DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
№ 4 , 2011
Electronic Journal,
reg.Эл № ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/user/diffjournal
e-mail: jodiff@mail.ru

Integral transform method for solving non homogenous time fractional partial differential equations

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Abstract

In this article, the authors solved certain non-homogeneous time fractional heat equations which is a generalization to the problem of spin- up effects on the geo strophic and quasi – geostrophic drags on a slowly rising products or drops in a rotating fluid. In the last three decades, transform methods have been used for solving fractional differential equations, singular integral equations.
The result reveals that the transform method is very convenient and effective.

Mathematics Subject Classification 2010: 26A33, 34A08, 34K37, 35R11

Keywords: Caputo fractional derivative; Non-homogeneous time fractional heat equation; Laplace transform; Singular integral equations.

1. Introduction

Engineering and other areas of sciences can be successfully modeled by the use of fractional derivatives. That is because of the fact that, a realistic modeling of physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus.
In this work, the authors implement transform method for solving the partial fractional diffusion equation which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1] , [2] , [3], [11] , the Fourier transform method [10], the iteration method [17] and operational method [10]. However most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. More detailed information about some of these results can be found in a survey paper by Kilbas and Trujillo [9]. Atanackovic and Stankovic [4],[5]and Stankovic [19] used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for a visco elastic rod. Oldham and Spanier I , [12] and [13] , respectively, by reducing a boundary value problem involving Fick's second low in electro analytic chemistry to a formulation based on the

partial Riemann – Liouville fractional with half derivative. Oldham and Spanier [13] gave other application of such equations for diffusion problems. Uchaikin [20],[21] in which the connection between solution of linear equation of fractional order and solution of the first order has been established.Wyss [23] and Schneider [18] considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions.

1.1 Definitions and notations

The left Caputo fractional derivatives of order $\alpha > 0$ $(n-1 < \alpha \le n\ ,\ n \in \mathbb{N})$ is defined by

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx .$$

Laplace transform of function $f(t)$ is as follows

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt := F(s).$$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds$$

where $\mathrm{Re}(s) > c$. For $n-1 < \alpha \le n$, one gets

$$L\{{}_0^C D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

The differential equation $x^2 y'' + xy' - (\lambda^2 x^2 + \nu^2) y = 0$ is called modified Bessel equation.
Its general solution is $y(x) = c_1 I_\nu(\lambda x) + c_2 K_\nu(\lambda x)$,
where $I_\nu(.)$, $K_\nu(.)$ are ν - order modified Bessel functions of first and second kind respectively.
On the other hand $K_\nu(.)$ is called Macdonald function.
The simplest Wright function is given by the series

$$W(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n + \beta)}$$

For $\alpha, \beta, z \in \mathbb{C}$. We have the following relationship

$$L\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp a} \quad (\mathrm{Re}(s) > |a|^{\frac{1}{\alpha}}).$$

Example 1.1: Let $L\{f(t)\} = F(s)$ and $L\{u(t,\tau)\} = U(s)\exp(-\tau q(s))$
and assuming $\phi(s)$, $q(s)$ are analytic, then one has

$$L\left(\int_0^\infty f(\tau)\, u(t,\tau)\, d\tau\right) = U(s)\, F(q(s)).$$

In special case, let us assume that, $U(s) = \dfrac{1}{s^\alpha}$ and $q(s) = s^\alpha$, then one has

$$L\{u(t,\tau)\} = \frac{\exp(-\tau s^\alpha)}{s^\alpha},$$

which leads to

$$u(t,\tau) = t^{\alpha-1} W(-\alpha,\alpha;-\tau t^{-\alpha}).$$

Then, we obtain

$$L\left(\frac{1}{t^{1-\alpha}} \int_0^\infty f(\tau) W(-\alpha,\alpha;-\tau t^{-\alpha})\, d\tau\right) = \frac{F(s^\alpha)}{s^\alpha}.$$

Provided that the integral in bracket converges absolutely.

1.2 : **Solution to singular integral equations with trigonometric kernel,**

Laplace transform can be used to solve certain types of singular integral equations.

Lemma 1.1: Solving the following singular integral equation of the form,

$$f(x) = g(x) + \lambda \int_0^\infty \frac{\sin\sqrt{xt}}{\sqrt{\pi t}} f(t)\, dt \qquad (1.1)$$

Solution: Let $L(f(t) = F(s)$, $L(g(t)) = G(s)$ be the Laplace transforms of $f(t), g(t)$ respectively,

then one gets the following relation,

$$F(s) = G(s) + \lambda \frac{1}{s\sqrt{s}} F(\frac{1}{s}), \qquad (1.2\ )$$

now, in relation (1.2) we replace s by $\frac{1}{s}$, to obtain

$$F(\frac{1}{s}) = G(\frac{1}{s}) + \lambda s\sqrt{s}F(s), \quad (1.3).$$

Combination of (1.3) and (1.2) and calculation of $F(s)$ leads to the following,

$$F(s) = \frac{G(s) + \frac{\lambda}{s\sqrt{s}} G(\frac{1}{s})}{1-\lambda^2} \quad , \quad (1.4)$$

upon using complex inversion formula, relation (1.4) leads to the following,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s) + \frac{\lambda}{s\sqrt{s}} G(\frac{1}{s})}{1-\lambda^2} e^{st}\, ds\,. \quad (1.5)$$

Example 1.2: Solve the following singular integral equation.

$$f(x) = \exp(-ax) + \lambda \int_0^\infty \frac{\sin\sqrt{xt}}{\sqrt{\pi t}} f(t)\, dt \ , \ a>0 \quad (\ 1.6)$$

Solution: Laplace-transform of the above integral equation, leads to the following

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\lambda e^{as}}{(1-\lambda^2)\sqrt{s}(1+as)}\ ds\,, \qquad (1.7)$$

the formal solution is

$$f(t) = \frac{\lambda e^{-\frac{t}{a}}}{a\sqrt{\pi}(1-\lambda^2)} \int_0^t \frac{e^{\frac{y}{a}}}{\sqrt{y}}\, dy$$

2. Main results

In this section, the authors considered certain non-homogeneous time fractional heat equations which is a generalization to the problem of spin- up effects on the geostrophic and quasi – geostrophic drags on a slowly rising products or drops in a rotating fluid studied by M. Ungarish [22].
In [20] the fundamental solution for the fractional homogenous diffusion – wave equation using joint transform method was obtained.
In this work, only the Laplace transformation is considered as it is easily understood and being popular among engineers and scientists.The basic goal of this work has been to employ the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving exact analytical solution.The transform - method introduces a significant improvement in this field over existing techniques.

2.1 Non – homogenous time fractional Heat equation.

Problem 2.1: Solve the non-homogeneous time fractional heat equation,

$$ {}_0^cD_t^\alpha u(r,t) = 2a^2\left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} - \frac{u(r,t)}{r^2}\right) - bu(r,t) - r + \frac{c}{\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)g(x)dx $$

$$ 0 \le r < 1, \quad t > 0, \quad 0 < \alpha \le 1 \quad (2.1) $$

with the boundary conditions: $\lim_{r\to 0}|u(r,t)| < \infty$, $u(1,t) = 0$, and the initial condition $u(r,0) = \lambda$, $0 \le r < 1$, $\lambda \in R$. Let $g(x)$ be Laplace transformable function.

Solution.

Step1: By taking the Laplace transform with respect to variable t of relationship (2.1), we get

$$ s^\alpha U(r,s) - s^{\alpha-1}u(r,0) = 2a^2\left(\frac{\partial^2 U(r.s)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,s)}{\partial r} - \frac{U(r,s)}{r^2}\right) - bU(r,s) - \frac{r}{s} + c\frac{G(\sqrt{s})}{\sqrt{s}} \quad . \qquad (2.2) $$

Then, by using the fact that $u(r,0) = \lambda$ in relation (1.2.2), we obtain

$$ \frac{\partial^2 U(r.s)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,s)}{\partial r} - \frac{U(r,s)}{r^2} - \left(\frac{b+s^\alpha}{2a^2}\right)U(r,s) = \frac{r}{2a^2 s} - \frac{\lambda}{2a^2}s^{\alpha-1} - \frac{c}{2a^2}\frac{G(\sqrt{s})}{\sqrt{s}} \qquad (2.3) $$

Moreover, we conclude from $u(1,t) = 0$ that

$$ U(1,s) = \int_0^\infty u(1,t)\exp(-st)dt = 0, \qquad (2.4) $$

also, $\lim_{r\to 0}|u(r,t)| < \infty$ leads to $\lim_{r\to 0}|U(r,s)| < \infty$. (2.5)

Step2: The relation(2.3) is the boundary-value problem with the conditions (2.4) and (2.5)

that can be re written as follows

$$ \frac{\partial^2 U(r.s)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,s)}{\partial r} - U(r,s)\left(\frac{1}{r^2} + \frac{b+s^\alpha}{2a^2}\right) = \frac{r}{2a^2 s} - \frac{\lambda}{2a^2}s^{\alpha-1} - \frac{c}{2a^2}\frac{G(\sqrt{s})}{\sqrt{s}}. \qquad (2.6) $$

Then we obtain its general solution as following

$$ r^2\frac{\partial^2 U(r,s)}{\partial r^2} + r\frac{\partial U(r,s)}{\partial r} - U(r,s)\left(1 + \left(\frac{s^\alpha + b}{2a^2}\right)r^2\right) = 0\,. \qquad (2.7) $$

Introducing the change of variable $\frac{\sqrt{s^\alpha + b}}{\sqrt{2a^2}}r = x$ (2.8)

we get

$$r\frac{\partial U}{\partial r}=x\frac{\sqrt{2a^2}}{\sqrt{s^\alpha+b}}\frac{\partial U}{\partial x}\frac{dx}{dr}=x\frac{\partial U}{\partial x} \qquad (2.9),$$

$$r^2\frac{\partial^2 U}{\partial r^2}=r^2\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\right)=\frac{2a^2}{s^\alpha+b}x^2\frac{\partial\left(\frac{\partial U}{\partial x}\frac{\sqrt{s^\alpha+b}}{\sqrt{2a^2}}\right)}{\partial x}\frac{\sqrt{s^\alpha+b}}{\sqrt{2a^2}}=x^2\frac{\partial^2 U}{\partial x^2}. \qquad (2.10)$$

Replacing (2.8)(2.9),(2.10) in (2.7) leads to the following equation

$$x^2\frac{\partial^2 U}{\partial x^2}+x\frac{\partial U}{\partial x}-(1+x^2)U=0. \qquad (2.11)$$

That is the modified Bessel equation with $\nu=1$. Its general solution is

$$U=c_1 I_1(x)+c_2 K_1(x). \qquad (2.12)$$

Where I_1 is first order, modified Bessel function of the first kind that is equal to

$$I_1(x)=i^{-1}J_1(ix)=\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{n!(n+1)!} \qquad (2.13)$$

and K_1 is first order, modified Bessel function of the second kind that is equal to

$$K_1(x)=\frac{\pi}{2}\frac{I_{-1}(x)-I_1(x)}{\sin\pi}=-\frac{\pi}{2}\left\{J_1(ix)+iY_1(ix)\right\}. \qquad (2.14)$$

With $x\to 0$ then $K_1\to\infty$.

Now, $\frac{\sqrt{s^\alpha+b}}{\sqrt{2a^2}}r=x,$ so that if $x\to 0$ then $r\to 0$ and in this case whereas $U(r,s)$ is bounded, therefore coefficient of $K_1(x)$ must be equal to zero which is $c_2=0$.

Then its general solution becomes

$$U=c_1 I_1(r\sqrt{s^\alpha+b}/\sqrt{2a^2}), \qquad (2.15)$$

and the complementary solution of the equation (2.6) is equal to the sum of the solutions of the following equations

$$\frac{\partial^2 U(r.s)}{\partial r^2}+\frac{1}{r}\frac{\partial U(r,s)}{\partial r}-U(r,s)\left(\frac{1}{r^2}+\frac{b+s^\alpha}{2a^2}\right)=\frac{r}{2a^2 s} \qquad (2.16)$$

$$\frac{\partial^2 U(r.s)}{\partial r^2}+\frac{1}{r}\frac{\partial U(r,s)}{\partial r}-U(r,s)\left(\frac{1}{r^2}+\frac{b+s^\alpha}{2a^2}\right)=-\frac{\lambda}{2a^2}s^{\alpha-1}-\frac{c}{2a^2}\frac{G(\sqrt{s})}{\sqrt{s}}. \qquad (2.17)$$

Now, for solving non - homogenous equation (2.16), we assume that $U_{p1}=\gamma\, r$ is the solution to the above equation. we have

$$\frac{1}{r}\gamma-\gamma\left(\frac{1}{r^2}+\frac{b+s^\alpha}{2a^2}\right)r=\frac{r}{2a^2 s} \quad\Rightarrow\quad \gamma=-\frac{1}{s(b+s^\alpha)}, \qquad (2.18)$$

therefore, the complementary solution becomes $\quad U_{p1}=-\frac{r}{s(b+s^\alpha)}. \qquad (2.19)$

Similarly, in equation (2.17), we assume that $U_{p2}=\beta$ is the complementary solution of the equation.Therefore, one has

$$-\beta\left(\frac{1}{r^2}+\frac{b+s^\alpha}{2a^2}\right)=-\frac{\lambda}{2a^2}s^{\alpha-1}-\frac{c}{2a^2}\frac{G(\sqrt{s})}{\sqrt{s}} \qquad (2.20)$$

$$\Rightarrow \beta=\frac{\lambda r^2 s^{\alpha-1}}{2a^2+r^2(b+s^\alpha)}+cr^2\frac{G(\sqrt{s})}{\sqrt{s}}\frac{1}{2a^2+r^2(b+s^\alpha)}=U_{p2}, \qquad (2.21)$$

using the relations (2.19), (2.21), the complementary solution of equation (2.6) reads

$$U_p=U_{p1}+U_{p2}=-\frac{r}{s(b+s^\alpha)}+\frac{\lambda r^2 s^{\alpha-1}}{2a^2+r^2(b+s^\alpha)}+cr^2\frac{G(\sqrt{s})}{\sqrt{s}\left\{2a^2+r^2(b+s^\alpha)\right\}}, \qquad (2.22)$$

finally, the general solution to equation (2.6) becomes

$$U(r,s)=-\frac{r}{s(b+s^\alpha)}+\frac{\lambda r^2 s^{\alpha-1}}{2a^2+r^2(b+s^\alpha)}+\frac{cr^2G(\sqrt{s})}{\sqrt{s}\left\{2a^2+r^2(b+s^\alpha)\right\}}+c_1I_1(r\sqrt{b+s^\alpha}/\sqrt{2a^2}) \qquad (2.23)$$

but, since $U(1,s)=0$ we get

$$c_1=\frac{2a^2+b+s^\alpha-\lambda s^\alpha(b+s^\alpha)-c\sqrt{s}(b+s^\alpha)G(\sqrt{s})}{s(b+s^\alpha)(2a^2+b+s^\alpha)I_1(\sqrt{b+s^\alpha}/\sqrt{2a^2})}. \qquad (2.24)$$

Consequently,

$$U(r,s)=\frac{2a^2+(b+s^\alpha)(1-\lambda s^\alpha)-c(b+s^\alpha)\sqrt{s}G(\sqrt{s})}{s(b+s^\alpha)(2a^2+b+s^\alpha)I_1(\sqrt{b+s^\alpha}/\sqrt{2a^2})}I_1(r\sqrt{b+s^\alpha}/\sqrt{2a^2})$$
$$-\frac{r}{s(b+s^\alpha)}+\frac{\lambda r^2 s^{\alpha-1}}{2a^2+r^2(b+s^\alpha)}+\frac{cr^2G(\sqrt{s})}{\sqrt{s}(2a^2+r^2(b+s^\alpha))}. \qquad (2.25)$$

Step3: Replacing $\alpha=0.5$ in (2.25) results in

$$U(r,s)=\frac{1}{\sqrt{s}}\left\{\frac{\left\{2a^2+(b+\sqrt{s})(1-\lambda\sqrt{s})\right\}I_1(r\sqrt{b+\sqrt{s}}/\sqrt{2a^2})}{\sqrt{s}(b+\sqrt{s})(2a^2+b+\sqrt{s})I_1(\sqrt{b+\sqrt{s}}/\sqrt{2a^2})}\right.$$
$$-\frac{r}{\sqrt{s}(b+\sqrt{s})}+\frac{\lambda r^2}{2a^2+r^2(b+\sqrt{s})}$$
$$\left.+cG(\sqrt{s})\left\{\frac{r^2}{2a^2+r^2(b+\sqrt{s})}-\frac{I_1(r\sqrt{b+\sqrt{s}}/\sqrt{2a^2})}{(2a^2+b+\sqrt{s})I_1(\sqrt{b+\sqrt{s}}/\sqrt{2a^2})}\right\}\right\}=\frac{F(r,\sqrt{s})}{\sqrt{s}}. \qquad (2.26)$$

Now, if we replace $\sqrt{s}$ with s, we obtain

$$F(r,s)=\frac{\left\{2a^2+(s+b)(1-\lambda s)\right\}I_1(r\sqrt{s+b}/\sqrt{2a^2})}{s(s+b)(2a^2+s+b)I_1(\sqrt{s+b}/\sqrt{2a^2})}-\frac{r}{s(s+b)}+\frac{\lambda r^2}{2a^2+r^2(s+b)}$$
$$+cG(s)\left\{\frac{r^2}{2a^2+r^2(b+s)}-\frac{I_1(r\sqrt{s+b}/\sqrt{2a^2})}{(2a^2+b+s)I_1(\sqrt{s+b}/\sqrt{2a^2})}\right\}. \qquad (2.27)$$

Step 4: We may rewrite $F(r,s)$ as following

$$F(r,s)=F_1(r,s)-F_2(r,s)+F_3(r,s)+cG(s)F_4(r,s)-cG(s)F_5(r,s) \qquad (2.28)$$

where, $$F_1(r,s)=\frac{\left\{2a^2+(s+b)(1-\lambda s)\right\}I_1(r\sqrt{s+b}/\sqrt{2a^2})}{s(s+b)(2a^2+b+s)I_1(\sqrt{s+b}/\sqrt{2a^2})}$$

$$F_2(r,s)=\frac{r}{s(s+b)}, \qquad F_3(r,s)=\frac{\lambda r^2}{2a^2+r^2(s+b)},$$

$$F_4(r,s)=\frac{r^2}{2a^2+r^2(b+s)}, \quad ,F_5(r,s)=\frac{I_1(r\sqrt{s+b}/\sqrt{2a^2})}{(2a^2+b+s)I_1(\sqrt{s+b}/\sqrt{2a^2})}, \tag{2.29}$$

F_1, F_2, F_3, F_4, F_5 are not analytic in their roots of denominators, where these roots are simple poles. Now, we obtain these poles as follows,

In F_1, $s=0, s=-b, s=-b-2a^2$ are roots of denominators.

Further, if $\alpha_n, n=1,2,3,...$ are the zeros of $J_1(\alpha_n)=0, n=1,2,3,...$, moreover subject to (2.13), the zeros of $J_1(ix)$, $I_1(x)$ are simple, then one has, $ix=\alpha_n \Rightarrow x=-i\alpha_n$

where, $x=-i\alpha_n, n=1,2,3,...$ are zeros of $I_1(x)$.

Now if we set, $\dfrac{\sqrt{s+b}}{\sqrt{2a^2}}=-i\alpha_n \qquad \Rightarrow \dfrac{s+b}{2a^2}=-\alpha_n{}^2$.

Therefore $s_n=-2a^2\alpha_n{}^2-b, n=1,2,3,...$ are the zeros of $I_1(\sqrt{s+b}/\sqrt{2a^2})$.

It is obvious that, F_1 has the poles at $s=0, s=-b, s=-b-2a^2, s_n=-2a^2\alpha_n{}^2-b, n=1,2,3,...$.

Similarly, $s=0, s=-b$ are the poles of F_2 and also $s=-\dfrac{2a^2}{r^2}-b$ is the pole of F_3, F_4 and $s=-b-2a^2, s_n=-2a^2\alpha_n{}^2-b, n=1,2,3,...$ are the poles of F_5.

Step 5: Application of Bromwich's integral and the residue theorem, leads to

$$f(r,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(st)F(r,s)\,ds=\sum_k \operatorname{Res}(\exp(st)F(r,s), s=s_k). \tag{2.30}$$

Then subject to (2.28) and the linearity property of the inversion of Laplace transform

and the Laplace transform of convolution of two functions, one has

$$f(r,t)=f_1(r,t)-f_2(r,t)+f_3(r,t)+c\{g(t)*_t f_4(r,t)\}-c\{g(t)*_t f_5(r,t)\}. \tag{2.31}$$

Therefore, to obtain $f_m(r,t), m=1,2,3,4,5$ and then $f(r,t)$, it is sufficient that we evaluate the residue of $\exp(st)F_m(r,s), m=1,2,3,4,5$ at every poles of $F_m(r,s), m=1,2,3,4,5$.

Finally, after performing all the calculations we get,

$$f(r,t)=\frac{I_1(r\sqrt{b/2a^2})}{bI_1(\sqrt{b/2a^2})}-\lambda\frac{J_1(r)}{J_1(1)}\exp(-t(b+2a^2))-\frac{r}{b}+\lambda r^2\exp(-t(b+\frac{2a^2}{r^2}))+$$

$$+2e^{-tb}\sum_{n=1}^{\infty}\frac{J_1(r\alpha_n)\left(1-\alpha_n{}^2-\lambda\alpha_n{}^2(b+2a^2\alpha_n{}^2)\right)\exp(-2a^2\alpha_n{}^2t)}{J_2(\alpha_n)(b+2a^2\alpha_n{}^2)\alpha_n(1-\alpha_n{}^2)}+cr^2\{g(t)*_t\exp(-t(b+\frac{2a^2}{r^2}))\}$$

$$-c\frac{J_1(r)}{J_1(1)}\left\{g(t)*_t\exp(-t(b+2a^2)\right\}-2c\sum_{n=1}^{\infty}\frac{\alpha_nJ_1(r\alpha_n)}{J_2(\alpha_n)(1-\alpha_n{}^2)}\left\{g(t)*_t\exp(-t(b+2a^2\alpha_n{}^2)\right\} \tag{2.32}$$

Step 6: Utilizing Efros's theorem, we have

$$u(r,t)=\frac{1}{\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})f(r,x)dx=\left\{\frac{I_1(r\sqrt{b/2a^2})}{bI_1(\sqrt{b/2a^2})}-\frac{r}{b}\right\}\frac{1}{\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})dx$$

$$-\lambda\frac{J_1(r)}{J_1(1)\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\exp(-x(b+2a^2))dx+\lambda\frac{r^2}{\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\exp(-x(b+\frac{2a^2}{r^2}))dx$$

$$+2\sum_{n=1}^\infty\frac{J_1(r\alpha_n)}{J_2(\alpha_n)}\frac{\left(1-\alpha_n{}^2-\lambda\alpha_n{}^2(b+2a^2\alpha_n{}^2)\right)}{(b+2a^2\alpha_n{}^2)\alpha_n(1-\alpha_n{}^2)\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\exp(-x(b+2a^2\alpha_n{}^2))dx$$

$$-2c\sum_{n=1}^\infty\frac{\alpha_n J_1(r\alpha_n)}{J_2(\alpha_n)(1-\alpha_n{}^2)\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+2a^2\alpha_n{}^2)d\xi\right\}dx$$

$$+\frac{cr^2}{\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+\frac{2a^2}{r^2}))d\xi\right\}dx$$

$$-c\frac{J_1(r)}{J_1(1)\sqrt{\pi t}}\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+2a^2))d\xi\right\}dx\,. \qquad (2.33)$$

At this point, we calculate every one of the integrals on the right hand side of (2.33).

First integral: By the change of variable $\frac{x}{2\sqrt{t}}=w$, we obtain

$$\int_0^\infty \exp(-\frac{x^2}{4t})dx=2\sqrt{t}\int_0^\infty \exp(-w^2)dw=\sqrt{\pi t}. \qquad (2.34)$$

Second integral: By making the change of variable $\frac{x}{2\sqrt{t}}+\sqrt{t}(b+2a^2)=w,$ we get

$$\int_0^\infty \exp-(\frac{x^2}{4t}+x(b+2a^2))dx=\exp(t(b+2a^2)^2)\int_0^\infty \exp(-(\frac{x}{2\sqrt{t}}+\sqrt{t}(b+2a^2))^2)dx$$

$$=\exp(t(b+2a^2)^2)(2\sqrt{t})\int_{\sqrt{t}(b+2a^2)}^\infty \exp(-w^2)dw=\sqrt{\pi t}\exp(t(b+2a^2)^2)erfc(\sqrt{t}(b+2a^2))\ . \qquad (2.35)$$

Similarly, by change of variable in the third integral, we obtain

$$\frac{x}{2\sqrt{t}}+\sqrt{t}(b+\frac{2a^2}{r^2})=v$$

$$\int_0^\infty \exp(-\frac{x^2}{4t}-x(b+\frac{2a^2}{r^2}))dx=\sqrt{\pi t}\exp(t(\frac{2a^2}{r^2}+b)^2)erfc(\sqrt{t}(\frac{2a^2}{r^2}+b)), \qquad (2.36)$$

also, by change of variable $\frac{x}{2\sqrt{t}}+\sqrt{t}(b+2a^2\alpha_n{}^2)=v$ in the fourth integral, we get

$$\int_0^\infty \exp(-(\frac{x^2}{4t}+(b+2a^2\alpha_n{}^2)x)dx=\sqrt{\pi t}\exp\{t(b+2a^2\alpha_n{}^2)^2\}erfc(\sqrt{t}(b+2a^2\alpha_n{}^2))\ . \qquad (2.37)$$

Fifth integral: By changing the order of integration followed by the change of variable $x-\xi=w$ in the inner integral ,we get

$$\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+2a^2\alpha_n{}^2))d\xi\right\}dx$$

$$=\int_0^\infty \exp(-\xi(b+2a^2\alpha_n{}^2))\left\{\int_0^\infty \exp(-\frac{(w+\xi)^2}{4t})g(w)dw\right\}d\xi$$

$$=\int_0^\infty g(w)\exp(-\frac{w^2}{4t})\exp(t(b+2a^2\alpha_n{}^2+\frac{w}{2t})^2)\left\{\int_0^\infty \exp(-(\frac{\xi}{2\sqrt{t}}+\sqrt{t}(b+2a^2\alpha_n{}^2+\frac{w}{2t}))^2)d\xi\right\}dw$$

$$=\exp(t(b+2a^2\alpha_n{}^2)^2)\int_0^\infty g(w)\exp((b+2a^2\alpha_n{}^2)w)\left\{2\sqrt{t}\int_{\sqrt{t}(b+2a^2\alpha_n{}^2+\frac{w}{2t}}^\infty \exp(-v^2)dv\right\}dw$$

$$=\sqrt{\pi t}\exp(t(b+2a^2\alpha_n{}^2)^2)\int_0^\infty g(w)\exp((b+2a^2\alpha_n{}^2)w)\left\{erfc(\sqrt{t}(b+2a^2\alpha_n{}^2+\frac{w}{2t}))\right\}dw \qquad (2.38)$$

along with the change of variable

$$\frac{\xi}{2\sqrt{t}}+\sqrt{t}(b+2a^2\alpha_n{}^2+\frac{w}{2t})=v.$$

Similarly, by change of variable $x-\xi=w$, and then $\frac{\xi}{2\sqrt{t}}+\sqrt{t}(b+\frac{2a^2}{r^2}+\frac{w}{2t})=v$ in the sixth integral ,we obtain

$$\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+\frac{2a^2}{r^2}))d\xi\right\}dx=$$

$$=\sqrt{\pi t}\exp(t(b+\frac{2a^2}{r^2})^2)\int_0^\infty g(w)\exp((b+\frac{2a^2}{r^2})w)\left\{erfc(\sqrt{t}(b+\frac{2a^2}{r^2}+\frac{w}{2t}))\right\}dw. \qquad (2.39)$$

In the last integral, by the same procedure and using the change of variable $x-\xi=w$, we get

$$\frac{\xi}{2\sqrt{t}}+\sqrt{t}(b+2a^2+\frac{w}{2t})=v,$$

so that

$$\int_0^\infty \exp(-\frac{x^2}{4t})\left\{\int_0^x g(x-\xi)\exp(-\xi(b+2a^2))d\xi\right\}dx=$$

$$=\sqrt{\pi t}\exp(t(b+2a^2)^2)\int_0^\infty g(w)\exp((b+2a^2)w)\left\{erfc(\sqrt{t}(b+2a^2+\frac{w}{2t}))\right\}dw \qquad (2.40)$$

Finally, substitution of (2.34),(2.35),(2.36),(2.37),(2.38),(2.39),(2.40)in (2.33),leads to the following formal solution.

$$u(r,t)=\frac{I_1(r\sqrt{b/2a^2})}{bI_1(\sqrt{b/2a^2})}-\frac{r}{b}-\frac{\lambda J_1(r)}{J_1(1)}\exp(t(b+2a^2)^2)erfc(\sqrt{t}(b+2a^2))+$$

$$+\lambda r^2\exp(t(b+\frac{2a^2}{r^2})^2)erfc(\sqrt{t}(b+\frac{2a^2}{r^2}))$$

$$+2\sum_{n=1}^{\infty}\frac{J_1(r\alpha_n)\left(1-\alpha_n{}^2-\lambda\alpha_n{}^2(b+2a^2\alpha_n{}^2)\right)}{J_2(\alpha_n)(b+2a^2\alpha_n{}^2)\alpha_n(1-\alpha_n{}^2)}\exp(t(b+2a^2\alpha_n{}^2)^2)erfc(\sqrt{t}(b+2a^2\alpha_n{}^2))$$

$$+cr^2\exp(t(b+\frac{2a^2}{r^2})^2)\int_0^{\infty}g(u)\exp(u(b+\frac{2a^2}{r^2}))erfc(\sqrt{t}(b+\frac{2a^2}{r^2}+\frac{u}{2t}))du$$

$$-c\frac{J_1(r)}{J_1(1)}\exp(t(b+2a^2)^2)\int_0^{\infty}g(u)\exp(u(b+2a^2))erfc(\sqrt{t}(b+2a^2+\frac{u}{2t}))du$$

$$-2c\sum_{n=1}^{\infty}\frac{\alpha_n J_1(r\alpha_n)}{J_2(\alpha_n)(1-\alpha_n{}^2)}\exp(t(b+2a^2\alpha_n{}^2)^2)\int_0^{\infty}g(u)\exp((b+2a^2\alpha_n{}^2)u)erfc(\sqrt{t}(b+2a^2\alpha_n{}^2+\frac{u}{2t}))du \quad (2.41)$$

3. CONCLUSION

The paper is devoted to study and application of Laplace transform. The integral transform provides powerful method for analyzing linear systems. The main purpose of this work is to develop a method for finding formal solution of certain singular integral equation and analytic solution of the time fractional heat equation which is a generalization to the problem of spin up effects on the geo strophic and quasi -geostropic drags on a slowly rising products or drops in a rotating fluid.
We hope that it will also benefit many researchers in the disciplines of applied mathematics, mathematical physics and engineering.

4. Acknowledgement

The authors would like to thank Professor Nataly B.Amplyova and Professor V. A. Kostin – for their valuable comments and suggestions that lead to a vast improvement in the paper.

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