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Uniform stability, boundedness and asymptotic behaviour of solutions of some third order nonlinear delay differential equations

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Abstract

In this paper, sufficient conditions are established for uniform asymptotic stability of the trivial solutions, uniform boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of some third order nonlinear neutral delay differential equations. We employ Lyapunov's direct method by constructing a complete Lyapunov functional to obtain the results. Recent results on third order nonlinear delay differential equations are particular cases of our results.

Keywords: Third order; Nonlinear differential equation; Uniform stability and ultimate boundedness; Asymptotic behaviour; Complete Lyapunov functional

1 Introduction

The difficulties in obtaining analytical solutions of nonlinear differential equations resulting from mathematical models prompted researchers in the determination of the qualitative behaviour of solutions. Methods, such as Lyapunov's direct method, frequency domain approach, comparison theorems, adaptive methods, canonical transformation, solutions representations, the well known Cauchy formula, to mention few have been developed to obtain information on the qualitative behaviour of solutions of differential equations in the literature when there is no analytical expression for solutions.

Till now with respect to our observation in the relevant literature, the most effective method to determine qualitative behaviour of solutions of differential equations is still the Lyapunov's direct method, see for instance: Burton *et al* [12] - [15], Driver [17], Hale [21], [22], Reissig *et al* [28], Rouch *et al* [29] and Yoshizawa [45, 46] which contain the general results on the subject matter. Other notable authors include Ademola *et al* [2, 1, 3, 4, 5, 7, 8], Chukwu [16], Ezeilo [18, 19, 20], Hara [23], Ogundare [24], Omeike [25, 26], Swick [31, 32], Bereketoglu, H. and Karakoc, F. [11], Tejumola [33], Tunç [35, 38, 41], Yamamoto [43] on ordinary differential equations. Ademola and Arawomo [6], Afuwape and Omeike [9, 10], Omeike [27], Sadek [30], Tejumola [34], Tunç [36, 37, 39, 40, 42], Yao and Wang [44] and Zhu [47] on functional or delay differential equations.

For over four decades many authors have dealt with delay differential equations and obtained many interesting results as stated above. In 2010, in particular, Yao and Wang [44] discussed conditions for global asymptotic stability of the third order nonlinear delay differential equation

$$\dddot{x} + \varphi(\ddot{x}) + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = 0.$$

Moreover, Afuwape and Omeike [10], and Omeike [27] established criteria for stability and boundedness of solutions of the following third order delay differential equations

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)f(x(t-r(t))) = 0$$

and

$$\dddot{x} + h(\dot{x})\ddot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = p(t, x, \dot{x}, x(t-r(t)), \dot{x}(t-r(t)), \ddot{x}).$$

Finally, Tunç [42] studied conditions for stability and boundedness of solutions for the non autonomous differential equation

$$\dddot{x} + a(t)\ddot{x} + b(t)g_1(\dot{x}(t-r(t))) + g_2(\dot{x}) + h(x(t-r(t)))$$
$$= p(t, x, \dot{x}, x(t-r(t)), \dot{x}(t-r(t)), \ddot{x}).$$

However, the problem of uniform asymptotic stability of the trivial solution, uniform boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of delay differential equations, in general, and those of third order, in particular, is not still solved for more general nonlinearity. Using a complete Lyapunov functional, the purpose of this paper therefore is to obtain conditions for uniform asymptotic stability of the zero solution, uniform boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the third order nonlinear non autonomous neutral delay differential equation

$$\begin{aligned}\dddot{x}+&f(x(t-r(t)),\dot{x}(t-r(t)))\ddot{x}+g(x(t-r(t)),\dot{x}(t-r(t)))\\&+h(x(t-r(t)))=p(t,x,\dot{x},x(t-r(t)),\dot{x}(t-r(t)),\ddot{x})\end{aligned} \tag{1.1}$$

or its equivalent system derived by setting $\dot{x}=y$ $\ddot{x}=z$ i.e.

$$\begin{aligned}&\dot{x}=y,\dot{y}=z,\dot{z}=p(t,x,y,x(t-r(t)),y(t-r(t)),z)-f(x,y)z\\&-g(x,y)-h(x)+\int_{t-r(t)}^{t}f_x(x(s),y(s))y(s)z(t)ds\\&+\int_{t-r(t)}^{t}f_y(x(s),y(s))z(s)z(t)ds+\int_{t-r(t)}^{t}g_x(x(s),y(s))y(s)ds\\&+\int_{t-r(t)}^{t}g_y(x(s),y(s))z(s)ds+\int_{t-r(t)}^{t}h'(x(s))y(s)ds,\end{aligned} \tag{1.2}$$

where $0\le r(t)\le\gamma$, $\gamma>0$ is a constant which will be determined later, the functions f,g,h and p are continuous in their respective arguments and the derivatives $f_x(x,y)$, $f_y(x,y)$, $g_x(x,y)$, $g_y(x,y)$, $h'(x)$ exist and are continuous for all x,y,z with $h(0)=g(0,0)=g(x,0)=0$. The dots as usual stands for differentiation with respect to the independent variable t. Also, conditions for existence and uniqueness of solutions of (1.2) are assumed. The results obtained in this investigation improve and extend the existing results on the third order nonlinear delay differential equations in the literature.

2 Preliminaries

Consider the general autonomous delay differential system

$$\dot{X}=F(X_t),\quad X_t(\theta)=X(t+\theta),\quad -r\le\theta\le 0.\ \ t\ge 0 \tag{2.1}$$

where $F:C_H\to\mathbb{R}^n$ is a continuous mapping , $F(0)=0$, we suppose that F takes closed bounded set of $\mathbb{R}^n$. Here $(C,\|\cdot\|)$ is the Banach space of continuous function $\phi:[-r,0]\to\mathbb{R}^n$ with supremum norm, $r>0$, C_H is an open ball of radius H in C;

$$C_H:=\{\phi\in C([-r,0],\mathbb{R}^n):\|\phi\|<H\}.$$

It has been shown by Burton [15], that if $\phi \in C_H$, and $t \geq 0$, then there is at least one continuous solution $X(t, t_0, \phi)$ satisfying (2.1) for $t > t_0$ on the interval $[t_0, t_0 + \alpha)$, such that $X_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remain in B, then $\alpha = \infty$.

Definition 1 A continuous function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$, $W(s) > 0$ if $s \neq 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer).

Definition 2 The zero solution of (2.1) is asymptotically stable if it is stable and if for each $t_0 \geq 0$ there is an $\eta > 0$ such that $\|\phi\| \leq \eta$ implies that

$$X(t, t_0, \phi) \to 0 \text{ as } t \to \infty.$$

Definition 3 An element $\psi \in C_H$ is in the $\omega-$limit set of ϕ, say $\Omega(\phi)$, if $X(t, 0, \phi)$ is defined on $\mathbb{R}^+$ and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to +\infty$, with $\|X_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$, where $X_{t_n}(\phi) = X(t_n + \theta, 0, \phi)$ for $-r \leq \theta < 0$.

Definition 4 A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution $X(t, 0, \phi)$ of (2.1) is defined on $\mathbb{R}^+$ and $X_t(\phi) \in Q$ for $t \in \mathbb{R}^+$.

Next, consider the system

$$\dot{X} = F(t, X_t),\ X_t = X(t + \theta),\ -r \leq \theta \leq 0,\ t \geq 0, \tag{2.2}$$

Where $F : [0, \infty) \times C \to \mathbb{R}^n$ is continuous and takes bounded sets into bounded sets.

Definition 5 Let $V(t, \phi)$ be a continuous functional defined for $t \in \mathbb{R}^+$, $\phi \in C_H$. The derivative of this functional V along a solution of (2.2) is defined by the following relation

$$\dot{V}_{(2.2)}(t, \phi) = \limsup_{h \to 0} \frac{V(t + h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h}$$

where $x(t_0, \phi)$ is the solution of (2.2) with $x_{t_0}(t_0, \phi) = \phi$.

Lemma 2.1 [15, 22, 47] If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $\mathbb{R}^+$ and $\|X_t(\phi)\| \leq H_1 < H$ for $t \in \mathbb{R}^+$, then $\Omega(\phi)$ is nonempty, compact, invariant set, and

$$dist(X_t(\phi), \Omega(\phi)) \to 0, \quad \text{as } t \to \infty.$$

Lemma 2.2 [14, 47] Let $V(\phi) : C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$ and such that

(i) $W(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$ where $W_1(r)$ and $W_2(r)$ are wedges;

(ii) $\dot{V}_{(2.1)}(\phi) \le 0$ for $\phi \in C_H$,

then the zero solution of (2.1) is uniformly stable.
If we define $Z = \{\phi \in C_H : \dot{V}_{(2.1)}(\phi) = 0\}$, then $X_t = 0$ of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $M = \{\mathbf{0}\}$.

Lemma 2.3 [45] Let $V(\phi)$ be a continuous Liapunov functional on C_H and let U_l denote the region such that $V(\phi) < l$. Suppose that $V(\phi) \ge 0$ and $\dot{V}_{(2.1)}(\phi) \le 0$ for all $\phi \in U_l$ and that there exists a constant $K \ge 0$ such that $|\phi(0)| \le K$ for all $\phi \in U_l$. If E is the set of all points in U_l where $\dot{V}_{(2.1)}(\phi) = 0$ and M is the largest invariant set in E, then every solution of (2.1) with initial value in U_l approaches M as $t \to \infty$.

The following lemma is a well-known result obtained by Burton [14].

Lemma 2.4 [14] Let $V : \mathbb{R}^+ \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If

(i) $W_0(|X_t|) \le V(t, X_t) \le W_1(|X_t|) + W_2\left(\int\limits_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ and

(ii) $\dot{V}_{(2.2)}(t, X_t) \le -W_4(|X_t|) + N$, for some $N > 0$ where W_i $(i = 0, 1, 2, 3, 4)$ are wedges,
then X_t of (2.2) is uniformly bounded and uniformly ultimately bounded for bound B.

3 Main Results

The main tool in the proofs of our results is the continuously differentiable functional $V \equiv V(t, x_t, y_t, z_t)$ defined as

$$V = e^{-P(t)}U, \tag{3.1a}$$

where

$$P(t) = \int_0^t |p(\mu, x, y, x(\mu - r(\mu)), y(\mu - r(\mu)), z)|d\mu \tag{3.1b}$$

and $U \equiv U(t, x_t, y_t, z_t)$ is defined as

$$
\begin{aligned}
2U &= 2(\alpha + a)\int_0^x h(\xi)d\xi + 4\int_0^y g(x,\tau)d\tau + 4yh(x) \\
&+ 2(\alpha + a)yz + 2z^2 + 2(\alpha + a)\int_0^y \tau f(x,\tau)d\tau + \beta y^2 + b\beta x^2 \\
&+ 2a\beta xy + 2\beta xz + \int\limits_{-r(t)}^{0}\int\limits_{t+s}^{t}[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds
\end{aligned}
\tag{3.1c}
$$

where: $a > 0, b > 0$ and $c > 0$ are constants; λ_1 and λ_2 are positive constants which will be determined later; $\alpha > 0$ and $\beta > 0$ are fixed constants satisfying

$$
c < \alpha b < ab; \tag{3.1d}
$$

and

$$
0 < \beta < \min\left\{(ab - c)a^{-1}, (ab - c)A_0^{-1}, \frac{1}{2}(a - \alpha)A_1^{-1}\right\} \tag{3.1e}
$$

where $A_0 := 1 + a + \delta^{-1}(\frac{g(x,y)}{y} - b)^2$ and $A_1 := 1 + \delta^{-1}(f(x,y) - a)^2$.We have the following results.

Theorem 3.1 Further to the basic assumptions on the functions f, g, h and p appearing in (1.2), suppose that $a, a_1, b, c, \beta_0, \delta, \gamma, L_1, L_2$ and M_2 are positive constants and that:

(i) $a \le f(x,y) \le a_1$, $yf_x(x,y) \le 0$, $|zf_x(x,y)| \le L_1$, $|zf_y(x,y)| \le L_2$ for all x, y and $z \ne 0$;

(ii) $g(0,0) = g(x,0) = 0$, $b \le \frac{g(x,y)}{y}$ for all x and $y \ne 0$, $|g_x(x,y)| \le M_1$ for some $M_1 \ge 0$, and $|g_y(x,y)| \le M_2$ for all x, y;

(iii) $h(0) = 0$, $\delta \le \frac{h(x)}{x}$ for all $x \ne 0$, $|h'(x)| \le c$ for all x and $ab > c$;

(iv) $r(t) \le \gamma$, $r'(t) \le \beta_0$ $0 < \beta_0 < 1$;

(v) $\int_0^\infty |p(t, x, y, x(t-r), y(t-r), z)|dt < \infty$ for all $t \ge 0$, x, y and $r(t) \ge 0$;

then every solution (x_t, y_t, z_t) of the system (1.2) satisfies

$$
\lim_{t\to\infty} x_t = 0, \quad \lim_{t\to\infty} y_t = 0, \quad \lim_{t\to\infty} z_t = 0, \tag{3.2}
$$

provided that

$$\gamma < \min\left\{\delta A_2^{-1}, 2(1-\beta_0)(\alpha b - c)A_3^{-1}, \frac{1}{2}(1-\beta_0)(a-\alpha)A_4^{-1}\right\} \tag{3.3}$$

where

$$A_2 := c + M_1 + M_2 + L_1 + L_2; \quad A_3 := (1-\beta_0)(a+\alpha)(c + M_1 + M_2 + L_1 + L_2)$$
$$+(c + M_1 + L_1)(2 + \alpha + \beta + a)$$

and

$$A_4 := 2(c + M_1 + M_2 + L_1 + L_2)(1-\beta_0) + (M_2 + L_2)(2 + \alpha + \beta + a).$$

If the function $p(t, x, y, x(t-r), y(t-r), z)$ in (1.2) is replaced by $p(t, x, y, z)$ and $p(t)$, we have the following results:

Corollary 3.1 Suppose that hypotheses (i)-(iv) of Theorem 3.1 hold, and

(i) $\int_0^\infty |p(t, x, y, z)|dt < \infty$ for all $t \geq 0$, $p : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$, then the solution (x_t, y_t, z_t) of (1.2) satisfies (3.2) provided that (3.3) holds.

(ii) $\int_0^\infty |p(t)|dt < \infty$ for all $t \geq 0$, $p : \mathbb{R}^+ \to \mathbb{R}$, then the solution x_t its first and second derivatives satisfy (3.2) if (3.3) holds.

Remark 3.1 If (1.1) is a constant coefficients neutral delay differential equation

$$\dddot{x} + a\ddot{x}(t-r(t)) + b\dot{x}(t-r(t)) + cx(t-r(t)) = 0$$

then conditions (i)-(v) of Theorem 3.1 reduce to the Routh-Hurtwitz conditions $a > 0$, $ab > c$ and $c > 0$. To show this, we assume $f(x(t-r(t)), \dot{x}(t-r(t))) = a$, $g(x(t-r(t)), \dot{x}(t-r(t))) = b\dot{x}(t-r(t))$, $h(x(t-r(t))) = cx(t-r(t))$ and $p(t, x, \dot{x}, x(t-r(t)), \dot{x}(t-r(t)), \ddot{x}) = 0$.

Theorem 3.2 If the assumptions (i)-(iv) of Theorem 3.1 hold true and $|p(t, x, y, x(t-r(t)), y(t-r(t)), z)| \leq P_2$, then the solutions of the system (1.2) are uniformly bounded and uniformly ultimately bounded, if the inequality in (3.3) holds.

Theorem 3.3 If the assumptions (i)-(iv) of Theorem 3.1 hold true and

$$|p(t, x, y, x(t-r(t)), y(t-r(t)), z)| \leq \phi_1(t) + \phi_2(t)(|x| + |y| + |z|) \tag{3.4a}$$

for all $t \geq 0, x, y$ and z where $\phi_1(t)$ and $\phi_2(t)$ are continuous functions satisfying:

$$\phi_1(t) \leq \varphi_1, \tag{3.4b}$$

where $0 < \varphi_1 < \infty$; and there exists $\epsilon > 0$ such that

$$0 \leq \phi_2(t) \leq \epsilon; \tag{3.4c}$$

then the solutions of (1.2) are uniformly bounded and uniformly ultimately bounded provided that the inequality in (3.3) is satisfied.

Theorem 3.4 Suppose that assumptions (i)-(v) of Theorem 3.1 hold, then there exists a finite constant $D_1 = D_1(a, a_1, b, c, \alpha, \beta, \delta, M_2, \lambda_1, \lambda_2, x_0, y_0, z_0, P_0)$ such that the unique solution $(x_t, y_t, z_t,)$ of (1.1) defined by the initial functions

$$x_{t_0} = x(\phi), \quad y_{t_0} = y(\phi), \quad z_{t_0} = z(\phi) \tag{3.5a}$$

satisfies the inequalities

$$|x_t| \leq D_1, \quad |y_t| \leq D_1, \quad |z_t| \leq D_1 \tag{3.5b}$$

for all $t \geq 0$ where $\phi \in C([-r, 0], \mathbb{R}^3)$, provided that the inequality in (3.3) holds true.

Remark 3.2 (i) If $p(t, x, y, x(t-r(t)), y(t-r(t)), z)$ is replaced by $p(t, x, y, z)$ or $p(t)$, then the conclusion of Theorem 3.2 holds.

(ii) If $f(x(t-r(t)), \dot{x}(t-r(t))) = a$, $g(x(t-r(t)), \dot{x}(t-r(t))) = g(\dot{x}(t-r(t)))$ and $p(t, x, \dot{x}, x(t-r(t)), \dot{x}(t-r(t)), \ddot{x}) = p(t)$ (1.2) reduces to the case discussed by Sadek [30] and Tunç [36]. Our hypotheses and conclusions coincide with theirs, thus our results extend theirs.

(iii) Whenever $f(x(t-r(t)), \dot{x}(t-r(t))) = f(y)$, $g(x(t-r(t)), \dot{x}(t-r(t))) = g(\dot{x}(t-r(t)))$ and $p(t, x, \dot{x}, x(t-r(t)), \dot{x}(t-r(t)), \ddot{x}) = p(t, x, \dot{x}, \ddot{x})$. Our hypotheses and conclusion coincide with that of Afuwape and Omeike [9, 10]. Hence, our results include and extend theirs.

Next, if $p \equiv 0$ in (1.2), we have the following system of equations

$$\begin{aligned} &\dot{x} = y, \dot{y} = z, \dot{z} = -f(x, y)z - g(x, y) - h(x) + \int_{t-r(t)}^{t} h'(x(s))y(s)ds \\ &+ \int_{t-r(t)}^{t} f_x(x(s), y(s))y(s)z(t)ds + \int_{t-r(t)}^{t} f_y(x(s), y(s))z(s)z(t)ds \\ &+ \int_{t-r(t)}^{t} g_x(x(s), y(s))y(s)ds + \int_{t-r(t)}^{t} g_y(x(s), y(s))z(s)ds, \end{aligned} \tag{3.6}$$

with the following result.

Theorem 3.5 Suppose that assumptions (i)-(iv) of Theorem 3.1 hold true, then the trivial solution of (3.6) is uniformly asymptotically stable provided that estimate (3.3) is satisfied.

Remark 3.3 In [9, 30, 36, 37] and [44] an incomplete Lyapunov's functionals were constructed and used to obtain stability results compare with a complete Lyapunov's functional used in this investigation. Thus our results generalize theirs.

The next lemma establishes the validity and reliability of the functional used in this investigation.

Lemma 3.1 If all assumptions of Theorem 3.1 hold true, then there exist positive constants D_2, D_3 and D_4 such that along a solution of (1.2)

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t,z_t) = \dot{V} \leq -D_2(x^2(t)+y^2(t)+z^2(t)) - [D_3(x^2(t)\\ + y^2(t)+z^2(t)) - D_4(|x|+|y|+|z|)]|p(t,x,y,x(t-r(t)),y(t-r(t)),z)|\end{aligned} \tag{3.7a}$$

for all x, y and z. Moreover, $V(t,0,0,0) = 0$ and there exist positive constants D_5, D_6 and D_7 such that

$$\begin{aligned}D_5(x^2(t)+y^2(t)+z^2(t)) \leq V(t,x_t,y_t,z_t) \leq D_6(x^2(t)+y^2(t)+z^2(t))\\ + D_7\int_{-r(t)}^{0}\int_{t+s}^{t}[x^2(\theta)+y^2(\theta)+z^2(\theta)]d\theta ds\end{aligned} \tag{3.7b}$$

for all x, y and z.

Proof: Let (x_t, y_t, z_t) be any solution of (1.2), the derivative of the functional V defined in (3.1a) along a solution of (1.2) is

$$\dot{V}_{(1.2)} = -e^{-P(t)}[U\dot{P}(t) - \dot{U}_{(1.2)}] \tag{3.8a}$$

where

$$\dot{P}(t) = |p(t,x,y,x(t-r(t)),y(t-r(t)),z)| \tag{3.8b}$$

and

$$
\begin{aligned}
\dot{U}_{(1.2)} &= 2y\int_0^y g_x(x,\tau)d\tau + (\alpha+a)y\int_0^y \tau f_x(x,\tau)d\tau + a\beta y^2 + 2\beta yz \\
&+ (\beta x + (\alpha+a)y + 2z)\Bigg\{p(t,x,y,x(t-r(t)),y(t-r(t)),z) \\
&+ \int_{t-r(t)}^t \Big[z(t)f_x(x(s),y(s))y(s) + z(t)f_y(x(s),y(s))z(s) \\
&+ g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s) + h'(x(s),y(s))y(s)\Big]ds\Bigg\} \\
&+ (\lambda_1 y^2 + \lambda_2 z^2)r(t) - (1-\dot{r}(t))\int_{t-r(t)}^t [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta \\
&- \beta\frac{h(x)}{x}x^2 - [(\alpha+a)\frac{g(x,y)}{y} - 2h'(x)]y^2 - [2f(x,y) - (\alpha+a)]z^2 \\
&- \beta(\frac{g(x,y)}{y} - b)xy - \beta(f(x,y)-a)xz
\end{aligned} \tag{3.8c}
$$

Eq. (3.8c) can be rearranged in the form

$$
\begin{aligned}
\dot{U}_{(1.2)} &= U_1 + U_2 + U_3 - U_4 - U_5 + (\lambda_1 y^2 + \lambda_2 z^2)r(t) \\
&- (1-\dot{r}(t))\int_{t-r(t)}^t [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta
\end{aligned} \tag{3.9}
$$

where:

$$U_1 := a\beta y^2 + 2\beta yz + 2y\int_0^y g_x(x,\tau)d\tau + (\alpha+a)y\int_0^y \tau f(x,\tau)d\tau;$$

$$U_2 := (\beta x + (\alpha+a)y + 2z)p(t,x,y,x(t-r(t)),y(t-r(t)),z);$$

$$
\begin{aligned}
U_3 := (\beta x + (\alpha+a)y + 2z)\int_{t-r(t)}^t \Big[&z(t)f_x(x(s),y(s))y(s) + z(t)f_y(x(s),y(s))z(s) \\
&+ g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s) + h'(x(s))y(s)\Big]ds;
\end{aligned}
$$

$$U_4 = \beta\frac{h(x)}{x}x^2 + [(\alpha+a)\frac{g(x,y)}{y} - 2h'(x)]y^2 + [2f(x,y) - (\alpha+a)]z^2;$$

and

$$U_5 = -\beta(\frac{g(x,y)}{y} - b)xy - \beta(f(x,y)-a)xz.$$

Applying the hypotheses of the Theorem 3.1 and the fact that $2pq \leq p^2 + q^2$ we have the following estimates for $U_i (i = 1, 2, 3, 4, 5)$:

$$U_1 \leq \beta[(1+a)y^2 + z^2],$$

$$U_2 \leq \max\{\beta, \alpha + a, 2\}(|x| + |y| + |z|)|p(t, x, y, x(t - r(t)), y(t - r(t)), z)|,$$

$$U_3 \leq \frac{1}{2}(c + M_1 + M_2 + L_1 + L_2)(\beta x^2 + (\alpha + a)y^2 + 2z^2)r(t)$$
$$+\frac{1}{2}(c + M_1 + L_1)(\alpha + \beta + a + 2)\int_{t-r(t)}^{t} y^2(s)ds$$
$$+\frac{1}{2}(M_2 + L_2)(\alpha + \beta + a + 2)\int_{t-r(t)}^{t} z^2(s)ds,$$

$$U_4 \geq \beta\delta x^2 + [(\alpha + a)b - 2c]y^2 + (a - \alpha)z^2$$

and

$$U_5 = \frac{1}{4}\beta\delta\left[x + 2\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)y\right]^2 + \frac{1}{4}\beta\delta\left[x + 2\delta^{-1}\Big(f(x,y) - a\Big)z\right]^2$$
$$-\frac{1}{2}\delta\beta x^2 - \beta\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)^2 y^2 - \beta\delta^{-1}\Big(f(x,y) - a\Big)^2 z^2.$$

Employing estimates U_i $i = 1, \cdots, 5$ in (3.9), we obtain

$$\dot{U}_{(1.2)} \leq \beta[(1+a)y^2 + z^2] + \beta\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)^2 y^2 + \beta\delta^{-1}\Big(f(x,y) - a\Big)^2 z^2$$
$$+ \max\{\beta, \alpha + a, 2\}(|x| + |y| + |z|)|p(t, x, y, x(t - r(t)), y(t - r(t)), z)|$$
$$\frac{1}{2}(c + M_1 + M_2 + L_1 + L_2)(\beta x^2 + (\alpha + a)y^2 + 2z^2)r(t) - (a - \alpha)z^2$$
$$+ \frac{1}{2}(c + M_1 + L_1)(\alpha + \beta + a + 2)\int_{t-r(t)}^{t} y^2(s)ds + (\lambda_1 y^2 + \lambda_2 z^2)r(t)$$
$$+ \frac{1}{2}(M_2 + L_2)(\alpha + \beta + a + 2)\int_{t-r(t)}^{t} z^2(s)ds - \frac{1}{2}\delta\beta x^2 - (\alpha b + ab - 2c)y^2$$
$$- \frac{1}{4}\beta\delta\left[x + 2\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)y\right]^2 - \frac{1}{4}\beta\delta\left[x + 2\delta^{-1}\Big(f(x,y) - a\Big)z\right]^2$$
$$- (1 - \dot{r}(t))\int_{t-r(t)}^{t} [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta.$$

Now, since $\left[x + 2\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)y\right]^2 \geq 0$, $\left[x + 2\delta^{-1}\left(f(x,y) - a\right)z\right]^2 \geq 0$ for all x, y, z and $r'(t) \leq \beta_0$, it follows that

$$\begin{aligned}
\dot{U}_{(1.2)} &\leq \max\{\beta, \alpha + a, 2\}(|x| + |y| + |z|)|p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \\
&- \left\{ab - c - \beta\left[1 + a\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]\right\}y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - \beta\left[1 + \delta^{-1}\left(f(x,y) - a\right)^2\right]\right\}z^2 \\
&- \left\{(1 - \beta_0)\lambda_1 - \frac{1}{2}(c + M_1 + L_1)(\alpha + \beta + a + 2)\right\}\int_{t-r(t)}^{t} y^2(s)ds \\
&- \left\{(1 - \beta_0)\lambda_2 - \frac{1}{2}(M_2 + L_2)(\alpha + \beta + a + 2)\right\}\int_{t-r(t)}^{t} z^2(s)ds \\
&- \frac{1}{2}\beta\left\{\delta - (c + M_1 + M_2 + L_1 + L_2)r(t)\right\}x^2 - \\
&\left\{\alpha b - c - \left[\frac{(\alpha + a)}{2}(c + M_1 + M_2 + L_1 + L_2) + \lambda_1\right]r(t)\right\}y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - (c + M_1 + M_2 + L_1 + L_2 + \lambda_2)r(t)\right\}z^2.
\end{aligned}$$

In view of estimate (3.1e), the fact that $0 < \beta_0 < 1$, $r(t) \leq \gamma$, $\gamma > 0$ for all $t \geq 0$, choosing $\lambda_1 = \frac{1}{2}(c + M_1 + L_1)(\alpha + \beta + a + 2)(1 - \beta_0)^{-1} > 0$ and $\lambda_2 = \frac{1}{2}(M_2 + L_2)(\alpha + \beta + a + 2)(1 - \beta_0)^{-1} > 0$ we have

$$\begin{aligned}
\dot{U}_{(1.2)} &\leq \max\{\beta, \alpha + a, 2\}(|x| + |y| + |z|)|p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \\
&- \frac{\beta}{2}\left\{\delta - (c + M_1 + M_2 + L_1 + L_2)\gamma\right\}x^2 - \left\{\alpha b - c - \left[(\alpha + a)(1 - \beta_0)(c\right.\right. \\
&\left.\left.+ M_1 + M_2 + L_1 + L_2) + (M_2 + L_2)(\alpha + \beta + a + 2)\right](1 - \beta_0)^{-1}\gamma\right\}y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - \left[2(c + M_1 + M_2 + L_1 + L_2)(1 - \beta_0) + (M_2 + L_2)(\alpha + \beta\right.\right. \\
&\left.\left.+ a + 2)\right](1 - \beta_0)^{-1}\gamma\right\}z^2.
\end{aligned}$$

By estimate (3.3), there exists a positive constant δ_0 such that

$$\begin{aligned}
\dot{U}_{(1.2)} &\leq -\delta_0(x^2 + y^2 + z^2) + \delta_1(|x| + |y| + |z|) \\
&\times |p(t, x, y, x(t - r(t)), y(t - r(t)), z)|
\end{aligned} \tag{3.10}$$

for all $t \geq 0, x, y, z$ where $\delta_1 = \max\{\beta, \alpha + a, 2\}$.
Furthermore, the functional U defined in (3.1c) can be rearranged in the form

$$\begin{aligned}
U &= b^{-1}\int_0^x [((\alpha + a))b - 2h'(\xi)]h(\xi)d\xi + 2\int_0^y \left(\frac{g(x,\tau)}{\tau} - b\right)\tau d\tau \\
&+ b^{-1}(h(x) + by)^2 + \int_0^y [(\alpha + a)f(x,\tau) - (\alpha^2 + a^2)]\tau d\tau + \frac{1}{2}(\beta x + ay + z)^2 \\
&+ \frac{1}{2}(z + \alpha y)^2 + \frac{1}{2}y^2 + \frac{\beta}{2}(b - \beta)x^2 \\
&+ \frac{1}{2}\int_{-r(t)}^0 \int_{t+s}^t [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds.
\end{aligned}$$

Now, on applying the hypotheses of Theorem 3.1 and since the double integrals is non-negative, this equation yields

$$\begin{aligned}
U &\geq \frac{1}{2}[(\alpha + a)b - 2c]b^{-1}\delta x^2 + b^{-1}(\delta x + by)^2 + \frac{1}{2}[\alpha(a - \alpha) + \beta]y^2 \\
&+ \frac{1}{2}(\beta x + ay + z)^2 + \frac{1}{2}(z + \alpha y)^2.
\end{aligned} \tag{3.11}$$

In view of (3.1d) and (3.1e), we have $\alpha b > c$, $ab > c$, $a > c$ and $b > \beta$ so that the right hand sides of (3.11) is positive definite. Hence, there exists a positive constant $\delta_2 = \delta_2(a, b, c, \alpha, \beta, \delta)$ such that

$$U \geq \delta_2(x^2 + y^2 + z^2), \tag{3.12}$$

for all x, y and z. Also, by (3.1b) and hypothesis (v) of Theorem 3.1, there exists a constant $P_0 > 0$ such that

$$0 \leq P(t) \leq P_0 \tag{3.13}$$

for all $t \geq 0$. Using estimate (3.10), (3.12) and (3.13) in (3.8a), we obtain

$$\begin{aligned}
\dot{V}_{(1.2)} &\leq -\delta_3(x^2 + y^2 + z^2) - [\delta_4(x^2 + y^2 + z^2) \\
&- \delta_5(|x| + |y| + |z|)]|p(t, x, y, x(t - r(t)), y(t - r(t)), z)|
\end{aligned} \tag{3.14}$$

for all $t \geq 0, x, y$ and z, where $\delta_3 = \delta_0 e^{-P_0}$, $\delta_4 = \delta_2 e^{-P_0}$ and $\delta_5 = \delta_1 e^{-P_0}$. This establishes estimate (3.7a).
Moreover, using estimates (3.12) and (3.13) in (3.1a), we have

$$V \geq \delta_6(x^2 + y^2 + z^2) \tag{3.15}$$

for all $t \geq 0, x, y$ and z, where $\delta_6 = \delta_2 e^{-P_0} > 0$. Also, since $h'(x) \leq c$, $g_y(x,y) \leq M_2$, $h(0) = 0 = g(x,0)$ for all x and $f(x,y) \leq a_1$ for all x and y. It is not difficult to show that

$$V \leq \delta_7(x^2 + y^2 + z^2) + \delta_8 \int_{-r(t)}^{0} \int_{t+s}^{t} [x^2(\theta) + y^2(\theta) + z^2(\theta)] d\theta ds \tag{3.16}$$

for all $t \geq 0, x, y$ and z, where

$$\delta_7 = \frac{1}{2} \max \Big\{ 2 + \beta(1 + a + b) + c(\alpha + a), (\alpha + a)(1 + a_1) + 2(M_2 + 1)$$
$$+\beta(1 + a), 2 + \alpha + \beta + a \Big\}$$

and

$$\delta_8 = \frac{1}{2} \max\{1, \lambda_1, \lambda_2\}.$$

Combining the inequalities in (3.15) and (3.16) estimate (3.7b) holds, this completes the proof of Lemma 3.1.

Next, we shall show, from the following result, that boundedness of the functional V necessarily implies boundedness of solutions of the neutral delay differential equation (1.2).

Lemma 3.2 Suppose that a, b, c, δ are positive constants and for all $t \geq 0$:

(i) $f(x,y) \geq a$ for all x, y;

(ii) $g(x,0) = 0$, $g(x,y)/y \geq b$ $y \neq 0$;

(iii) $h(0) = 0$, $\frac{h(x)}{x} \geq \delta$ $(x \neq 0)$, $h'(x) \leq c$ for all x and $ab > c$;

then for any positive constant D_8 with

$$V(x_t, y_t, z_t) \leq D_8 \tag{3.17}$$

there exists a positive constant $D_9 = D_9(a, b, c, \alpha, \beta, \delta_0, P_0, D_8)$ such that

$$|x(t)| \leq D_9, |y(t)| \leq D_9, |z(t)| \leq D_9 \tag{3.18}$$

for all $t \geq 0, x, y$ and z.

Proof: Let (x_t, y_t, z_t) be any solution of (1.2). From estimate (3.15) and (3.17), we have

$$x^2 + y^2 + z^2 \leq \delta_6^{-1} V \leq \delta_6^{-1} D_8$$

for all $t \geq 0, x, y$ and z. Estimate (3.18) follows immediately where $D_9 = \delta_6^{-1} D_8$. This completes the proof of Lemma 3.2.

4 Proof of Theorems

Proof of Theorem 3.1. Let (x_t, y_t, z_t) be any solution of the system (1.2). Now since $(|x|+|y|+|z|)^2 \leq 3(x^2+y^2+z^2)$ and choosing $(x^2+y^2+z^2)^{1/2} \geq \delta_9$ where $\delta_9 = 3^{1/2}\delta_4^{-1}\delta_5 > 0$ is a constant, estimate (3.14) becomes

$$\dot{V}_{(1.2)} \leq -\delta_3(x^2+y^2+z^2) \leq 0 \tag{4.1}$$

for all $t \geq 0, x, y$ and z. Now by Lemma 3.2 equation (3.18) the boundedness of solutions of (1.2) is assured. Consider the set

$$\Omega_1 = \{X_t = (x_t, y_t, z_t) \in \mathbb{R}^3 | \dot{V}_{(1.2)}(t, X_t) = 0\},$$

and since (1.2) can be written in the form

$$\dot{X} = F(X_t) + G(t, X_t), \tag{4.2}$$

where

$$F(X_t) = \Bigg(y, z, -f(x,y)z - g(x,y) - h(x) + \int_{t-r(t)}^{t} f_x(x(s), y(s))y(s)z(t)ds$$
$$+ \int_{t-r(t)}^{t} f_y(x(s), y(s))z(s)z(t)ds + \int_{t-r(t)}^{t} g_x(x(s), y(s))y(s)ds$$
$$+ \int_{t-r(t)}^{t} g_y(x(s), y(s))z(s)ds + \int_{t-r(t)}^{t} h'(x(s))y(s)ds\Bigg)^T$$

and

$$G(t, X_t) = \Big(0, 0, p(t, x, y, x(t-r(t)), y(t-r(t)), z)\Big)^T.$$

By (4.1), the fact that $\dot{V}_{(1.2)}(t, X_t) = 0$ on Ω_1, implies that $x = y = z = 0$ and since $h(0) = g(0,0) = 0$, it follows that

$$\dot{X} = F(X_t)$$

has solution

$$X_t^T = K^T$$

where $X_t = (x_t, y_t, z_t) \in \mathbb{R}^3$ and $K = (k_1, k_2, k_3)$. For $X_t \in \mathbb{R}^3$ to remain in Ω_1, we must have $k_1 = k_2 = k_3 = 0$. The largest invariant set in Ω_1 is $\{0, 0, 0\}$ so that by estimates (3.15), (3.17), (3.18) and (4.1) all assumptions of Lemma 2.3 hold true, hence by Lemma 2.3, (3.2) is established. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2 Let (x_t, y_t, z_t) be any solution of the system (1.2), if $t = 0$ in (3.1b), then the proof of Theorem 3.2 depends on the functional U defined in (3.1c). By estimate (3.10) and the fact that $|p(t, x, y, x(t-r(t)), y(t-r(t)), z)| \leq P_2$ for all $t \geq 0, x, y$ and z, it follows that

$$\dot{U}_{(1.2)} \leq -\delta_0(x^2 + y^2 + z^2) + \delta_{10}(|x| + |y| + |z|) \tag{4.3}$$

for all x, y and z where $\delta_{10} = \delta_1 P_2 > 0$ is a constant. Since

$$(|x| - \delta_0^{-1}\delta_{10})^2 + (|y| - \delta_0^{-1}\delta_{10})^2 + (|z| - \delta_0^{-1}\delta_{10})^2 \geq 0$$

for all x, y and z, estimate (4.3) becomes

$$\dot{U}_{(1.2)} \leq -\delta_{11}(x^2 + y^2 + z^2) + \delta_{12} \tag{4.4}$$

for all x, y and z, where $\delta_{11} = \frac{1}{2}\delta_0$ and $\delta_{12} = \frac{3}{2}\delta_0^{-1}\delta_{10}^2$.
Moreover, by estimates (3.12) and (3.16), we obtain

$$\begin{aligned} \delta_2(x^2 + y^2 + z^2) \leq U \leq \delta_7(x^2 + y^2 + z^2) \\ + \delta_8 \int_{-r(t)}^{0} \int_{t+s}^{t} (x^2(\theta) + y^2(\theta) + z^2(\theta)) d\theta ds \end{aligned} \tag{4.5}$$

for all $t \geq 0, x, y$ and z. In view of estimates (4.4) and (4.5) the assumptions of Lemma 2.4 hold true, hence by Lemma 2.4 the solution (x_t, y_t, z_t) of (1.2) is uniformly bounded and uniformly ultimately bounded. This completes the proof of Theorem 3.2.

Proof of Theorem 3.3. Let (x_t, y_t, z_t) be any solution of the system (1.2), the proof of this result depends on the functional U defined in (3.1c). Using estimates (3.10) and (3.4a), we obtain

$$\dot{U}_{(1.2)} \leq -\delta_0(x^2 + y^2 + z^2) + \delta_1\phi_1(t)(|x| + |y| + |z|) + \delta_1\phi_2(t)(|x| + |y| + |z|)^2$$

for all $t \geq 0, x, y$ and z. The conclusion of remaining part of the proof follows the strategy in the proof of Theorem 3.2 in [6] and hence it is omitted.

Proof of Theorem 3.4] Let (x_t, y_t, z_t) be any solution of the system (1.2), from estimate (3.14), noting that $|x| < 1 + x^2$, we have

$$\begin{aligned} &\dot{V}_{(1.2)} \leq -\delta_3(x^2 + y^2 + z^2) - [(\delta_4 - \delta_5)(x^2 + y^2 + z^2) - 3\delta_5] \\ &\times |p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \end{aligned} \tag{4.6}$$

for all $t \geq 0, x, y$ and z. Since the integrals in (3.16) is positive, there exists a positive constant $\delta_{13} = \delta_{13}(\delta_7, \delta_8)$ such that

$$x^2 + y^2 + z^2 \geq \delta_{13}^{-1} V \tag{4.7}$$

for all $t \geq 0, x, y$ and z. Choosing $\delta_4 - \delta_5 = \delta_{14} > 0$, using estimate (4.7) in (4.6), we obtain

$$\dot{V}_{(1.2)} + \delta_{13}^{-1}\delta_{14}|p(t,x,y,x(t-r(t)),y(t-r(t)),z)|V$$
$$\leq 3\delta_5|p(t,x,y,x(t-r(t)),y(t-r(t)),z)|.$$

Solving this first order neutral delay differential inequality using the integrating factor
$\exp(\delta_{13}^{-1}\delta_{14}P(t))$ ($P(t)$ is defined in (3.1b)) and estimate (3.13), we find

$$V(t,x_t,y_t,z_t) \leq \delta_{15} \tag{4.8}$$

for all $t \geq 0, x_t, y_t$ and z_t, where $\delta_{15} = V(\vartheta) + 3\delta_5 P_0 \exp(\delta_{13}^{-1}\delta_{14}P_0)$, $V(\vartheta) = V(t_0, x_{t_0}, y_{t_0}, z_{t_0})$, $t_0 = 0$, $x_{t_0} = x(\vartheta)$, $y_{t_0} = y(\vartheta)$ and $z_{t_0} = z(\vartheta)$. Using (4.8) in (3.15), we obtain

$$|x_t| \leq \delta_{16}, |y_t| \leq \delta_{16}, |z_t| \leq \delta_{16}$$

for all $t \geq 0$. where $\delta_{16} = \delta_6^{-1}\delta_{15}$. This completes the proof of the theorem.

Proof of Theorem 3.5. Let (x_t, y_t, z_t) be any solution of the system (1.2), since $p \equiv 0$ in (1.2), Eq. (3.1a) coincides with (3.1c). Clearly, in view of (3.1c) $U(0,0,0) = 0$ and estimate (3.10) becomes

$$\dot{U}_{(1.2)} \leq -\delta_0(x^2 + y^2 + z^2) \leq 0$$

for all x, y, z. Also, by (3.12) and (4.7), we have

$$\delta_2(x^2 + y^2 + z^2) \leq U \leq \delta_{13}(x^2 + y^2 + z^2)$$

x, y and z. Now define a set

$$\Omega_2 = \{X_t = (x_t, y_t, z_t) \in \mathbb{R}^3 | \dot{U}_{(1.2)}(t, X_t) = 0\}.$$

The conclusion of remaining part of the proof follows the strategy in the proof of Theorem 3.1 and hence it is omitted.

5 A global example

As a particular case of (1.1), consider the following third order nonlinear non autonomous neural delay differential equation

$$
\begin{aligned}
&\dddot{x}(t) + \left[4 + \frac{1}{1 + \frac{1+|x(t-r(t))\dot{x}(t-r(t))|}{|\ddot{x}(t)|(1-\dot{x}^2(t-r(t)))}}\right]\ddot{x}(t) + 3\dot{x}(t-r(t)) \\
&\quad + \frac{\dot{x}(t-r(t))}{1 + |x(t-r(t))\dot{x}(t-r(t))| + \dot{x}^2(t-r(t))} \\
&\quad + 2x(t-r(t)) + \frac{x(t-r(t))}{1 + |x(t-r(t))|} \\
&\quad = \frac{1}{1 + t^2 + |x| + |\dot{x}| + |x(t-r(t))| + |\dot{x}(t-r(t))| + |\ddot{x}|}
\end{aligned}
\tag{5.1}
$$

or its equivalent system

$$
\begin{aligned}
&\dot{x} = y, \dot{y} = z, \dot{z} = \frac{1}{1 + t^2 + |x| + |y| + |x(t-r(t)| + |y(t-r(t)| + |z|} \\
&- \left[4 + \frac{1}{1 + \frac{1+|xy|}{|z|(1+y^2)}}\right] z - \left(3 + \frac{1}{1 + |xy| + y^2}\right) y - \left(2 + \frac{1}{1+|x|}\right) x \\
&- \int_{t-r(t)}^{t} \frac{y^2 z(t)}{|z(t)|(1+y^2(s))[1 + \frac{1+|x(s)y(s)|}{|z(t)|(1+y^2(s))}]^2} ds \\
&- \int_{t-r(t)}^{t} \frac{(|x(s)|(1-y^2(s)) - 2|y(s)|) z(t)}{|z(t)|(1+y^2(s))[1 + \frac{1+|x(s)y(s)|}{|z(t)|(1+y^2(s))}]^2} ds \\
&- \int_{t-r(t)}^{t} \frac{y^2(s) ds}{[1 + |x(s)y(s)| + y^2(s)]^2} \\
&+ \int_{t-r(t)}^{t} \left[3 + \frac{1 - y^2(s)}{[1 + |x(s)y(s)| + y^2(s)]^2}\right] z(s) ds \\
&+ \int_{t-r(t)}^{t} \left[2 + \frac{1}{(1 + |x(s)|)^2}\right] y(s) ds
\end{aligned}
\tag{5.2}
$$

From (1.2) and (5.2), we obtain the following relations:

(a) the function

$$
f(x, y) = \left[4 + \frac{1}{1 + \frac{1+|xy|}{|z|(1+y^2)}}\right]
$$

since $0 \le \dfrac{1}{1 + \frac{1+|xy|}{|z|(1+y^2)}} \le 1$ for all x, y and $z \neq 0$, we find that

$$
4 \le f(x, y) \le 5
$$

for all x, y and $z \neq 0$, where $a = 4 > 0$ and $a_1 = 5 > 0$. Also,

$$yf_x(x,y) = \frac{-y^2}{|z|(1+y^2)\left[1+\frac{1+|xy|}{|z|(1+y^2)}\right]^2} \leq 0$$

for all x, y and $z \neq 0$,

$$|zf_x(x,y)| \leq \left|\frac{|y|}{(1+y^2)\left[1+\frac{1+|xy|}{|z|(1+y^2)}\right]^2}\right| \leq 1$$

for all x, y and $z \neq 0$, where $L_1 = 1 > 0$, and for $y > 0$, $|x| \geq 2|y|[1-y^2]^{-1}$, we have

$$|zf_y(x,y)| \leq \left|\frac{|x|(1-y^2)-2|y|}{(1+y^2)^2\left[1+\frac{1+1/2(x^2+y^2)}{|z|(1+y^2)}\right]^2}\right| \leq 1$$

for all x, y and $z \neq 0$, where $L_2 = 1 > 0$;

(b) the function

$$g(x,y) = 3y + \frac{y}{1+|xy|+y^2}.$$

Clearly $g(0,0) = 0 = g(x,0)$ for all x. Also, since $0 \leq \dfrac{1}{1+|xy|+y^2}$, we have

$$\frac{g(x,y)}{y} \geq 3$$

for all x and $y \neq 0$, where $b = 3 > 0$. Moreover,

$$|g_x(x,y)| \leq \frac{y^2}{(1+|xy|+y^2)^2} \leq 1$$

for all x and y, where $M_1 = 1 > 0$ and $M_1 = 0$ when $y = 0$ and

$$g_y(x,y) = 3 + \frac{1-y^2}{(1+|xy|+y^2)^2}.$$

Since $\dfrac{1-y^2}{(1+|xy|+y^2)^2} \leq 1$ for all x and y, we have

$$|g_y(x,y)| \leq 4$$

for all x and y where $M_2 = 4 > 0$.

(c) the function
$$h(x) = 2x + \frac{x}{1+|x|}.$$
It is clear, from this relation, that $h(0) = 0$, also, since $0 \le \frac{1}{1+|x|} \le 1$ for all x, we have that
$$\frac{h(x)}{x} \ge 2$$
for all $x \ne 0$ where $\delta = 2 > 0$. Moreover, for $x > 0$, we obtain
$$h'(x) = 2 + \frac{1}{(1+|x|)^2}$$
this implies that
$$|h'(x)| \le 3$$
since $\frac{1}{(1+|x|)^2} \le 1$ for all x where $c = 3 > 0$, $ab > c$ implies that $4 > 1$.

(d) the function
$$p(t, x, y, x(t-r(t)), y(t-r(t)), z) = \frac{1}{1 + t^2 + |x| + |y| + |x(t-r(t)| + |y(t-r(t)| + |z|} \le 1$$
for all $t \ge 0, x, y, x(t-r(t)), y(t-r(t))$ and z. it follows that
$$\int_0^\infty \left| \frac{1}{1 + t^2 + |x| + |y| + |x(t-r(t)| + |y(t-r(t)| + |z|} \right| dt < \infty$$
for all $t \ge 0, x, y, x(t-r(t)), y(t-r(t))$ and z. Furthermore,
$$0 < \beta < \min\left\{ 2\frac{1}{4}, 1\frac{4}{5}, \frac{1}{2} \right\} = \frac{1}{2}.$$
Choosing $\alpha = 3, \beta = \frac{1}{3}$ and $\beta_0 = \frac{1}{2}$, we obtain
$$\gamma < \min\left\{ \frac{1}{10}, \frac{18}{245}, \frac{3}{680} \right\} = \frac{3}{680}.$$
All assumptions of the main results hold true, thus, the conclusions also follow.

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References

[1] Ademola, A.T. and Arawomo, P. O.; Asymptotic behaviour of solutions of third order nonlinear differential equations. *Acta Univ. Sapientiae, Mathematica,* **3,** 2 (2011) 197 - 211.

[2] Ademola, A.T. and Arawomo, P. O.; Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order. *Afr. Mat.* 23 (2012), no. 2, 261 - 271.

[3] Ademola, A.T. and Arawomo, P. O.; Stability and ultimate boundedness of solutions to certain third-order differential equations. *Applied Mathematics E-Notes.* **10**(2010), 61-69.

[4] Ademola, A.T. and Arawomo, P. O.; Stability and uniform ultimate boundedness of solutions of a third-order differential equation. *International Journal of Applied Mathematics.* **23** no. 1, (2010) 11-22.

[5] Ademola, A.T. and Arawomo, P. O.; Stability and uniform ultimate boundedness of solutions of some third order differential equations. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis* **27** (2011), 51-59.

[6] Ademola, A.T. and Arawomo, P. O.; Uniform stability and boundedness of solutions of nonlinear delay differential equations of the third order. *Mathematical Journal Okayama University* **55** (2013), 157166.

[7] Ademola, A.T., Ogundiran, M.O., Arawomo, P.O. and Adesina, O.A.; Boundedness results for a certain third-order nonlinear differential equations. *Appl. Math. Comput.* **216** (2010), 3044-3049.

[8] Ademola, A.T., Ogundiran, M.O., Arawomo, P.O. and Adesina, O.A.; Stability results for the solutions of a certain third-order nonlinear differential equation. *Mathematical Sciences Research Journal MSRJ.* **Vol. 12,** no. 6, (2008) 124-134.

[9] Afuwape, A.U. and Omeike, M.O.; On the stability and boundedness of solutions of a kind of third order delay differential equations. *Applied Mathematics and Computation,* **200** (2008), 444-451.

[10] Afuwape, A.U. and Omeike, M.O.; Stability and boundedness of solutions of a kind of third-order delay differential equations. *Computational & Applied Mathematics,* **29,** N. 3, (2010) 329-342.

[11] Bereketoğlu, H. and Karakoç, F., Some results on boundedness and stability of a third order differential equation with delay, *IASI, Tomul LI, s.I, Matematică,* **f.2,** 245-258, 2005.

[12] Burton, T.A. and Hering, R.; Liapunov theory for functional differential equations. *Rocky Mountain J. Math.* **24** (1994), 3-17.

[13] Burton, T.A. and Makey, G.; Asymptotic stability for functional differential equations. *Acta Math. Hungar,* **65** (1994), 243-251.

[14] Burton, T.A.; Stability and periodic solutions of ordinary and functional differential equations. *Mathematics in Science and Engineering,* **178** Academic Press. Inc., Orlando, FL, 1985.

[15] Burton, T.A.; Volterra integral and differential equations. *New York: Academic Press,* 1983.

[16] Chukwu, E.N.; On boundedness of solutions of third order differential equations, *Ann. Mat. Pura. Appl.,* **104** (4) (1975), 123-149.

[17] Driver, R.D.; Ordinary and delay differential equations. *Applied Mathematical Sciences,* Springer-Verlag New York Inc. (1976).

[18] Ezeilo, J.O.C.; Further results for the solutions of a third-order differential equation, *Proc. Camb. Phil. Soc.,* **59** (1963). 111-116.

[19] Ezeilo, J.O.C.; A boundedness theorem for a certain third-order differential equation, *Proc. London Math. Soc.* (3) **13** (1963), 99-124.

[20] Ezeilo, J.O.C.; On the stability of solutions of some third-order differential equations, *J. London Math. Soc.,* **43** (1968), 161-167.

[21] Hale, J.K.; Functional differential equations. *Applied Mathematical Sciences,* **3** Springer-verlag New York. Heidelberg. Berlin. (1971).

[22] Hale, J.K.; Theory of functional differential equations. Springer-verlag New York, (1977).

[23] Hara, T.; On the uniform ultimate boundedness of solutions of certain third-order differential equations, *J. Math. Anal. Appl.* (2) **80** (1981), 533-544.

[24] Ogundare, B.S.; On the boundedness and stability results for the solutions of certain third-order nonlinear differential equations. *Kragujevac J. Math.* **29** (2006) 37-48.

[25] Omeike, M.O.; New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation, *J. Inequal.Pure and Appl. Math.,* **9** (1) (2008), Art. 15, 8 pp.

[26] Omeike, M.O.; New results on the asymptotic behavior of a third-order nonlinear differential equations,*Differential Equations & Applications* **Vol. 2008** (2008) 1-13.

[27] Omeike, M.O.; New results on the stability of solution of some non-autonomous delay differential equations of the third-order. *Differential Equations and Control Processes* **2010** 1, (2010), 18-29.

[28] Reissig, R., Sansone, G. and Conti, R.; Nonlinear differential equations of higher order, *Noordhoff International Publishing Leyeden* (1974).

[29] Rouche, N., Habets, N. and Laloy, M.; Stability theory by Liapunov's direct method. Applied Mathematical Sciences 22, *Spriger-Verlag New York. Heidelberg. Berlin* (1977).

[30] Sadek, A.I.; Stability and boundedness of a kind of third-order delay differential system. *Applied Mathematics Letters* **16** (2003) 657-662.

[31] Swick, K.E.; Asymptotic behavior of the solutions of certain third-order differential equations, *SIAM J. Appl.,* **19** N0.1 (1970), 96-102.

[32] Swick, K.E.; Boundedness and stability for a nonlinear third-order differential equation, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.,* **56** (20), (1974), 859-865.

[33] Tejumola, H.O.; A note on the boundedness of solutions of some nonlinear differential equations of the third-order, *Ghana J. of Science,* **11** (2), (1970), 117-118.

[34] Tejumola, H.O.; A note on the boundedness and the stability of solutions of certain third-order differential equations. *Ann. Math. Pura. Appl.,* **92,** (1972) 65-75

[35] Tunç, C.; Boundedness of solutions of a third-order nonlinear differential equation, *J. Inequal. Pure and Appl. Math.,* **6** (1) (2005), 1-6.

[36] Tunç, C.; New results about stability and boundedness of solutions of certain nonlinear third-order delay differential equations. *The Arabian Journal for Science and Engineering,* **31.** 2A (2006), 185-196.

[37] Tunç, C.; On asymptotic stability of solutions to third-order nonlinear differential equations with retarded argument. *Communications in Applied Analysis* **11** (2007), 515-528.

[38] Tunç, C.; On the asymptotic behavior of solutions of certain third-order nonlinear differential equations, *J. Applied Math. and Stochastic Anal.* **1** (2005) 29-35.

[39] Tunç, C.; On the stability of solutions for non-autonomous delay differential equations of third-order, *Iranian Journal of Science & Technology, Transaction A,* **Vol. 32,** No. A4 (2008), 261-272.

[40] Tunç, C.; Stability and boundedness of solutions of nonlinear differential equations of third-order with delay. *Differential Equations and Control Processes* **2007** 3 (2007), 1-12.

[41] Tunç, C.; Some new results on the boundedness of solutions of a certain nonlinear differential equation of third-order. International *J. of Nonlinear Science,* **vol. 7** No.2 (2009) 246-256.

[42] Tunç, C.; Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments. *Electronic J. of Qualitative Theory of Differential Equations,* **2010** 1 (2010), 1-12.

[43] Yamamoto, M.; Further results for the solutions of certain third-order non-autonomous differential equations. *Proc. Japan Acad.,* No.5, **45** (1973), 317-321.

[44] Yao, H. and Wang, J.; Globally asymptotic stability of a kind of third order delay differential system. *International Journal of Nonlinear Science* **10,** No. 1, (2010), 82-87.

[45] Yoshizawa, T.; Stability theory by Liapunov's second method, *The Mathematical Society of Japan* (1966).

[46] Yoshizawa, T.; Stability theory and existence of periodic solutions and almost periodic solutions. *Spriger-Verlag, New York. Heidelberg. Berlin* (1975).

[47] Zhu, Y.F.; On stability, boundedness and existence of periodic solution of a kind of third-order nonlinear delay differential system. *Ann. Differential Equation* **8** (2) (1992), 249-259.