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Stohastic differential equations

STABILITY, BOUNDEDNESS AND UNIQUENESS OF SOLUTIONS TO CERTAIN THIRD ORDER STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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Abstract

The behaviour of solutions to certain third order nonlinear non autonomous stochastic delay differential equations with constant deviating arguments is considered. The main procedure lies on the construction of a perfect Lyapunov functional which is used to obtain suitable conditions which guarantee uniform stability, boundedness and uniqueness of global solution for $t > 0$. The obtained results are new and complement related second order stochastic differential equations that have appeared in the literature. Moreover, examples are given to illustrate the feasibility and correctness of the main results.

Keywords: Third order; Nonlinear stochastic delay differential equation; Uniform stability; Boundedness of solutions

1 Introduction

In the applications, the future behavior of many phenomena, in science, economy, medicine, information theory, engineering and so on, are assumed to be described by the solutions of an ordinary differential equation (ODE). In fact, the investigation of qualitative behavior of solutions, especially, the discussion of uniform stability, boundedness and existence of unique solutions are crucial problems in the theory and applications of ODEs. Many interesting work on qualitative behaviour of solutions of second-, third- and higher order ODEs, have been discussed and are still receiving the attention of authors, see for instance the book of Burton [21, 22], Yoshizawa [60, 61] which contain the background to the study and the papers of Ademola *et al.* [3]-[10],[13, 16], Burton and Hatvani [20], Ogundare *et al.* [36, 37], Raffoul [40], Tunç [49], Yoshizawa [59], Wang and Zhu [55] and the references cited therein.

A more general type of differential equations often called a "functional differential equation" is one in which the unknown functions occur with various different arguments. In the Russian literature these are refereed to as "differential equations with deviating argument." The simplest and perhaps the most natural type of functional differential equations is a "delay differential equation" (DDE) (or "differential equation with retarded argument"), (see Driver [26]). In this direction many authors have proposed different approach to discuss qualitative behaviour of solutions of second-, third- and higher order DDEs with deviating arguments (single or multiple, constant or variable). We can mention the survey books of Burton [21], Hale [25], Yoshizawa [61], and the papers of Ademola *et al.* [11, 12, 14, 17, 18], Cahlon and Schmidt [23], Caraballo *et al.* [24], Domoshnitsky [27], Mahmoud [33], Ogundare *et al.* [35], Olutimo and Adams [39], Remili *et al.* [41, 42, 43], Tunç *et al.* [46]-[48], [50]-[54], Xianfeng and Wei [56], Yeniçerioğlu [57, 58], Zhu [63] and the references cited therein to mention few.

"Since Itô introduced his stochastic calculus about 50 years ago, the theory of stochastic differential equation (SDE) has been developed very quickly," (see Mao [34]). In particular, the randomness or stochastic effects (a concept beyond the scope of ODE and DDE) introduced to ODE and DDE gives birth to stochastic differential equations (SDE) and stochastic delay differential equation (SDDE) respectively. In this direction, interesting articles have been published by authors using different approach, see for instance Arnold [19], Kolmanovskii and Shaikhet [30], Oksendal [38], Shaikhet [45], which contain the general results on the subject matters and the dazzling papers of Abou-El-Ela *et al.* [1, 2], Ademola *et al.* [15], Ivanov *et al.* [28], Kolarova [29], Kolmanovskii and Shaikhet [30, 31], Liu and Raffoul [32], Mao [34], Zhu *et al.* [62] and the references cited therein.

In their contribution the authors in [1] discussed the stability of solutions for certain second order SDDE

$$\ddot{x}(t) + a\dot{x}(t) + bx(t-h) + \sigma x(t)\dot{\omega}(t) = 0$$

and

$$\ddot{x}(t) + a\dot{x}(t) + f(x(t-h)) + \sigma x(t-\tau)\dot{\omega}(t) = 0,$$

where a, b and σ are positive constants, h and τ are two positive constant delays, $\omega(t) \in \mathbb{R}$ is a standard Wiener Process, the function f is continuous with respect to x with $f(0) = 0$.

Moreover, the authors in [2] considered a more general problem (including and extending the results in [1]) by introducing more nonlinear functions and they discussed stochastic stability and boundedness of solutions for the SDDE

$$\ddot{x}(t) + g(\dot{x}(t)) + bx(t-h) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), x(t-h)),$$

where a and b are positive constants and $h > 0$ is a constant delay, g and p are continuous functions with $g(0) = 0$, $\omega(t) = (\omega_1(t), \cdots, \omega_n(t)) \in \mathbb{R}^m$ an $m-$dimensional standard Brownian motion defined on the probability space.

Recently, the authors in [15] investigated stability and boundedness of solutions to a certain second order nonautonomous stochastic differential equation

$$x''(t) + g(x(t), x'(t))x'(t) + f(x(t)) + \sigma x(t)\omega'(t) = p(t, x(t), x'(t)),$$

where σ is a positive constant, the functions g, f and p are continuous in their respective arguments on $\mathbb{R}^2, \mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^2$.

Motivated by the above discussion, using Lyapunov second method, we proceed to study the problems of stability, boundedness and uniqueness of solutions of certain third order stochastic delay differential equations

$$\dddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t) + h(x(t-\tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)), \quad (1.1)$$

where a, b, σ are positives constants, h, p are nonlinear continuous functions in their respective arguments with $h(0) = 0$, $\tau > 0$ is a constant delay whose value will be determined later and $\omega(t)$ is defined above. The continuity and local Lipschitz conditions on the functions h and p are sufficient for the existence and uniqueness of solutions of equation (1.1) respectively. Also, the dots as usual stands for the differentiation with respect to the independent variable t. Suppose that $\dot{x}(t) = y(t)$ and $\ddot{x}(t) = z(t)$, then equation (1.1) is equivalent to system

$$\begin{aligned} \dot{x}(t) \ &= y(t), \\ \dot{y}(t) \ &= z(t), \\ \dot{z}(t) \ &= \ -h(x) - by - az - \sigma x(t)\dot{\omega}(t) + \int_{t-\tau}^{t} h'(x(s))y(s)ds \\ &\quad + p(t, x(t), \dot{x}(t), \ddot{x}(t)), \end{aligned} \tag{1.2}$$

where h' (the derivative of the function h with respect to x) exists and continuous for all x. However, to the best of our knowledge, there is no previous literature on stochastic stability and stochastic boundedness of solutions of nonlinear non autonomous third order SDDE (1.1). The rest of this paper is organized as follows. In Section 2, we present definition of terms and some preliminary results on SDDEs. Results and their proofs on stability, boundedness and uniqueness of solutions of the SDDE (1.1) are discussed in Section 3, while illustrative special cases of the SDDE (1.1) are presented in the last section.

2 Preliminaries

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t>0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t>0}$ satisfying the usual conditions (i.e. it is right continuous and $\{\mathfrak{F}_0\}$ contains all $\mathbb{P}-$null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m-$dimensional Brownian motion defined on the probability space. Let $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by

$$||A|| = \sqrt{\text{trace } (A^T A)}.$$

For more information see Arnold [19] and Mao [34]. Consider a non autonomous $n-$dimensional SDDE

$$dx(t) = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dB(t) \tag{2.1}$$

on $t > 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\}, x_0 \in C([-\tau, 0]; \mathbb{R}^n)$. Here $F : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are measurable functions. Suppose that the functions F, G satisfy the local Lipschitz condition, given any $b > 0,\ p \geq 2,\ F(t, 0, 0) \in C^1([0, b]; \mathbb{R}^n)$ and $g(t, 0, 0) \in C^p([0, b]; \mathbb{R}^{m\times n})$. Then there must be a stopping time $\beta = \beta(\omega) > 0$ such that equation (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ [class of $\mathfrak{F}_\text{t}$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ_t and

$E\|\xi_t\|^p < \infty]$ has a unique maximal solution on $t \in [t_0, \beta)$ which is denoted by $x(t, x_0)$. Assume further that

$$F(t, 0, 0) = G(t, 0, 0) = 0$$

for all $t \geq 0$. Hence, the SDDE admits zero solution $x(t, 0) \equiv 0$ for any given initial value $x_0 \in C([-\tau, 0]; \mathbb{R}^n)$.

Definition 2.1 *The zero solution of the stochastic differential equation* (2.1) *is said to be* stochastically stable *or* stable in probability, *if for every pair* $\epsilon \in (0, 1)$ *and* $r > 0$, *there exists a* $\delta_0 = \delta_0(\epsilon, r) > 0$ *such that*

$$Pr\{\|x(t; x_0)\| < r \quad \textit{for all} \quad t \geq 0\} \geq 1 - \epsilon \quad \textit{whenever} \quad \|x_0\| < \delta_0.$$

Otherwise, it is said to be stochastically unstable.

Definition 2.2 *The zero solution of the stochastic differential equation* (2.1) *is said to be* stochastically asymptotically stable *if it is stochastically stable and in addition if for every* $\epsilon \in (0, 1)$ *and* $r > 0$, *there exists a* $\delta = \delta(\epsilon) > 0$ *such that*

$$Pr\{\lim_{t\to\infty} x(t; x_0) = 0\} \geq 1 - \epsilon \quad \textit{whenever} \quad \|x_0\| < \delta.$$

Definition 2.3 *A solution* $x(t, x_0)$ *of the SDDE* (2.1) *is said to be* stochastically bounded *or* bounded in probability, *if it satisfies*

$$E^{x_0}\|x(t, x_0)\| \leq N(t_0, \|x_0\|), \ \forall \ t \geq t_0 \tag{2.2}$$

where E^{x_0} *denotes the expectation operator with respect to the probability law associated with* x_0, $N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a constant function depending on* t_0 *and* x_0.

Definition 2.4 *The solutions* $x(t_0, x_0)$ *of the SDDE* (2.1) *is said to be* uniformly stochastically bounded *if* N *in* (2.2) *is independent of* t_0.

Let $\mathbb{K}$ denote the family of all continuous non-decreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$ if $r \neq 0$. In addition, $\mathbb{K}_\infty$ denotes the family of all functions $\rho \in \mathbb{K}$ with

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Suppose that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, denotes the family of all non negative functions $V = V(t, x_t)$ (Lyapunov functional) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t. By Itô's formula we have

$$dV(t, x_t) = LV(t, x_t)dt + V_x(t, x_t)G(t, x_t)dB(t),$$

where

$$\begin{aligned} LV(t,x_t) \ &= \frac{\partial V(t,x_t)}{\partial t} + \frac{\partial V(t,x_t)}{\partial x_i} F(t,x(t)) \\ &+ \frac{1}{2}\text{trace } [G^T(t,x_t)V_{xx}(t,x_t)G(t,x_t)] \end{aligned} \tag{2.3}$$

with

$$V_{xx}(t,x_t) = \left(\frac{\partial^2 V(t,x_t)}{\partial x_i \partial x_j} \right)_{n\times n}, \ i,j = 1,\cdots,n$$

In this study we will use the diffusion operator $LV(t,x_t)$ defined in (2.3) to replace $V'(t,x(t)) = \frac{d}{dt}V(t,x(t))$. We now present the basic results that will be used in the proofs of the main results.

Lemma 2.1 *(See [19]) Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and $\phi \in \mathbb{K}$ such that*

(i) $V(t,0) = 0$, for all $t \geq 0$;

(ii) $V(t,x_t) \geq \phi(\|x(t)\|)$, $\phi(r) \to \infty$ as $r \to \infty$; and

(iii) $LV(t,x_t) \leq 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of SDDE (2.1) *is* stochastically stable. *If conditions (ii) and (iii) hold then* (2.1) *with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ has a unique global solution for $t > 0$ denoted by $x(t;x_0)$.*

Lemma 2.2 *(See [19]) Suppose that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and $\phi_0, \phi_1, \phi_2 \in \mathbb{K}$ such that*

(i) $V(t,0) = 0$, for all $t \geq 0$;

(ii) $\phi_0(\|x(t)\|) \leq V(t,x_t) \leq \phi_1(\|x(t)\|)$, $\phi_0(r) \to \infty$ as $r \to \infty$; and

(iii) $LV(t,x_t) \leq -\phi_2(\|x(t)\|)$ for all $(t,x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of SDDE (2.1) *is* uniformly stochastically asymptotically stable in the large

Assumption 2.1 *(See [32, 40]) Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, suppose that for any solutions $x(t_0,x_0)$ of SDDE* (2.1) *and for any fixed $0 \leq t_0 \leq T < \infty$, we have*

$$E^{x_0}\left\{ \int_{t_0}^{T} V^2_{x_i}(t,x_t)G^2_{ik}(t,x_t)dt \right\} < \infty, \ 1 \leq i \leq n, \ 1 \leq k \leq m. \tag{2.4}$$

Assumption 2.2 *(See [32, 40]) A special case of the general condition* (2.4) *is the following condition. Assume that there exits a function* $\sigma(t)$ *such that*

$$|V_{x_i}(t,x_t)G_{ik}(t,x_t)| < \sigma(t),\ x \in \mathbb{R}^n\ 1 \le i \le n,\ 1 \le k \le m, \tag{2.5}$$

for any fixed $0 \le t_0 \le T < \infty$,

$$\int_{t_0}^{T} \sigma^2(t)dt < \infty. \tag{2.6}$$

Lemma 2.3 *(See [32, 40]) Assume there exists a Lyapunov function* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, *satisfying Assumption 2.1, such that for all* $(t, x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^p \le V(t,x_t) \le \|x(t)\|^q$,

(ii) $LV(t,x_t) \le -\alpha(t)\|x(t)\|^r + \beta(t)$,

(iii) $V(t,x_t) - V^{r/q}(t,x_t) \le \gamma$,

where $\alpha, \beta \in C(\mathbb{R}^+;\mathbb{R}^+)$, p, q, r *are positive constants,* $p \ge 1$ *and* γ *is a non negative constant. Then all solutions of SDDE* (2.1) *satisfy*

$$E^{x_0}\|x(t,x_0)\| \le \left\{ V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + A \right\}^{1/p}, \tag{2.7}$$

for all $t \ge t_0$, *where*

$$A := \int_{t_0}^{t} \Big(\gamma\alpha(u) + \beta(u) \Big) e^{-\int_u^t \alpha(s)ds} du.$$

Lemma 2.4 *(See [32, 40]) Assume there exists a Lyapunov function* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, *satisfying Assumption 2.1, such that for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^p \le V(t,x_t)$,

(ii) $LV(t,x_t) \le -\alpha(t)V^q(t,x_t) + \beta(t)$,

(iii) $V(t,x_t) - V^q(t,x_t) \le \gamma$,

where $\alpha, \beta \in C(\mathbb{R}^+;\mathbb{R}^+)$, p, q *are positive constants,* $p \ge 1$ *and* γ *is a non negative constant. Then all solutions of SDDE* (2.1) *satisfy* (2.7) *for all* $t \ge t_0$.

Corollary 2.1 *(See [32, 40])*

(i) Assume that hypotheses (i) to (iii) of Lemma 2.3 hold. In addition

$$\int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)e^{-\int_u^t\alpha(s)ds}du\le M,\forall\ t\ge t_0\ge 0, \tag{2.8}$$

for some positive constant M, then all solution of SDDE (2.1) *are* uniformly stochastically bounded.

(ii) Assume the hypotheses (i) to (iii) of Lemma 2.4 hold. If condition (2.8) *is satisfied, then all solutions of SDDE* (2.1) *are* stochastically bounded.

3 Main Results

Let $x(t)=x, y(t)=y, z(t)=z$ and $X_t=(x_t,y_t,z_t)\in\mathbb{R}^3$ be any solution of the SDDE (1.2). The Lyapunov functional employed in the proofs of our results is $V(t,X_t)=V(t,x_t,y_t,z_t)$ defined as

$$\begin{aligned}2V(t,X_t)\ &=2(\alpha+a)\int_0^x h(\xi)d\xi+2by^2+4yh(x)+2(\alpha+a)yz+2z^2\\&+(\alpha^2+a^2+\beta)y^2+\beta bx^2+2a\beta xy+2\beta xz\\&+\int_{-\tau}^{0}\int_{t+s}^{t}\Big(\lambda_1y^2(\theta)+\lambda_2z^2(\theta)\Big)d\theta ds,\end{aligned}\tag{3.1}$$

where a,b are positive constants, α,β are constants satisfying

$$b^{-1}l<\alpha<a,\quad l>0, \tag{3.2}$$

$$0<\beta<\min\left\{b,\ \frac{ab-l}{2(1+a)},\ \frac{a-\alpha}{4}\right\}, \tag{3.3}$$

λ_1 and λ_2 are non-negative constants which will be determined later. In what follows we will state the main results of the paper and give their proofs.

Theorem 3.1 *If $a,b,c,l,\delta,\sigma,\tau$ and M_0 are positive constants such that*

(i) $h(0)=0$,

(ii) $\delta x\le h(x)\le cx$ *for all* $x\ne0$, $|h'(x)|\le l$ *for all* x,

(iii) $\sigma^2<\beta\delta$, $l<ab$ *and* $(\alpha+\beta+a+2)c<ab-l$,

(iv) $|p(t,x,y,z)|\le M_0$,

then the solution (x_t, y_t, z_t) *of the SDDE* (1.2) *is uniformly stochastically bounded provided that*

$$\tau < \min\left\{\frac{2(\beta\delta - \sigma^2)}{\beta c},\ \frac{(ab-l)-(\alpha+\beta+a+2)c}{(\alpha+a)c},\ \frac{a-\alpha}{4c}\right\}. \tag{3.4}$$

Remark 3.1 *We give the following observations*

(i) Whenever $h(x(t-\tau)) = cx$ *and* $\sigma x\dot{\omega} = p(t,x,\dot{x},\ddot{x}) = 0$, *then equation* (1.1) *specializes to a linear first order homogenous ODE*

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0, \tag{3.5}$$

and assumptions (i) to (iv) of Theorem 3.1 reduces to Routh Hurwitz criteria $a > 0, b > 0, c > 0, ab > c$ *for asymptotic stability of the trivial solution of the equation* (3.5).

(ii) If $\sigma x\dot{\omega} = 0 = p(t,x,\dot{x},\ddot{x})$, *then equation* (1.1) *specializes to a DDE discussed in [63]. Thus Theorem 3.1 includes and extends the stability result discussed in [63].*

(iii) The term $\sigma x\dot{\omega}$ *in SDDE* (1.1) *extends all results on third order differential equation with or without delay.*

(iv) The obtained results in [1, 2] and [15] are on second order stochastic differential equations.

(v) This is the first paper on nonlinear non-autonomous SDDE as there is no previous literature on third order.

Next, we will state and proof a result that will be helpful in the proof of Theorem 3.1 and the subsequent results.

Lemma 3.1 *Subject to the assumptions of Theorem 3.1, there exist positive constants* $D_0 = D_0(a,b,l,\alpha,\beta,\delta)$ *and* $D_2 = D_2(a,b,c,\alpha,\beta,\lambda_1,\lambda_2)$ *such that*

$$D_0(x^2(t) + y^2(t) + z^2(t)) \le V(t, X_t) \le D_1(x^2(t) + y^2(t) + z^2(t)), \tag{3.6}$$

for all $t \ge 0, x, y$ *and* z. *Furthermore, there exist positive constants* $D_2 = D_2(a,b,c,\alpha,\beta,\delta,\tau)$ *and* $D_3 = D_3(a,\alpha,\beta)$ *such that*

$$\begin{aligned} LV(t, X_t) \le\ & -D_2(x^2(t) + y^2(t) + z^2(t)) + D_3(|x(t)| + |y(t)| + |z(t)|)\times \\ & |p(t,x,y,z)|, \end{aligned} \tag{3.7}$$

for all $t \ge 0, x, y$ *and* z.

Proof. Suppose that (x_t, y_t, z_t) is any solution of the SDDE (1.2). Putting $X_t = \mathbf{0} = (0, 0, 0) \in \mathbb{R}^3$ in equation (3.1) clearly

$$V(t, \mathbf{0}) = V(t, 0, 0, 0) = 0, \tag{3.8}$$

for all $t \geq 0$. Furthermore, since $h(0) = 0$ equation (3.1) becomes

$$\begin{aligned} V(t, X_t) &= b^{-1}\int_0^x \Big[(a+\alpha)b - 2h'(\xi)\Big]h(\xi)d\xi + b^{-1}\Big(h(x) + by\Big)^2 \\ &+ \frac{1}{2}\Big(\beta x + ay + z\Big)^2 + \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}\beta(b-\beta)x^2 + \frac{1}{2}\beta y^2 \\ &+ \int_{-\tau}^0 \int_{t+s}^t \Big(\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big)d\theta ds. \end{aligned} \tag{3.9}$$

Now since the double integrals in equation (3.9) is non-negative, the fact that $\delta x \leq h(x)$ for all $x \neq 0$, $h'(x) \leq l$ for all x, applying the inequalities (3.2) and (3.3), there exists a constant $\delta_0 > 0$ such that

$$V(t, X_t) \geq \delta_0(x^2 + y^2 + z^2), \tag{3.10}$$

for all $t \geq 0, x, y$ and z, where

$$\begin{aligned} \delta_0 := \ \min\Big\{ &\frac{\delta}{2b}[(\alpha + a)b - 2l] + \frac{1}{2}\beta(b-\beta) + \frac{1}{b}\min\{\delta, b\} + \frac{1}{2}\min\{a, \beta, 1\}, \\ &\frac{1}{b}\min\{\delta, b\} + \frac{1}{2}\min\{a, \beta, 1\} + \frac{1}{2}\min\{\alpha, 1\} + \frac{1}{2}\beta, \\ &\frac{1}{2}\min\{a, \beta, 1\} + \frac{1}{2}\min\{\alpha, 1\}\Big\}. \end{aligned}$$

Inequality (3.10) gives rise to equation

$$V(t, X_t) = 0 \quad \Leftrightarrow x^2 + y^2 + z^2 = 0 \tag{3.11}$$

and the inequality

$$V(t, X_t) > 0 \quad \Leftrightarrow x^2 + y^2 + z^2 \neq 0, \tag{3.12}$$

and that

$$V(t, X_t) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{3.13}$$

Estimate (3.13) shows that the functional $V(t, X_t)$ is radially unbounded. Moreover, using assumption (ii) of Theorem 3.1 and the fact that $2x_1x_2 \leq x_1^2 + x_2^2$, in equation (3.1) there exists a constant $\delta_1 > 0$ such that

$$V(t, X_t) \leq \delta_1(x^2 + y^2 + z^2), \tag{3.14}$$

for all $t \geq 0, x, y$ and z, where

$$\delta_1 := \max\Big\{(a+b+1)\beta + (\alpha+\beta)c + 2\delta,\ \alpha(\alpha+1) + (a+\beta)(a+1) + 2(b+\delta) + \frac{1}{2}\tau^2\max\{\lambda_1, \lambda_2\},\ \alpha+\beta+a+2 + \frac{1}{2}\tau^2\max\{\lambda_1, \lambda_2\}\Big\},$$

thus the inequality (3.14) implies that the functional $V(t, X_t)$ defined by (3.1) is decrescent. Inequalities (3.10) and (3.14) combined give the following inequalities

$$\delta_0(x^2+y^2+z^2) \leq V(t, X_t) \leq \delta_1(x^2+y^2+z^2), \tag{3.15}$$

for all $t \geq 0, x, y$ and z. Inequalities (3.15) satisfy (3.6) with δ_0 and δ_1 equivalent to D_0 and D_1 respectively.

Next, applying Itô's formula (2.3) in equation (3.1) using system (1.2), we find that

$$\begin{aligned} LV(t, X_t) \ =\ & -\beta x^2\frac{h(x)}{x} - \Big[(\alpha+a)b - 2h'(x)\Big]y^2 - (a-\alpha)z^2 \\ & - \alpha(a-\alpha)yz - \int_{t-\tau}^{t}\Big(\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big)d\theta + \sigma^2 x^2 \\ & + \lambda_1 y^2 + \lambda_2 z^2 + a\beta y^2 + 2\beta yz + \Big(\beta x + (\alpha+a)y + 2z\Big)\times \\ & \Big[p(t,x,y,z) + \int_{t-\tau}^{t} h'(x(s))y(s)ds\Big]. \end{aligned} \tag{3.16}$$

In view of assumption (ii) of Theorem 3.1 $|h'(x)| \leq l$ for all x and employing the inequality

$$\alpha^2(a-\alpha)^2 < (\alpha b - l)(a-\alpha)$$

it follows that

$$\begin{aligned} (\alpha b - l)y^2 + \alpha(a-\alpha)yz + \frac{1}{4}(a-\alpha)z^2 \geq \\ \Big(\sqrt{(\alpha b - l)}|y| - \frac{1}{2}\sqrt{(a-\alpha)}|z|\Big)^2 \geq 0 \end{aligned} \tag{3.17}$$

for all y and z. Using inequality (3.17) and the fact that $h(x) \geq \delta x$ for all $x \neq 0$, equation (3.16) becomes

$$
\begin{aligned}
LV(t,X_t) \le & -\beta\delta x^2 - \frac{1}{2}(ab-l)y^2 - \frac{1}{4}(a-\alpha)z^2 \\
& - \left(\frac{1}{2}(ab-l) - (a+1)\beta\right)y^2 - \left(\frac{1}{4}(a-\alpha) - \beta\right)z^2 \\
& - \left[\lambda_1 - \frac{c}{2}(\alpha+\beta+a+2)\right]\int_{t-\tau}^{t} y^2(\theta)d\theta + \lambda_2\int_{t-\tau}^{t} z^2(\theta)d\theta \\
& + \sigma^2 x^2 + \lambda_1 y^2 + \lambda_2 z^2 + \frac{c}{2}\left(\beta x^2 + (\alpha+a)y^2 + 2z^2\right)\tau \\
& + (\beta|x| + (\alpha+a)|y| + 2|z|)|p(t,x,y,z)|,
\end{aligned} \tag{3.18}
$$

for all $t \ge 0, x, y$ and z. Choose $\lambda_1 := 2^{-1}c(\alpha+\beta+a+2)$, $\lambda_2 := 0$, using the inequalities (3.2), (3.3) and (3.4) in estimate (3.18), there exist constants $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$
LV(t,X_t) \le -\delta_2(x^2+y^2+z^2) + \delta_3(|x|+|y|+|z|)|p(t,x,y,z)|, \tag{3.19}
$$

for all $t \ge 0, x, y$ and z, where

$$
\begin{aligned}
\delta_2 := \min\Bigg\{ & \beta\delta - \sigma^2 - \frac{1}{2}\beta c\tau,\ \frac{1}{2}\Bigg[ab - l - (\alpha+\beta+a+2)c \\
& - (\alpha+a)c\tau\Bigg],\ \frac{1}{4}(a-\tau) - c\tau\Bigg\}
\end{aligned} \tag{3.20}
$$

and

$$
\delta_3 := \max\{\alpha+a,\ \beta,\ 2\}.
$$

Inequality (3.19) satisfies inequality (3.7) with δ_2 and δ_3 equivalent to D_2 and D_3 respectively. This completes the proof of Lemma 3.1.

Proof of Theorem 3.1. Let (x_t, y_t, z_t) be any solution of the SDDE (1.2). Using assumption (iii) in estimate (3.19), noting that δ_2, δ_3 are positive constants and the fact that

$$
\left[|x| + \delta_2^{-1}\delta_3 M_0\right]^2 + \left[|y| + \delta_2^{-1}\delta_3 M_0\right]^2 + \left[|z| + \delta_2^{-1}\delta_3 M_0\right]^2 \ge 0
$$

for all x, y and z, there exist constants $\delta_4 > 0$ and $\delta_5 > 0$ such that

$$
LV(t,X_t) \le -\delta_4(x^2+y^2+z^2) + \delta_5 \tag{3.21}
$$

for all $t \ge 0, x, y, z$ where $\delta_4 := 2^{-1}\delta_2$ and $\delta_5 := 3 \times 2^{-1}\delta_2^{-1}\delta_3^2 M_0^2$. Inequalities (3.15) and (3.21) fulfill assumptions (i) to (iii) of Lemma 2.3 with $p = q = r = 2$,

$\alpha(t) = \delta_4$, $\beta(t) = \delta_5$ and $\gamma = 0$. It follows from the inequality (2.8) that

$$\delta_5 \int_{t_0}^{t} \left[e^{-\delta_4 \int_u^t ds} \right] du \leq \delta_4^{-1}\delta_5, \tag{3.22}$$

where $M := \delta_4^{-1}\delta_5$. Furthermore, from system (1.2) and equation (3.1) there exists a constant $\delta_6 > 0$ such that

$$\left| V_{x_i}(t,X) G_{ik}(t,X) \right| \leq \delta_6 (x^2 + y^2 + z^2), \quad X \in \mathbb{R}^3, \ 1 \leq i,k \leq 3, \tag{3.23}$$

where

$$\delta_6 := \frac{\sigma}{2} \max\{\alpha + a, \beta, \alpha + \beta + a + 4\}$$

and for any fixed $0 \leq t_0 \leq T < \infty$

$$\int_{t_0}^{T} \sigma^2(t) dt = \int_{t_0}^{T} \delta_6^2 (x^2 + y^2 + z^2)^2 dt < \infty. \tag{3.24}$$

In view of inequalities (3.23) and (3.24) inequalities (2.5) and (2.6) of Assumption 2.2 hold with $\sigma(t) = \delta_6 < \infty$, and hence Assumption 2.1 follows immediately. Finally, from inequalities (3.14) and (3.22), we have

$$E^{x_0} \|x(t,x_0)\| \leq \left(\delta_1 X_0^2 + \delta_4^{-1}\delta_5 \right)^{1/2}, \tag{3.25}$$

for all $t \geq t_0$, where $X_0 = \left(x_0^2 + y_0^2 + z_0^2 \right) \in \mathbb{R}^3$. Assumption of Corollary 2.1 (i) hold, hence by Corollary 2.1 (i) all solutions (x_t, y_t, z_t) of the SDDE (1.2) are uniformly stochastically bounded. This completes the proof of Theorem 3.1.

Theorem 3.2 *If assumptions (i) to (iv) of Theorem 3.1 and the inequality* (3.4) *hold, then the solutions* (x_t, y_t, z_t) *of the SDDE* (1.2) *are stochastically bounded.*

Proof. Let (x_t, y_t, z_t) be any solution of the SDDE (1.2). In view of (3.14) and (3.19) there exists a constant $\delta_* > 0$ such that

$$LV(t,X_t) \leq -\delta_* V(t,X_t) + \delta_5, \tag{3.26}$$

for all $t \geq 0, x, y, z$ where

$$\delta_* := \delta_1^{-1}\delta_4.$$

From inequalities (3.10) and (3.26), assumptions (i) to (iii) of Lemma 2.4 hold with $p = 2$, $q = 1$, $\gamma = 0$, $\alpha(t) = \delta_4$ and $\beta(t) = \delta_5$. Also, the inequalities (3.22), (3.23), (3.24) and (3.25) hold, so that assumption (ii) of Corollary 2.1 hold, then by assumption (ii) of Corollary 2.1 all solutions of the SDDE (1.2) are stochastically bounded. This completes the proof of Theorem 3.2.

Next, if $p(t, x, \dot{x}, \ddot{x}) = 0$ and $p(t, x, y, z) = 0$ in equations (1.1) and system (1.2) respectively, we have

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x(t-\tau)) + \sigma x(t)\dot{\omega}(t) = 0 \tag{3.27}$$

and equation (3.27) is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -h(x) - by - az - \sigma x(t)\dot{\omega}(t) + \int_{t-\tau}^{t} h'(x(s))y(s)ds \end{aligned} \tag{3.28}$$

where h and ω are defined in Section 1.

Theorem 3.3 *Suppose that assumptions (i) to (iii) of Theorem 3.1 and inequality* (3.4) *hold, then the trivial solution of the SDDE* (3.28) *is uniformly stochastically asymptotically stable in the large.*

Proof. Let (x_t, y_t, z_t) be any solution of the SDDE (3.28). Equation (3.8), estimate (3.13) and the inequalities (3.15) satisfy assumptions (i) and (ii) of Lemma 2.2. Next, using Itô's formula (2.3) and the system (3.28), we have

$$LV(t, X_t) \leq -\delta_2(x^2 + y^2 + z^2) \tag{3.29}$$

for all $t \geq 0, x, y$ and z, where δ_2 is defined in (3.20). Inequality (3.29) satisfies assumption (iii) of Lemma 2.2, thus by Lemma 2.2 the trivial solution of the SDDE (3.28) is uniformly stochastically asymptotically stable in the large. This completes the proof of Theorem 3.3.

Theorem 3.4 *Suppose that assumptions (i) to (iii) of Theorem 3.1 and the inequality* (3.4) *are satisfied, then the trivial solution of the SDDE* (3.28) *is stochastically stable.*

Proof. Suppose that (x_t, y_t, z_t) be any solution of the SDDE (3.28). From equation (3.8) and inequality (3.10), assumptions (i) and (ii) of Lemma 2.1 are satisfied. Also, from inequality (3.29) we have

$$LV(t, X_t) \leq 0, \tag{3.30}$$

for all $t \geq 0$, $X = (x, y, z) \in \mathbb{R}^3$. Inequality (3.30) fulfills assumption (iii) of Lemma 2.1, hence by Lemma 2.1 the trivial solution of the SDDE (3.28) is stochastically stable. This completes the proof of Theorem 3.4.

Theorem 3.5 *If assumption (i) to (iii) of Theorem 3.1 and the inequality* (3.4) *are satisfied, then system* (3.28) *with* $X_0 \in C^p_{\mathfrak{F}_{t_0}}$ *has a unique global solution for* $t > 0$.

Proof. Let (x_t, y_t, z_t) be any solution of the SDDE (3.28), in view of inequalities (3.10) and (3.30) assumptions (ii) and (iii) of Lemma 2.1 are satisfied, thus by Lemma 2.1 assumptions (ii) and (iii) system (3.28) with $X_0 \in C^p_{\mathfrak{F}_{t_0}}$, has a unique global solution for $t > 0$. This completes the proof of Theorem 3.5.

Next, if the function $p(t, x, \dot{x}, \ddot{x})$ is replaced by $p(t)$ defined on $\mathbb{R}^+$, equation (1.1) specializes to

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x(t-\tau)) + \sigma x(t)\dot{\omega}(t) = p(t) \tag{3.31}$$

and equation (3.31) is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -h(x) - by - az - \sigma x(t)\dot{\omega}(t) + \int_{t-\tau}^{t} h'(x(s))y(s)ds + p(t). \end{aligned} \tag{3.32}$$

We have the following results

Corollary 3.1 *Suppose that assumptions (i) to (iii) of Theorem 3.1 are satisfied, assumption (iv) is replaced by*

$$|p(t)| \leq M_1, \quad 0 < M_1 < \infty$$

and inequality (3.4) *holds, then the solutions of the SDDE* (3.32) *are*

(i) uniformly stochastically bounded, and

(ii) stochastically bounded.

Proof. The proof of Corollary 3.1 is similar to the proofs of Theorem 3.1 and Theorem 3.2, hence it is omitted. This completes the proof of Corollary 3.1.

4 Examples

In this section, we present examples to check the validity and effectiveness of our results obtained in the previous section.

Example 4.1 *Consider a third order scalar SDDE*

$$\dddot{x} + \ddot{x} + 20\dot{x} + x(t-\tau) + \frac{\sin x^2(t-\tau)}{x(t-\tau)} + \frac{1}{11}x\dot{\omega} = \frac{2+t^2+x^2+\dot{x}^2+\ddot{x}^2}{1+t^2+x^2+\dot{x}^2+\ddot{x}^2}. \quad (4.1)$$

Equation (4.1) *can be written in the equivalent form*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\left(x + \frac{\sin x^2}{x}\right) - 20y - z - \frac{1}{11}x\dot{\omega} + \frac{2+t^2+x^2+y^2+z^2}{1+t^2+x^2+y^2+z^2} \\ &\quad + \int_{t-\tau}^{t}\left[1 + 2\cos x^2(s) - \frac{\sin x^2(s)}{x^2(s)}\right]y(s)ds. \end{aligned} \quad (4.2)$$

Comparing system (1.2) *with system* (4.2) *we have the following relations*

(i) $a = 1$, $b = 20$;

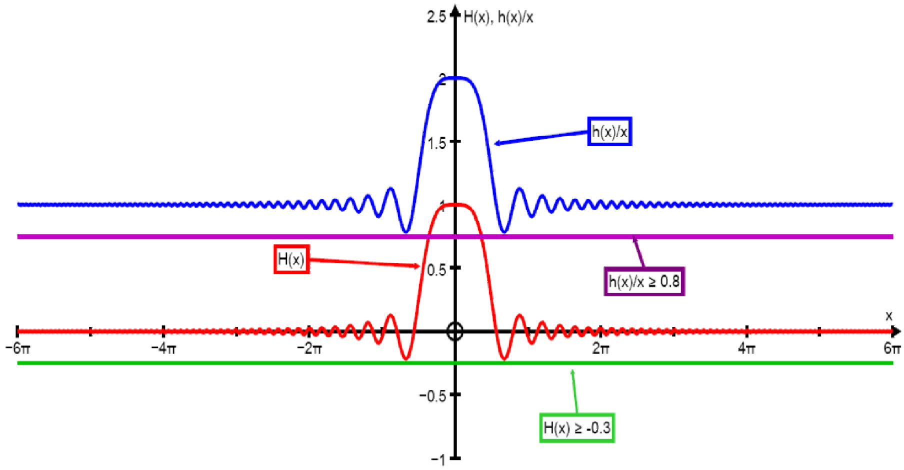

Figure 1: The Behaviour of functions $H(x)$ and $x^{-1}h(x)$

(ii) the nonlinear function

$$h(x) := x + \frac{\sin x^2}{x} \quad or \quad \frac{h(x)}{x} = 1 + \frac{\sin x^2}{x^2},$$

for all $x \neq 0$. *Let*

$$H(x) = \frac{\sin x^2}{x^2}$$

for all $x \neq 0$, *since*

$$-0.3 \leq H(x) \leq 1$$

for all $x \neq 0$, *it follows that*

$$0.7 \leq \frac{h(x)}{x} \leq 2$$

for all $x \neq 0$, *where* $\delta := 0.7$ *and* $c := 2$. *See the behaviour of and bounds on functions* $H(x)$ *and* $\frac{h(x)}{x}$ *in Figure 1. Furthermore, the derivative of the function* $h(x)$ *with respect to* x *is*

$$h'(x) := 1 + 2\cos x^2 - \frac{\sin x^2}{x^2}.$$

It is not difficult to show that

$$|h'(x)| \leq 3.5$$

for all x, *where* $l := 3.5$. *The behaviour of* $|h'(x)|$ *is shown in Figure 2.*

The estimated values of a, b, c, δ *and* l *facilitate the determination of* α, β *in* (3.2), (3.3), λ_1 *and* λ_2 *as*

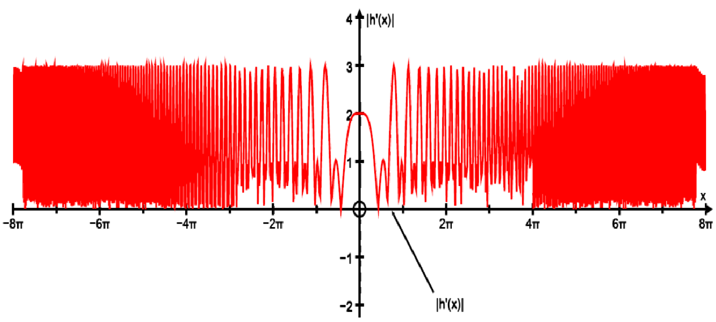

Figure 2: The Behaviour of function $|h'(x)|$ for all x

(iii) $0.2 < \alpha < 1$, *we choose* $\alpha = 0.3$;

(iv) $0 < \beta < \min\{20, 4.1, 0.2\} = 0.2$, *we choose* $\beta = 0.1$;

(v) $\lambda_1 := 3.4$, $\lambda_2 := 0$;

Substituting the values of $a, b, c, l, \alpha, \beta$ *and* δ, *inequality* (3.4) *becomes* *(vii)* $\tau < \min\{0.65, 0.09, 3.78\} = 0.09$ *and we choose* $\tau = 0.01$.

(vi) Finally, the function

$$p(t,x,y,z) := 1 + \frac{1}{1+t^2+x^2+y^2+z^2}.$$

Clearly

$$|p(t,x,y,z)| \le 2 = M_0 < \infty,$$

for all $t \ge 0, x, y$ *and* z.

Next, the functional $V(t,X_t)$ *defined in equation* (3.1) *becomes*

$$\begin{aligned} 2V(t,X_t) = 2.6\int_0^x (\xi + \sin\xi^2)d\xi + 2x^2 + 41.19y^2 + 2z^2 + 4(x+\sin x^2)y \\ + 2.6yz + 0.2(xy+xz) + 3.4\int_{-0.01}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \end{aligned} \tag{4.3}$$

From (4.3) *it is not difficult to show that*

$$0.2(x^2+y^2+z^2) \le V(t,X_t) \le 44(x^2+y^2+z^2) \tag{4.4}$$

for all x, y, z *where* $\delta_0 := 0.2$ *and* $\delta_1 = 44$. *Inequalities* (4.4) *satisfy the inequalities* (3.15). *It must be noted that the first inequality in* (4.4) *satisfies estimates* (3.11), (3.12) *and* (3.13) *in the proof of Lemma 3.1.*

Moreover, applying Itô's formula using system (4.2), *we have*

$$\begin{aligned} LV(t,X_t) = {} & [1.3(x+\sin x^2) + 2y(1+2\cos x^2 - x^{-2}\sin x^2) + 2x \\ & + 0.1(y+z)]y + [0.1x + 41.19y + 1.3z + 2(x+\sin x^2)]z \\ & + \frac{1}{121}x^2 + 0.034y^2 - 3.4\int_{t-0.01}^{t} y^2(\theta)d\theta \\ & + (0.1x+1.3y+2z)\Bigg[\frac{2+t^2+x^2+y^2+z^2}{1+t^2+x^2+y^2+z^2} \\ & - \Bigg((x+\sin x^2) + 20y + z\Bigg) \\ & + \int_{t-0.01}^{t}\left(1 + 2\cos x^2(s) - \frac{\sin x^2(s)}{x^2(s)}\right)ds\Bigg] \end{aligned} \tag{4.5}$$

Simplifying equation (4.5), *noting that*

$$1 + \frac{\sin x^2}{x^2} \ge 0.7$$

for all $x \neq 0$ *and*

$$1 + 2\cos x^2 - \frac{\sin x^2}{x^2} \leq \left| 1 + 2\cos x^2 - \frac{\sin x^2}{x^2} \right| \leq 3.5$$

for all x, *we obtain*

$$LV(t, X_t) \leq -0.06(x^2 + y^2 + z^2) + 2(|x| + |y| + |z|) \left| \frac{2 + t^2 + x^2 + y^2 + z^2}{1 + t^2 + x^2 + y^2 + z^2} \right| \quad (4.6)$$

for all $t \geq 0, x, y$ *and* z. *Inequality* (4.6) *satisfies inequality* (3.19) *with* $\delta_2 := 0.06$ *and* $\delta_3 := 2$. *Next, using item (vi) above, inequality* (4.6) *becomes*

$$LV(t, X_t) \leq -0.03(x^2 + y^2 + z^2) + 24000 \quad (4.7)$$

for all $t \geq 0, x, y$ *and* z. *Inequality* (4.7) *satisfies* (3.21) *with* $\delta_4 := 0.03$ *and* $\delta_5 := 24000$. *Also, with the values of* δ_4 *and* δ_5 *inequality* (3.22) *becomes*

$$24000 \int_{t_0}^{t} \left[e^{-0.03 \int_u^t ds} \right] du = 8.0 \times 10^5 \left[1 - e^{-0.03(t - t_0)} \right] < 8.0 \times 10^5, \quad (4.8)$$

where $8.0 \times 10^5 = \delta_4^{-1} \delta_5$. *Furthermore, inequality* (3.23) *specializes to*

$$|V_{x_3}(t, X_t) G_{33}(t, X_t)| \leq 0.24(x^2 + y^2 + z^2), \quad (4.9)$$

for all $t \geq 0, x, y, z$ *where* $\delta_6 = 0.24$. *Inequality* (3.24) *likewise follows from* (4.9) *for any fixed* $0 \leq t_0 \leq T < \infty$. *In view of estimate* (4.9) *Assumptions 2.2 and 2.1 hold immediately. By Corollary 2.1 (i), all solutions of system* (4.2) *are uniformly stochastically bounded and satisfy*

$$E^{X_0} \|X(t, X_0)\| \leq \left\{ 44X_0^2 + 8.0 \times 10^5 \right\}^{1/2}. \quad (4.10)$$

Also, to apply Theorem 3.2, since the functional $V(t, X_t)$ *defined by* (4.3) *is positive definite (i.e.* $V(t, 0) = 0$ *and* $V(t, X_t) \geq 0.2(x^2 + y^2 + z^2)$*). It follows from the upper inequality* (4.4) *that*

$$V(t, X_t) \leq 44(x^2 + y^2 + z^2),$$

for all $t \geq 0, x, y$ *and* z. *Using this inequality in* (4.7) *we obtain*

$$LV(t, X_t) \leq -6.8 \times 10^{-4} V(t, X_t) + 2.4 \times 10^4, \quad (4.11)$$

for all $t \geq 0, x, y, z$ *where* $\delta_* := 6.8 \times 10^{-4}$ *and* $\delta_5 = 2.4 \times 10^4$. *In view of inequalities* (4.8), (4.9) *and* (4.11) *assumptions of Lemma 2.4 and Corollary 2.1 (ii) hold, thus by Corollary 2.1 (ii), all solutions of system* (4.2) *are stochastically bounded and satisfy estimate* (4.10).

Example 4.2 *As an application of Theorems 3.3 and 3.4, we consider the third order SDDE*

$$\dddot{x} + \ddot{x} + 20\dot{x} + x(t-\tau) + \frac{\sin x^2(t-\tau)}{x(t-\tau)} + \frac{1}{11}x\dot{\omega} = 0. \tag{4.12}$$

Equation (4.12) *is equivalent to system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -(x + \frac{\sin x^2}{x}) - 20y - z - \frac{1}{11}x\dot{\omega} \\ &\quad + \int_{t-\tau}^{t} \left[1 + 2\cos x^2(s) - \frac{\sin x^2(s)}{x^2(s)}\right] y(s)ds. \end{aligned} \tag{4.13}$$

Items (i) to (vi) of Example 4.1 and the Lyapunov functional $V(t, X_t)$ *of equation* (4.3) *are still valid for system* (4.13)*, and that*

$$V(t, \boldsymbol{0}) = 0, \tag{4.14}$$

for all $t \geq 0$ *where* $\boldsymbol{0} = (0, 0, 0) \in \mathbb{R}^3$*. Furthermore, inequalities* (4.4) *hold and from the first inequality in* (4.4) *we have*

$$V(t, X_t) \to +\infty \quad as \quad x^2 + y^2 + z^2 \to \infty,$$

so that from estimates (4.4) *and* (4.14)*, conditions (i) and (ii) of Lemma 2.2 hold.*

Applying Itô's formula using equation (4.3) *and system* (4.13)*, we have*

$$LV(t, X_t) \leq -0.06(x^2 + y^2 + z^2) \tag{4.15}$$

for all $t \geq 0, x, y$ *and* z*. Inequality* (4.15) *fulfills condition (iii) of Lemma 2.2, thus by Lemma 2.2 the trivial solution of the SDDE* (4.13) *is uniformly stochastically stable in the large. This completes the verification of Theorem 3.3.*

Next, to apply Theorem 3.4 we employed the functional $V(t, X_t)$ *defined by equation* (4.3)*, this function is positive definite (see* (4.14) *and the first inequality in* (4.4)*). In addition, from inequality* (4.15) *we have*

$$LV(t, X_t) \leq 0, \tag{4.16}$$

for all $t \geq 0, x, y$ *and* z*. Inequality* (4.16) *satisfies* (3.30)*. All conditions of Lemma 2.1 hold hence by Lemma 2.1 the trivial solution of system* (4.13) *is stochastically stable. This completes the verification of Theorem 3.4.*

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