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Control problems in nonlinear systems

## STABILITY OF DIFFUSIVE COUPLED NETWORKS AND TECHNICAL SYSTEMS UNDER RANDOM NOISE

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### Abstract.

We consider stability of special dynamical systems under a random noise. These systems (networks) are important in many applications. We focus our attention on supporting of homeostasis of these systems with respect to fluctuations of an external medium (the problem is posed by M. Gromov, A. Carbone [7]). Using a measure of stochastic stability we show that a network with fixed parameters is always unstable, i.e., the probability to support homeostasis converges to zero as time  $T \rightarrow \infty$ .

## 1 Introduction

We consider stability of special dynamical systems under a random noise. These systems (networks) are important in many applications. We focus our attention on supporting of homeostasis of these systems with respect to fluctuations of an external medium (the problem is posed by M. Gromov, A.Carbone [7]). Using a measure of stochastic stability we show that a network with fixed parameters is always unstable, i.e., the probability to support homeostasis converges to zero as time  $T \rightarrow \infty$ . We consider stability of special dynamical systems under a random noise. These systems (networks) are important in many applications. We focus our attention on supporting of homeostasis of these systems with respect to fluctuations of an external medium (the problem is posed by M. Gromov, A.Carbone [7]). Using a measure of stochastic stability we show that a network with fixed parameters is always unstable, i.e., the probability to support homeostasis converges to zero as time  $T \rightarrow \infty$ .

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## 2 Network models

In last decade, large attention is given to problems of global organization, stability and evolution of complex networks such as protein and gene networks, networks of metabolic reactions, neural and economical circuits, Internet etc. (see [1] for a review). The simplest mathematical model of such network is a (directed) graph. For example, for a gene network we can associate with this network a graph where a node describes a gene, the  $i$ -th node is connected with the  $j$ -th one if the corresponding genes interact. Stability of these graphs can be examined in different contexts. For example, we can examine how much edges (or nodes) must be eliminated in order to destroy connectivity of the graph. In this paper our attention is focused on stability of the networks with respect to fluctuations ( an internal noise and oscillations of an environment). To study this problem, we extend simple graph models. In fact, metabolic reaction networks or gene networks are complex dynamical system, where a scheme of interaction of substrats,ferments or genes can be associated with a graph. A part of the substrats enters this system from an external medium (input) and another part can be considered as an output (products). It is well known that

these systems successfully support an output independent of fluctuating input [9, 2].

It is difficult to describe in details such dynamical systems. Genetic circuit models ( [5, 4, 10] among many others, see [12] for a review) to take into account theoretical ideas and experimental information on gene interactions. We consider a model proposed by [10, 11]

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + R_i \sigma \left( \sum_{j=1}^m K_{ij} u_j + \theta_i - \xi_i(t) \right) - \lambda_i u_i, \quad (2.1)$$

where  $m$  is the number of genes included in the circuit,  $u_i(x, t)$  is the concentration of the  $i$ -th gene,  $x \in \Omega$ , where  $\Omega$  is a bounded open domain with a regular boundary  $\partial\Omega$ ,  $\lambda_i$  are the gene decay rates,  $d_i$  are diffusion coefficients, the parameters  $\theta_i$  are activation thresholds,  $\xi_i(t)$  describe random fluctuations, and  $\sigma$  is so-called sigmoidal function (see below). We assume that the  $\xi_i$  are random processes with piecewise continuous trajectories. Real number  $K_{ij}$  measures the influence of the  $j$ -th gene on the  $i$ -th one,  $R_i > 0$  are coefficients.

We set the zero Neumann conditions

$$\frac{\partial u_i}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0.$$

The initial data are

$$u_i(x, 0) = s_i, \quad s_i \geq 0, \quad (2.2)$$

where  $s_i$  are constants. The function  $\sigma$  is a smooth, strictly monotone increasing function satisfying

$$\lim_{z \rightarrow -\infty} \sigma(z) = 0, \quad \lim_{z \rightarrow \infty} \sigma(z) = 1. \quad (2.3)$$

The well known example is  $\sigma(z) = \frac{1 + \tanh(z)}{2}$ .

Model (2.1) takes into account the following fundamental processes: a) the decay (degradation) of gene products (the term  $-\lambda_i u_i$ ); and b) gene regulation and synthesis, c) diffusion [10, 11]. Notice that if  $s_i$  are constants, we can set  $d_i = 0$ , since then the solution depends only on  $t$ . Then system (2.1) reduces to the famous Hopfield model of the attractor neural network [8].

Another possible model is a dynamical system with discrete time, for example, defined by the following iterative process

$$u_i^{t+1} = r_i \sigma \left( \sum_{j=1}^m K_{ij} u_j^t + \theta_i - \xi_i^t \right) - \lambda_i u_i^t, \quad (2.4)$$

$$u_i^0 \equiv s_i, \quad (2.5)$$

where  $t = 0, 1, 2, \dots, T$ ,  $T$  is an integer,  $\xi_i^t$  are random functions of discrete time  $t$ . Numerical procedures solving (2.1) (with  $d_i = 0$ ) lead to models similar to (2.4). This system was studied in [15, 16]. In this paper we consider system (2.1), more complicated, than systems (2.4).

To conclude this section, let us make the two remarks. Notice that  $\xi_i$  are involved nonlinearly in (2.1). In this case it is difficult to study strongly discontinuous processes  $\xi_i$ . However, we can consider a variant of system (2.1) with a multiplicative noise, which can have  $\delta$ -like contributions (see Section 4). Notice that there also exists a case, where  $x \in \mathbf{Z}^d$ , i.e.,  $x$  runs over a grid and  $\Delta$  is a discrete approximation of the Laplace operator [11]. Our results hold for this case.

Let us formulate now main results. We consider the question on the stochastic stability of genetic circuits (2.1). The stochastic stability can be defined following the well known ideas (see below and [17, 16]). We extend on the case (2.1) the result obtained in [16] that the more is the valency of a node the stabler is the circuit with respect to perturbations in this node. We also prove that the survival probability of each circuit of a fixed structure tends to zero as  $T \rightarrow \infty$ . Therefore, "homeostasis" generated by a fixed circuit will be broken as time tends to infinity. We give an estimate of the survival probability.

### 3 Stochastic Stability for Circuits

The important meaning has the problem of stability of networks under random perturbations of different parameters. We obtain in this section some estimates on stability of (2.1) under noises.

For  $s_i$  independent of  $x$  we can consider problem (2.1) with  $d_i = 0$ . We assume that random processes  $\xi_i(t)$  are independent for different  $i$ . Different choices of the distributions for  $\xi_i^t$  and  $S_i$  may correspond to different "ecological conditions" (random environments).

Let us introduce functions  $\Psi_i$  by

$$Prob\{\xi_i(t) > a \text{ for all } t \in [T_1, T_2]\} = \Psi_i(a, T_1, T_2). \quad (3.1)$$

The following assumptions play a key role. Let us suppose first that

$$\Psi_i(a, T_1, T_2) > 0 \quad (3.2a)$$

for any  $i$ ,  $a$  and  $T_1, T_2$  such that  $T_2 > T_1$ . Moreover, let us assume that for any fixed  $\tau > 0$ , an index  $i$  and a number  $a$  the probability  $P_i(a, \tau, T)$  that there exists a moment  $t_0 \in [0, T]$  such that  $\xi_i(t) > a$  for any  $t \in [t_0, t_0 + \tau]$  satisfies

$$P_i(a, \tau, T) \rightarrow 1 \quad (T \rightarrow \infty). \quad (3.2b)$$

Roughly speaking, this means that  $\xi_i(t)$  can take any large values within any bounded time periods with non-zero probabilities.

Notice that trajectories  $\eta_i(t)$  of the independent standard Wiener process satisfy (3.2a, b). One can check that the property (3.2a) holds in this case [6]. To show (3.2b), we can use the well known results [6]. First we notice that in this case the probability  $P_T(b)$  to attain any fixed  $b$  within  $[0, T]$  converges to 1 as  $T \rightarrow \infty$ . In fact, due to Theorem 2 from [6], Ch. VI, part 3, we have

$$P_T(b) = Prob\{\max_{t \in [0, T]} \eta(t) > b\} = 2 \frac{1}{\sqrt{2\pi T}} \int_b^\infty \exp(-\frac{x^2}{2T}) dx.$$

This relation shows that  $P_T(b) \rightarrow 1$  as  $T \rightarrow \infty$ . Given  $a, \tau$ , we choose a sufficiently large  $b = b(a, \tau)$ . The process attains, within a sufficiently large time interval  $[0, T]$ , this value  $b$  at some  $t_0(b) \in [0, T]$ . Moreover, let us notice that the process is a regular one, i.e., the probability to return on the level  $a$  starting from  $b$  within a time interval of a fixed length  $\tau$  converges to zero as  $b \rightarrow \infty$  [6]. This regularity property shows that the Wiener process satisfies (3.2b).

Let  $\Pi$  be a closed domain in the  $u$  - phase space. We say that a system (a circuit (2.1)) "survives" in  $\Pi$  (supports homeostasis in  $\Pi$ ) if the concentrations  $u_i$  lie in  $\Pi$ . Notice that our conditions (2.3) on  $\sigma$  entail the dissipativity of (2.1), i.e., there is a box  $B \subset \mathbf{R}^m$  such that each solution with initial data from this box cannot leave the box  $B$ . Moreover, one can show that concentrations  $s_i$  stay non-negative. It is natural, thus, to suppose that  $\Pi$  is contained in  $B \cap \{u : u_i \geq 0\}$ . As a measure of the stochastic stability of the circuit homeostasis on the time interval  $[T_1, T_2]$ , we consider the probability

$$P(\Pi, T_1, T_2) = Prob\{\mathbf{u}(t) \in \Pi \text{ for each } t \in [T_1, T_2]\}, \quad (3.3)$$

where  $\mathbf{u} = (u_1, \dots, u_m)$  (see [16]). This probability depends on the circuit parameters  $\mathcal{P}$  and the homeostasis domain  $\Pi$ . We name it the survival probability on the time interval  $[T_1, T_2]$  and denote by  $P(T_1, T_2)$  omitting the dependence on the domain  $\Pi$ .

We estimate the stability via the following parameters: the valency,  $|K_*|$ , the maximum  $b$  of  $|\theta_i|$  and some parameter  $N_{key}$  that will be introduced below.

It is important to take into account the valency since it is well known that biological circuits are far from being completely connected: for each fixed node  $i$  we have a valency  $V_i < m$ : only  $V_i$  among the entries  $K_{ij}$  are not equal zero. In economical, technical and biological applications one has typically  $V_i \ll m$  [1].

To define  $N_{key}$ , let us observe first that

$$\inf_{\mathbf{u} \in \Pi} u_i = W_i \geq 0. \tag{3.4}$$

Denote  $U_i = \sigma^{-1}(W_i)$ . Some  $W_i$  and  $U_i$  may be positive. The corresponding indices  $i_1, \dots, i_s \in [1, \dots, m]$  will be named key indices and the corresponding genes will be named the key ones. In fact, assumption  $W_i > 0$  means that the organism cannot survive if the concentration of  $i$ -th gene is small enough. The number of the key genes is denoted by  $N_{key}$ . We denote by  $I$  the set of the key indices corresponding to the key genes.

The first step is a priori estimate of solutions of system (2.1). We obtain

$$|u_i| < \mu_i = \max\{R_i \lambda_i^{-1}, s_i\}. \tag{3.5}$$

Let us fix now a key index  $i \in I$  and consider (2.1). Using (3.5) one obtains the following simple inequality

$$\sum_{j=1}^m K_{ij} u_j(t) + \theta_i - \xi_i(t) \leq V_i K_* \mu_i + b - \xi_i(t). \tag{3.6}$$

Suppose  $t_0, \tau > 0$ . Let us take a sufficiently large  $a$  and, using hypothesis (3.2), consider a trajectory  $\xi_i(t)$  such that

$$\xi_i(t) > a, \quad t \in [t_0, t_0 + \tau] \tag{3.7}$$

The probability to find such a trajectory is a positive, due to (3.2a). Consider system (2.1) on the interval  $[t_0, t_0 + \tau]$ . We choose a number  $a$  such that

$$V_i K_* \mu_i + b - \xi_i(t) < U_i - \delta, \tag{3.8}$$

where  $\delta > 0$  is a small positive number. Since  $\sigma \in (0, W_i)$  on this interval, and  $u_i(t_0)$  satisfies estimate (3.7), one derives a priori estimate of  $u_i$

$$0 < u_i(t) < \mu_i \exp(-\lambda_i(t - t_0)) + (W_i - \delta_1)(1 - \exp(-\lambda_i(t - t_0))), \tag{3.9}$$

where  $\delta_1 > 0$  is a small number,  $t \in [t_0, t_0 + \tau]$ . For sufficiently large  $\tau$  we have then  $u_i(t_0 + \tau) < W_i$ . So, we conclude that the event

$$\xi_i(t) > V_i K_* \mu_i + b - U_i = \gamma_i, \quad t \in [t_0, t_0 + \tau], \tag{3.10}$$

entails the event that the concentration  $u_i(t_0 + \tau)$  is less than the critical value  $W_i$ . Thus, the set  $A_i$  of the random trajectories  $\xi_i$  such that  $\xi_i(t) > \gamma_i$  on  $[t_0, t_0 + \tau]$  is contained in the set  $B_i$  of the random trajectories such that the corresponding solution  $\mathbf{u}(t)$  satisfies  $u_i(t_0 + \tau) < W_i$ . Therefore, the complements  $\bar{A}_i, \bar{B}_i$  of  $A_i$  and  $B_i$  satisfy  $\bar{B}_i \subset \bar{A}_i$ . The same holds for the intersections:

$$\bar{B} = (\cap_{i \in I} \bar{B}_i) \subset (\cap_{i \in I} \bar{A}_i) = \bar{A}.$$

Moreover, if at least one  $u_i(t)$  is less than  $W_i$ , the state  $\mathbf{u}(t)$  is outside of this domain  $\Pi$ . This shows that the condition  $\mathbf{u}(t_0 + \tau) \in \Pi$  is equivalent to  $\mathbf{u}(t_0 + \tau) \in \bar{B}$ . We obtain thus that  $Prob \bar{B} \leq Prob \bar{A}$ . Since we suppose that the random processes  $\xi_i$  are independent, all the events  $\{\xi_i(t) > \gamma_i \text{ on } [t_0, t_0 + \tau]\}$  are independent. This implies  $Prob \bar{A} = Prob \bar{A}_1 Prob \bar{A}_2 \dots Prob \bar{A}_{N_{key}}$ . Each probability  $Prob \bar{A}_i$  can be expressed through  $\Psi_i(\gamma_i, t_0, t_0 + \tau)$ . Hence, we have

$$Prob\{\mathbf{u}(t) \in \Pi\} \leq \prod_{i \in I} (1 - \Psi_i(V_i K_* \mu_i + b - U_i, t - \tau, t)) \equiv R(t, \tau). \quad (3.11)$$

Therefore, we have proved

**Proposition** *The survival probability satisfies*

$$P(T_1, T_2) \leq R(T_1, \tau). \quad (3.12)$$

This estimate implies the following consequence. Notice that the function  $R$  is a monotone increasing function of the valencies  $V_i$ . So, to increase the stability, we can increase the valencies. Moreover, one can prove, by the same arguments and assumption (3.2b), that all circuits are stochastically unstable as the time  $T \rightarrow \infty$ . In fact, let us consider the arbitrary key index  $i \in I$ . The probability that  $\xi_i(t) > \gamma_i$  on some  $[t, t + \tau] \subset [0, T]$  tends to 1 as  $T \rightarrow \infty$ . Thus, the probability that for some  $t + \tau \in [0, T]$  the value  $u_i(t + \tau) < W_i$  also tends to 1 as  $T \rightarrow \infty$ .

## 4 Stability under jump-like noises

We consider a network under jump-like multiplicative noise

$$\frac{du_i}{dt} = R_i \sigma \left( \sum_{j=1}^m K_{ij} u_j + \theta_i \right) - \lambda_i u_i + g_i(u) \xi_i(t), \quad (4.1)$$

where  $g_i$  are smooth functions,  $\xi_i$  are processes defined by

$$\xi_i = \sum_k \gamma_{ik} \delta(t - t_k),$$

where  $t_k$  are random time moments,  $0 < t_1 < t_2 < \dots < t_k < \dots$ ,  $\gamma_{ik}$  are random numbers. We suppose that the processes  $\xi_i$  are time homogeneous.

As above, we are going to estimate the stochastic stability of the network defined by (4.1), i.e, the probability  $P(T_1, T_2) = P(0, T)$ ,  $T = T_2 - T_1$ .

First we observe that the noise influence reduces to jump-like changes of the system trajectories. Suppose  $u(0) = v^0$ . On the interval  $[0, t_1]$  we have the trajectory  $u(t, u^0)$ ,  $t \in (0, t_1]$  defined by (4.1) with  $\xi_i \equiv 0$ . The jump at  $t_1$  gives the trajectory  $u(t, v^1)$ , where  $t \in (t_1, t_2]$  and where initial data are

$$u(t_1) = v^1 + w^1, \quad v^1 = \lim_{t \rightarrow t_1 - 0} u(t), \quad w^1 = g_i(v^1) \gamma_{i1}, \quad (4.2)$$

etc. In general, an analysis of such jump dynamics is a difficult problem. In order to simplify the situation, we suppose the following.

We suppose that  $\Pi$  contains a unique local attractor  $\mathcal{A}$  of (4.1), which is a hyperbolic set with a Bowen-Ruelle-Sinai invariant measure  $\rho(u)$ . The support of this measure lies in  $\mathcal{A}$ . The results of [14] shows that this situation is quite possible in the networks: the Hopfield networks can have any structurally stable attractors.

The second hypothesis is that the jumps are, in a sense, seldom in average. This means that the mathematical expectation

$$\mathbf{E}(t_k - t_{k-1}) = \tau_{rand} \gg \tau_{attr}, \quad (4.3)$$

where  $\tau_{attr}$  is a characteristic time, which describes the rate of trajectory approaching to the attractor  $\mathcal{A}$  (physically, it is the relaxation time). Clearly, we use a rough estimate  $\tau_{attr}^{-1} \approx \min\{\lambda_s\}$ , where  $\lambda_s$  are negative Lyapunov exponents associated with the dynamics on the attractor. Since we assume that the random process is time homogeneous, the distributions of  $\gamma_{ik}$  are independent of  $k$ .

The hypothesis (4.3) yields the following result. Suppose  $[0, T]$  time interval such that  $\tau_{attr} \ll T \ll \tau_{rand}$ . Let us introduce the formal small parameter  $\epsilon = \tau_{attr}/\tau_{rand}$ . Then one can expect that

$$P(0, T) = \int_{\mathcal{A}} \rho(u) q_{g, \gamma}(u) du + o(1) \quad (\epsilon \rightarrow 0), \quad (4.4)$$



where  $q_{g,\gamma}(u)$  is the probability that, making a jump from the point  $u$ , the system state leaves the domain  $\Pi$ :

$$q_{g,\gamma}(u) = \text{Prob}\{u + g^\gamma(u) \notin \Pi\}, \quad (4.5)$$

where  $g^\gamma$  is a random vector with components  $g_i^\gamma(u) = g_i \gamma_{ik}$ . This vector is defined by the multiplicative factor  $g$  and the probabilistic law for  $\gamma_{ik}$ .

Relation (4.5) allows us to define an optimal attractor structure. Under above hypothesis the attractor  $\mathcal{A}$  giving maximal survival probability is an equilibrium  $\mathcal{A} = \{u^*\}$ . Here  $u^*$  is a point, where the function  $q_{g,\gamma}(u)$  has the global minimum. Then  $\rho(u) = \delta(u - u^*)$ .

So, we see that the optimal attractor structure is trivial. The situation changes dramatically if the system may be under different random perturbations  $(g, \gamma) \in G$  from some family  $G$  of perturbations. Suppose it is impossible to foresee which from them influences on our system (*unpredictable environment*). In this case we obtain (as  $\epsilon \rightarrow 0$ )

$$\min_{(g,\gamma) \in G} P_T(\Pi) = \min_{(g,\gamma) \in G} \int_{\mathcal{A}} \rho(u) q_{g,\gamma}(u) du.$$

This relation leads to the following minimax problem for the optimal measure  $\rho$ :

$$\max_{\rho} \min_{(g,\gamma) \in G} \int_{\mathcal{A}} \rho(u) q_{g,\gamma}(u) du.$$

One can show [3] that, in general, the support of the measure  $\rho$  is localized on a certain set. Then the attractor must be chaotic or periodic, and the dynamics on this attractor must be complex.

Another method to obtain a larger stability in an unpredictable environment is a simple dynamics under a small internal noise. This noise helps to stabilize the system in the non-predictable environment, because there also is an invariant measure  $\rho(u)$  localized on a set [17]. There occurs, therefore, an interesting problem: that is better, for stabilization, an internal noise or chaos? Under which conditions the chaos is stabler?

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