldots, j_q, \ldots, j_k$ ($q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}$). Since

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l} = g_{2l-1} + 1\}} = 0 \tag{149}$$

for Case 3, then from (143) we get (148) for $k > 2r$ (recall that the left-hand side of (149) is a constant for the quantities (139) for all $d = 1, 2, \ldots, 2^{k-2r}$). From (104) for $k = 2r$ and (148) for $k > 2r$ (Case 3) we obtain (148) for $k \geq 2r$ (Case 3). The equality (134) is proved for Case 3. The equality (134) is proved.

3.3 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$). The Case of an Arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$

Consider the following condition

$$\exists \lim_{p,q\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2 < \infty \tag{150}$$

for all $r = 1, 2, \ldots, [k/2]$ and for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)), where

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty} \stackrel{\mathsf{def}}{=} \lim_{q\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q};$$

another notations are the same as in Theorem 2.

This section is devoted to the following two theorems.

Theorem 5 [14]. *Suppose that the condition* (150) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ ($k \in \mathbb{N}$) *of iterated Itô stochastic integrals defined by* (11) *the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (8), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_{\tau}^{(0)} = \tau$.

Theorem 6 [14]. *Suppose that the condition* (150) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* (3) *of multiplicity* k $(k \in \mathbb{N})$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \qquad (151)$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (8), l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_{\tau}^{(0)} = \tau$.

Note that Theorem 6 is a corollary of Theorem 5 (see Theorem 1).

Proof of Theorems 5, 6. It is easy to see that Theorem 5 will be proved if we prove that (see Theorem 2 for the case $p_1 = \ldots = p_k = p$)

$$\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}} \right)^2 = 0$$

(152)

for all $r = 1, 2, \ldots, [k/2]$ and for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)).

Using the condition (150), theorem on reducing of a limit to iterated one and the equality (134) ($k > 2r$), we obtain

$$\lim_{p,q\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}} \right)^2 =$$

$$= \lim_{q\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q} \lim_{p\to\infty} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}} \right)^2 = 0.$$

Thus, we get

$$\lim_{p,q\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}} \right)^2 = 0. \tag{153}$$

Substituting $p = q$ in (153), we obtain (152). Theorems 5, 6 are proved.

3.4 Another Approach to the Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$)

We will start this section with an example. Let us assume that $h_1(\tau), \ldots, h_{12}(\tau) \in L_2([t, T])$ and consider the following integral

$$I \stackrel{\mathsf{def}}{=} \int\limits_t^T h_{12}(t_{12}) \int\limits_t^{t_{12}} h_{11}(t_{11}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{11} dt_{12}.$$

We want to transform the integral I in such a way that

$$I=\int\limits_t^T h_{10}(t_{10})\int\limits_t^{t_{10}} h_6(t_6)\int\limits_t^{t_6} h_4(t_4)\int\limits_t^{t_4} h_3(t_3)\,(\ldots)\,dt_3dt_4dt_6dt_{10},$$

where $(\ldots)$ is some expression.

Using Fubini's Theorem, we obtain

$$I=\int\limits_t^T h_{12}(t_{12})\int\limits_t^{t_{12}} h_{11}(t_{11})\int\limits_t^{t_{11}} \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} h_9(t_9)\int\limits_t^{t_9} h_8(t_8)\int\limits_t^{t_8} h_7(t_7)\int\limits_t^{t_7} \underline{h_6(t_6)}\times$$

$$\times\int\limits_t^{t_6} h_5(t_5)\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)dt_1dt_2dt_3dt_4dt_5dt_6dt_7dt_8\times$$

$$\times dt_9dt_{10}dt_{11}dt_{12}=$$

$$=\int\limits_t^T \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} h_9(t_9)\int\limits_t^{t_9} h_8(t_8)\int\limits_t^{t_8} h_7(t_7)\int\limits_t^{t_7} \underline{h_6(t_6)}\int\limits_t^{t_6} h_5(t_5)\times$$

$$\times\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)dt_1dt_2dt_3dt_4dt_5dt_6dt_7dt_8dt_9\times$$

$$\times\left(\int\limits_{t_{10}}^T h_{11}(t_{11})\int\limits_{t_{11}}^T h_{12}(t_{12})dt_{12}dt_{11}\right)dt_{10}=$$

$$=\int\limits_t^T \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} \underline{h_6(t_6)}\int\limits_t^{t_6} h_5(t_5)\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4dt_5\left(\int\limits_{t_6}^{t_{10}} h_7(t_7)\int\limits_{t_7}^{t_{10}} h_8(t_8)\int\limits_{t_8}^{t_{10}} h_9(t_9)dt_9dt_8dt_7\right)dt_6\times$$

$$\times\left(\int\limits_{t_{10}}^T h_{11}(t_{11})\int\limits_{t_{11}}^T h_{12}(t_{12})dt_{12}dt_{11}\right)dt_{10}=$$

$$= \int\limits_t^T \underline{h_{10}(t_{10})} \int\limits_t^{t_{10}} \underline{h_6(t_6)} \int\limits_t^{t_6} \underline{h_4(t_4)} \int\limits_t^{t_4} \underline{h_3(t_3)} \left(\int\limits_t^{t_3} h_2(t_2) \int\limits_t^{t_2} h_1(t_1) dt_1 dt_2 \right) dt_3 \times$$

$$\times \left(\int\limits_{t_4}^{t_6} h_5(t_5) dt_5 \right) dt_4 \left(\int\limits_{t_6}^{t_{10}} h_7(t_7) \int\limits_{t_7}^{t_{10}} h_8(t_8) \int\limits_{t_8}^{t_{10}} h_9(t_9) dt_9 dt_8 dt_7 \right) dt_6 \times$$

$$\times \left(\int\limits_{t_{10}}^{T} h_{11}(t_{11}) \int\limits_{t_{11}}^{T} h_{12}(t_{12}) dt_{12} dt_{11} \right) dt_{10} =$$

$$= \int\limits_t^T \underline{h_{10}(t_{10})} \int\limits_t^{t_{10}} \underline{h_6(t_6)} \int\limits_t^{t_6} \underline{h_4(t_4)} \int\limits_t^{t_4} \underline{h_3(t_3)} \left(\int\limits_t^{t_3} h_2(t_2) \int\limits_t^{t_2} h_1(t_1) dt_1 dt_2 \right) \times$$

$$\times \left(\int\limits_{t_4}^{t_6} h_5(t_5) dt_5 \right) \left(\int\limits_{t_6}^{t_{10}} h_9(t_9) \int\limits_{t_6}^{t_9} h_8(t_8) \int\limits_{t_6}^{t_8} h_7(t_7) dt_7 dt_8 dt_9 \right) \times$$

$$\times \left(\int\limits_{t_{10}}^{T} h_{12}(t_{12}) \int\limits_{t_{10}}^{t_{12}} h_{11}(t_{11}) dt_{11} dt_{12} \right) dt_3 dt_4 dt_6 dt_{10}. \qquad (154)$$

Further, let us suppose that $h_l(\tau) = \psi_l(\tau)\phi_{j_l}(\tau)$ $(l = 1, \ldots, 12)$ in (154) (here $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_{12}(\tau) \in L_2([t,T])$). Thus, we get

$$C_{j_{12}j_{11}\underline{j_{10}}j_9j_8j_7\underline{j_6}j_5\underline{j_4}\underline{j_3}j_2j_1} = \int\limits_t^T \psi_{10}(t_{10})\phi_{j_{10}}(t_{10}) \int\limits_t^{t_{10}} \psi_6(t_6)\phi_{j_6}(t_6) \int\limits_t^{t_6} \psi_4(t_4)\phi_{j_4}(t_4) \times$$

$$\times \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) C_{j_{12}j_{11}}^{\psi_{12}\psi_{11}}(T, t_{10}) C_{j_9j_8j_7}^{\psi_9\psi_8\psi_7}(t_{10}, t_6) C_{j_5}^{\psi_5}(t_6, t_4) C_{j_2j_1}^{\psi_2\psi_1}(t_3, t) \times$$

$$\times dt_3 dt_4 dt_6 dt_{10}, \qquad (155)$$

where (here and further)

$$C_{j_k \ldots j_1}^{\psi_k \ldots \psi_1}(s, \tau) = \int\limits_\tau^s \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_\tau^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k,$$

where $t \le \tau < s \le T$.

Suppose that $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (7) and $k > 2r$, $r \ge 1$. Consider $d_1, e_1, \ldots, d_f, e_f, f \in \mathbb{N}$ such that

$$1 \le d_1 - e_1 + 1 < \ldots < d_1 < \ldots < d_f - e_f + 1 < \ldots < d_f \le k,$$

$$e_1 + e_2 + \ldots + e_f = 2r,$$

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} = \{d_1 - e_1 + 1, \ldots, d_1\} \cup \ldots \cup \{d_f - e_f + 1, \ldots, d_f\},$$

$$\{1, \ldots, k\} \setminus \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} = \{q_1, \ldots, q_{k-2r}\}.$$

We will say that the condition (A) *is satisfied if* $\forall \ \{g_l, g_{l+1}\}$ $(l = 1, \ldots, 2r-1)$ $\exists \ h \in \{1, \ldots, f\}$ *such that*

$$\{g_l, g_{l+1}\} \subset \{d_h - e_h + 1, \ldots, d_h\}. \tag{156}$$

Moreover, $\forall \ h \in \{1, \ldots, f\}$ $\exists \ \{g_l, g_{l+1}\}$ $(l = 1, \ldots, 2r-1)$ *such that* (156) *is fulfilled.*

If the condition (A) is satisfied, then $e_1, \ldots, e_f$ are even and we can write

$$\{d_1 - e_1 + 1, \ldots, d_1\} = \left\{g_1^{(1)}, g_2^{(1)}, \ldots, g_{2r_1-1}^{(1)}, g_{2r_1}^{(1)}\right\},$$

$$\ldots$$

$$\{d_f - e_f + 1, \ldots, d_f\} = \left\{g_1^{(f)}, g_2^{(f)}, \ldots, g_{2r_f-1}^{(f)}, g_{2r_f}^{(f)}\right\},$$

$$\left\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\right\} =$$

$$= \left\{g_1^{(1)}, g_2^{(1)}, \ldots, g_{2r_1-1}^{(1)}, g_{2r_1}^{(1)}, \ldots, g_1^{(f)}, g_2^{(f)}, \ldots, g_{2r_f-1}^{(f)}, g_{2r_f}^{(f)}\right\}.$$

If the condition (A) is not fulfilled, then some of $e_1, \ldots, e_f$ can be uneven.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Using (104) and a modification of Algorithm 1 from Sect. 3.2

(see below for details) it can be proved that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),\,j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}} \qquad (157)$$

if the condition (A) is satisfied, and

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=0 \qquad (158)$$

if the condition (A) is not fulfilled, where $t_{k+1}\stackrel{\sf def}{=}T$, $t_0\stackrel{\sf def}{=}t$, $r_1+\ldots+r_f=r$ in (157), (158) and $e_h=2r_h$ $(h=1,\ldots,f)$ in (157).

Note that the series on the left-hand sides of (157) and (158) converge absoluty since their sums do not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$). Recall that any permutation of basis functions in a Hilbert space forms a basis in this Hilbert space [58].

Let us prove the formulas (157) and (158).

1. Suppose that the condition (A) is satisfied and

$$\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}=1 \qquad (159)$$

for all $h = 1, \ldots, f$. In this case we can use the results from Sect. 3.2. We have (see (104))

$$\lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \Biggr) \Biggr|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \lim_{p\to\infty} \sum_{j_{g_1^{(1)}}, j_{g_3^{(1)}}, \ldots, j_{g_{2r_1-1}^{(1)}}=0}^{p} C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \Biggr|_{j_{g_1^{(1)}}=j_{g_2^{(1)}}, \ldots, j_{g_{2r_1-1}^{(1)}}=j_{g_{2r_1}^{(1)}}} \times$$

$$\ldots$$

$$\times \lim_{p\to\infty} \sum_{j_{g_1^{(f)}}, j_{g_3^{(f)}}, \ldots, j_{g_{2r_f-1}^{(f)}}=0}^{p} C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \Biggr|_{j_{g_1^{(f)}}=j_{g_2^{(f)}}, \ldots, j_{g_{2r_f-1}^{(f)}}=j_{g_{2r_f}^{(f)}}} =$$

$$= \prod_{h=1}^{f} \frac{1}{2^{r_h}} \prod_{l=1}^{r_h} \mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}} \times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1}, t_{d_h-e_h}) \Biggr|_{(j_{g_2^{(h)}} j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}} j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),\, j_{g_1^{(h)}}=j_{g_2^{(h)}}, \ldots, j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}.$$

Thus, we get the formula (157).

2. Suppose that the condition (A) is satisfied and for some $h = 1, \ldots, f$

$$\prod_{l=1}^{r_h} \mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}} = 0. \tag{160}$$

In this case, we act the same as in the previous case. Applying (104), we obtain

$$\lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \Biggr) \Biggr|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$
= \lim_{p\to\infty} \sum_{j_{g_1^{(1)}}, j_{g_3^{(1)}}, \ldots, j_{g_{2r_1-1}^{(1)}}=0}^{p} C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1})\Bigg|_{j_{g_1^{(1)}}=j_{g_2^{(1)}}, \ldots, j_{g_{2r_1-1}^{(1)}}=j_{g_{2r_1}^{(1)}}} \times
$$

$$
\ldots
$$

$$
\times \lim_{p\to\infty} \sum_{j_{g_1^{(f)}}, j_{g_3^{(f)}}, \ldots, j_{g_{2r_f-1}^{(f)}}=0}^{p} C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f})\Bigg|_{j_{g_1^{(f)}}=j_{g_2^{(f)}}, \ldots, j_{g_{2r_f-1}^{(f)}}=j_{g_{2r_f}^{(f)}}} = 0 \qquad (161)
$$

(al least one of the multipliers is equal to zero on the right-hand side of (161)).

The equality (157) is proved in our case (the right-hand side of (157) is equal to zero for the considered case (see (160))).

3. Suppose that the condition (A) is not satisfied. In this case, we act according to Algorithm 1 from Sect. 3.2. More precisely, let us select blocks in the multi-index $j_{d_h}\ldots j_{d_h-e_h+1}$ $(h=1,\ldots,f)$ that correspond to the fulfillment of the condition

$$
\prod_{l=1}^{r_{m,h}} \mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}} = 1,
$$

where $r_{m,h}$ is the number of pairs $\{g_{2l-1}^{(h)}, g_{2l}^{(h)}\}$ (from the set $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\}$) in the block with number m that corresponds to the multi-index $j_{d_h}\ldots j_{d_h-e_h+1}$.

Let us save multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the Volterra–type kernels corresponding to the Fourier coefficients

$$
C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}), \ldots, C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \qquad (162)
$$

and corresponding to the above blocks.

At that, we remove the remaining multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the Volterra–type kernels corresponding to the Fourier coefficients (162).

As a result, we get a modified left-hand side of the equality (158). For definiteness, let us denote this expression by $(^-)$.

Using generalized Parseval's equality (Parseval's equality for two functions) and (97), we represent the expression $(^{-})$ as an integral over the hypercube $[t,T]^r$.

It is not difficult to see that the indicated integral over the hypercube $[t,T]^r$ is represented as a product of integrals over hypercubes of smaller dimentions. At that, at least one of these integrals is equal to zero due to the generalized Parseval equality (Parseval's equality for two functions) and the fulfillment of the condition $t \le t_{d_1-e_1} \le t_{d_1+1} \le \ldots \le t_{d_f-e_f} \le t_{d_f+1} \le T$ (see the above example and (154) and (155)). For definiteness, let us denote the equality of $(^{-})$ to zero by $(\bar{K})$. We interpret the above zero as the zero functional in $L_2([t,T]^r)$.

Further, transformations and passages to the limit in the equality $(\bar{K})$ are performed iteratively in such a way as to restore the removed multipliers $\mathbf{1}_{\{t_n<t_{n+1}\}}$ on the left-hand side of $(\bar{K})$ (for more details, see Sect. 3.2). As a result, we obtain the equality (158). The equalities (157) and (158) are proved.

For definiteness, suppose that $q_1 < \ldots < q_{k-2r}$, $k > 2r$, $r \ge 1$ (recall that the case $k = 2r$ is proved earlier). Using Fubini's Theorem (as in the above example (see (154)), we obtain

$$
\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$
= \int_t^T \psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}) \ldots \int_t^{t_{q_1+1}} \psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times
$$

$$
\times \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} \Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f}) \ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}}, \tag{163}
$$

$$\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}=$$

$$=\int\limits_t^T\psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times$$

$$\times\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\times$$

$$\times dt_{q_1}\ldots dt_{q_{k-2r}}. \tag{164}$$

Suppose that

$$\Biggl|\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr|\le K<\infty, \tag{165}$$

where constant K does not depend on p and $t_{d_1+1},t_{d_1-e_1},\ldots,t_{d_f+1},t_{d_f-e_f}$. In (165): $t_{k+1}\stackrel{\sf def}{=}T$, $t_0\stackrel{\sf def}{=}t$, $r_1+\ldots+r_f=r$; another notations as above in this section.

Applying (157), (158), (163), (164), we obtain ($k>2r$, $r\ge 1$)

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2\le$$

$$\le\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$
\left.\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\right|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2=
$$

$$
=\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\Biggl(\int\limits_t^T\psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times
$$

$$
\times\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Bigl(C^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}_{j_{d_f}\ldots j_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\left.\ldots C^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}_{j_{d_1}\ldots j_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Bigr)\right|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\left.\times C^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}_{j_{d_h}\ldots j_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\right|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\Biggr)\times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}}\Biggr)^2= \qquad (166)
$$

$$
=\lim_{p\to\infty}\int\limits_t^T\psi^2_{q_{k-2r}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi^2_{q_1}(t_{q_1})\times
$$

$$
\times\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Bigl(C^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}_{j_{d_f}\ldots j_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\left.\ldots C^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}_{j_{d_1}\ldots j_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Bigr)\right|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\times C^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}_{j_{d_h}\ldots j_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\Biggr)^2\times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}}= \tag{167}
$$

$$
=\int\limits_t^T \psi^2_{q_{k-2r}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi^2_{q_1}(t_{q_1})\times
$$

$$
\times\lim_{p\to\infty}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Bigl(C^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}_{j_{d_f}\ldots j_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}_{j_{d_1}\ldots j_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Bigr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\times C^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}_{j_{d_h}\ldots j_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\Biggr)^2\times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}}=0, \tag{168}
$$

where the transition from (166) to (167) is based on the Parseval equality and the transition from (167) to (168) is based on Lebesgue's Dominated Convergence Theorem (see (13)–(15), (157), (158), (165)) and also on convergence to zero (almost everywhere on $X=\{(t_{q_1},\ldots,t_{q_{k-2r}}): t\le t_{q_1}\le\ldots\le t_{q_{k-2r}}\le T\}$ with respect to Lebesgue's measure) of the integrand function in (167).

Thus, we have the following theorem.

Theorem 7 [14]. *Suppose that the condition* (165) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$*. Then, for the* *sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (11) *the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (8), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$), $\mathbf{w}_\tau^{(0)} = \tau$; *another notations are the same as in Theorem* 2.

Using Theorem 1, we obtain the following corollary of Theorem 7.

Theorem 8 [14]. *Suppose that the condition* (165) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of multiplicity* k $(k \in \mathbb{N})$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid; *another notations are the same as in Theorem* 7.

Consider application of Thorem 8 to the expansion of the following iterated Stratonovich stochastic integrals

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \int\limits_t^{*T} (t_3-t)^{l_3} \int\limits_t^{*t_3} (t_2-t)^{l_2} \int\limits_t^{*t_2} (t_1-t)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}, \qquad (169)$$

$$I_{l_1 l_2 l_3 l_4 T,t}^{*(i_1 i_2 i_3 i_4)} = \int\limits_t^{*T} (t_4 - t)^{l_4} \ldots \int\limits_t^{*t_2} (t_1 - t)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_4}^{(i_4)}, \qquad (170)$$

where $i_1, \ldots, i_4 = 0, 1, \ldots, m$; $l_1, \ldots, l_4 = 0, 1, 2, \ldots$

Denote

$$C_{j_1 j_1}^{\psi_{i+1}\psi_i}(s,\tau) = \int\limits_\tau^s \psi_{i+1}(y)\phi_{j_1}(y) \int\limits_\tau^y \psi_i(x)\phi_{j_1}(x)dxdy, \;\; C_{j_1}^{\psi_q}(s,\tau) = \int\limits_\tau^s \psi_q(x)\phi_{j_1}(x)dx,$$

where (here and further) $i = 1, 2, 3$, $t \leq \tau < s \leq T$, $\psi_q(x) = (x - t)^{l_q}$, $l_q = 0, 1, 2, \ldots$, $q = 1, \ldots, 4$, $x \in [t, T]$.

Using Fubini's Theorem and the technique that leads to the formulas (154), (155), we obtain

$$C_{j_3 j_1 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) C_{j_1 j_1}^{\psi_2\psi_1}(t_3, t) dt_3,$$

$$C_{j_3 j_3 j_1} = \int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) C_{j_3 j_3}^{\psi_3\psi_2}(T, t_1) dt_1,$$

$$C_{j_1 j_2 j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) C_{j_1}^{\psi_1}(t_2, t) C_{j_1}^{\psi_3}(T, t_2) dt_2,$$

$$C_{j_4 j_3 j_1 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) C_{j_1 j_1}^{\psi_2\psi_1}(t_3, t) dt_3 dt_4,$$

$$C_{j_4 j_1 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) C_{j_1}^{\psi_1}(t_2, t) C_{j_1}^{\psi_3}(t_4, t_2) dt_2 dt_4,$$

$$C_{j_1 j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) C_{j_1}^{\psi_1}(t_2, t) C_{j_1}^{\psi_4}(T, t_3) dt_2 dt_3,$$

$$C_{j_4 j_2 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) C_{j_2 j_2}^{\psi_3\psi_2}(t_4, t_1) dt_1 dt_4,$$

$$C_{j_2j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)C_{j_2}^{\psi_2}(t_3,t_1)C_{j_2}^{\psi_4}(T,t_3)dt_1dt_3,$$

$$C_{j_3j_3j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)C_{j_3j_3}^{\psi_4\psi_3}(T,t_2)dt_1dt_2,$$

where $C_{j_3j_2j_1}$, $C_{j_4j_3j_2j_1}$ are defined by (8).

It is easy to see (based on the above equalities) that the condition (165) will be satisfied if

$$\left|\sum_{j_1=0}^{p}C_{j_1j_1}^{\psi_{i+1}\psi_i}(t_2,t_1)\right|\le K, \tag{171}$$

$$\left|\sum_{j_1=0}^{p}C_{j_1}^{\psi_k}(t_2,t_1)C_{j_1}^{\psi_q}(t_4,t_3)\right|\le K, \tag{172}$$

where $p\in\mathbb{N}$, $i=1,2,3$, $k,q=1,\ldots,4$, $t\le t_1<t_2\le T$, $t\le t_3<t_4\le T$, constant K does not depend on $p,t_1,\ldots,t_4$.

Let us prove (171), i.e. we will prove the following equality

$$\left|\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t_1}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right|\le K<\infty, \tag{173}$$

where $l,m=0,1,2,\ldots$, $t\le t_1<t_2\le T$, constant K does not depend on p,t_1,t_2.

Using Fubini's Theorem and Parseval's equality, we have for $m>l$ ($l,m=0,1,2,\ldots$)

$$\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^l(\tau-t)^{m-l}\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^l\phi_j(\tau)\int\limits_{t}^{\tau}(\theta-t)^{m-l-1}(m-l)d\theta d\tau ds=$$

$$= (m-l)\sum_{j=0}^{p}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\int\limits_{\theta}^{t_2}(\tau-t)^{l}\phi_j(\tau)\int\limits_{\tau}^{t_2}(s-t)^{l}\phi_j(s)dsd\tau d\theta =$$

$$= (m-l)\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\frac{1}{2}\sum_{j=0}^{p}\left(\int\limits_{\theta}^{t_2}(\tau-t)^{l}\phi_j(\tau)d\tau\right)^2 d\theta \le$$

$$\le \frac{m-l}{2}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\sum_{j=0}^{\infty}\left(\int\limits_{\theta}^{t_2}(\tau-t)^{l}\phi_j(\tau)d\tau\right)^2 d\theta =$$

$$= \frac{m-l}{2}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\int\limits_{\theta}^{t_2}(\tau-t)^{2l}d\tau d\theta \le K_1 \le \infty, \tag{174}$$

where constant K_1 does not depend on p, t_2.

For $l > m$ $(l, m = 0, 1, 2, \ldots)$ we get

$$\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^{l}\phi_j(s)\int\limits_{t}^{s}(\tau-t)^{m}\phi_j(\tau)d\tau ds =$$

$$= \sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^{l}\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^{m}\phi_j(\tau)d\tau -$$

$$- \sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^{l}\phi_j(s)\int\limits_{s}^{t_2}(\tau-t)^{m}\phi_j(\tau)d\tau ds =$$

$$= \sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^{l}\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^{m}\phi_j(\tau)d\tau -$$

$$- \sum_{j=0}^{p}\int\limits_{t}^{t_2}(\tau-t)^{m}\phi_j(\tau)\int\limits_{t}^{\tau}(s-t)^{l}\phi_j(s)dsd\tau. \tag{175}$$

Applying Cauchy–Bunyakovsky's inequality and Parseval's equality, we obtain

$$\left(\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^{l}\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^{m}\phi_j(\tau)d\tau\right)^2 \le$$

$$\leq \sum_{j=0}^{p}\left(\int_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2\sum_{j=0}^{p}\left(\int_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau\right)^2 \leq$$

$$\leq \sum_{j=0}^{\infty}\left(\int_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2\sum_{j=0}^{\infty}\left(\int_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau\right)^2 =$$

$$= \int_{t}^{t_2}(s-t)^{2l}ds\int_{t}^{t_2}(\tau-t)^{2m}d\tau \leq K_2 \leq \infty, \qquad (176)$$

where constant K_2 does not depend on p, t_2.

Using (174)–(176), we obtain

$$\left|\sum_{j=0}^{p}\int_{t}^{t_2}(s-t)^l\phi_j(s)\int_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right| \leq K_3 < \infty, \qquad (177)$$

where $l > m$ $(l, m = 0, 1, 2, \ldots)$, constant K_3 does not depend on p, t_2.

For the case $l = m$ we get

$$\sum_{j=0}^{p}\int_{t}^{t_2}(s-t)^l\phi_j(s)\int_{t}^{s}(\tau-t)^l\phi_j(\tau)d\tau ds =$$

$$= \sum_{j=0}^{p}\frac{1}{2}\left(\int_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2 \leq \sum_{j=0}^{\infty}\frac{1}{2}\left(\int_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2 =$$

$$= \frac{1}{2}\int_{t}^{t_2}(s-t)^{2l}ds \leq K_4 < \infty, \qquad (178)$$

where constant K_4 does not depend on p, t_2.

Combining (174), (177), (178), we have

$$\left|\sum_{j=0}^{p}\int_{t}^{t_2}(s-t)^l\phi_j(s)\int_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right| \leq K_5 < \infty, \qquad (179)$$

where $l, m = 0, 1, 2, \ldots$, constant K_5 does not depend on p, t_2.

Note that

$$\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t_1}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds-$$

$$-\sum_{j=0}^{p}\int\limits_{t}^{t_1}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds-$$

$$-\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_1}(\tau-t)^m\phi_j(\tau)d\tau, \tag{180}$$

where $l,m=0,1,2,\ldots$ and $t\le t_1<t_2\le T$.

By analogy with (176) we get

$$\left|\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_1}(\tau-t)^m\phi_j(\tau)d\tau\right|\le K_6<\infty, \tag{181}$$

where $l,m=0,1,2,\ldots$, constant K_6 does not depend on p,t_2.

Combining (180), (179), and (181), we obtain (173). The equality (171) is proved. Obviously, the relation (172) is proved in complete analogy with (176).

Thus, the condition (165) of Theorem 8 is fulfilled for $k=3,4$ and $\psi_q(x)=(x-t)^{l_q}$, $l_q=0,1,2,\ldots$, $q=1,\ldots,4$. Then, using Theorem 8, we obtain the following expansions of iterated Stratonovich stochastic integrals (169), (170)

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{182}$$

$$I_{l_1l_2l_3l_4T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converge in the mean-square sense, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$, $i_1,\ldots,i_4=0,1,\ldots,m$; $l_1,\ldots,l_4=0,1,2,\ldots,$

$$C_{j_3j_2j_1}=\int\limits_t^T(t_3-t)^{l_3}\phi_{j_3}(t_3)\int\limits_t^{t_3}(t_2-t)^{l_2}\phi_{j_2}(t_2)\int\limits_t^{t_2}(t_1-t)^{l_1}\phi_{j_1}(t_1)dt_1dt_2dt_3,$$

$$C_{j_4j_3j_2j_1}=\int\limits_t^T(t_4-t)^{l_4}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}(t_1-t)^{l_1}\phi_{j_1}(t_1)dt_1\ldots dt_4,$$

another notations are the same as in Theorem 8.

Note that the iterated Stratonovich stochastic integrals (169), (170) are important for applications [14] (Chapter 4).

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