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# The Generalized Laplace Transform and Fractional Differential Equations of Distributed Orders

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**Abstract**

In this article, we introduce a new generalized Laplace transform and derive the complex inversion formula , convolution theorem and generalized product theorem for the transform. Furthermore, the fundamental solutions of two Cauchy type fractional diffusion equation of single and distributed order are given by means of new transform in terms of the Wright functions. Also, applicability of this transform in evaluation of improper integrals of special functions is stated.

**Mathematics Subject Classification:**26A33; 44A10; 44A15; 44A35

**Keywords:** Laplace transform, Fractional derivatives, Fractional diffusion equation, Wright functions.

## 1 Introduction

We consider the Laplace-type integral transform as follows

$$\mathcal{L}_{\mathcal{E}}\{f(x);p\} = \int_0^\infty \mathcal{E}'(x) e^{-\Phi(p)\mathcal{E}(x)} f(x) dx, \tag{1-1}$$

where, $f(x)$ is piecewise continuous and of the exponential order (i.e. $|f(x)| \leq Me^{\Phi(c)\mathcal{E}(x)}$) for some constants $c, M$, complex parameter $p$ and invertible function $\Phi(p)$. Also, $\mathcal{E}(x) = \int e^{-a(x)} dx$ is an integration of exponential function with invertible function $a(x)$ .

By the definition (1-1), it is obvious that, the exponential Laplace transform is a generalization of the following well known transforms.

**i)** The Laplace transform (in the case $\Phi(p) = p, \mathcal{E}(x) = x$, and the abscissa of convergence $c_0$)

$$\begin{aligned} \mathcal{L}\{f(x);p\} &= F(p) = \int_0^\infty e^{-px} f(x)dx, \\ f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)e^{px}dp, \qquad c = \Re(p) > c_0 \end{aligned} \tag{1-2}$$

**ii)** The Mellin transform [18] (in the case $\Phi(p) = -p, \mathcal{E}(x) = \ln(x)$)

$$\begin{aligned} \mathcal{M}\{f(x);p\} &= F(p) = \int_0^\infty x^{p-1} f(x)dx, \\ f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)x^{-p}dp, \qquad c = \Re(p). \end{aligned} \tag{1-3}$$

**iii)** The $\mathcal{L}_2$-transform [1] [21-23] (in the case $\Phi(p) = \mathcal{E}(p) = p^2$)

$$\begin{aligned} \mathcal{L}_2\{f(x);p\} &= F(p) = \int_0^\infty 2xe^{-p^2x^2} f(x)dx, \\ f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\sqrt{p})e^{px^2}dp, \qquad \Re(p^2) > c. \end{aligned} \tag{1-4}$$

Since the integral transform methods are applicable tools for analyzing boundary value problems , many researchers use these methods for studing the linear partial fractional differential equations (LPFDEs) to find the fundamental solutions of these equations in terms of higher transcendental functions such as Fox H-function and the Wright function. For example see works of Gorenflo [9,10] and Mainardi [12-17].

In this paper we introduce the exponential Laplace transform ($\mathcal{L}_\mathcal{E}$-transform) which is technical merit to solve LPFDEs with non-constant coefficients by choosing the appropriate function $\mathcal{E}(x)$.

In this regard, in section 2, we derive a new inversion formula for the $\mathcal{L}_\mathcal{E}$-transform in terms of Bromwich's integral. Three theorems in the $\mathcal{L}_\mathcal{E}$-transform of the ${}_\mathcal{E}\delta_x$-derivatives , convolution and generalized product properties are also established. These properties can be useful for obtaining the solutions of some fractional diffusion on fractals. In sections 3 and 4 , we find the fundamental solutions of single and distributed order fractional diffusion equation on fractals introduced by Giona and Roman [8,18] by applying the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform and $\mathcal{L}_{\sinh^{-1}(x)}$-transform. These solutions can be expressed in terms of the higher transcendental functions of the Wright type.

In section 5, by using the generalized product theorem another application of the $\mathcal{L}_\mathcal{E}$-transform in evaluation of infinite integrals with special function is discussed. Finally, in section 6, the main conclusions are drawn .

[1] At first , Yurekli and Sadek [23] introduced this transform and used the Parseval-Goldstein theorems involving the $\mathcal{L}_2$-transform and other transforms to obtain identities involving several infinite integrals of elementary and special functions. Later, Aghili and Ansari applied this transform to solve some systems of PFDEs and singular integral equations with the special kernels [1,2].

## 2 Some Properties of The Exponential Laplace Transform

In this section, we establish theorems on the exponential Laplace transform which can be useful for solving LPFDEs. First, we derive a complex inversion formula for this transform in terms of Bromwich's integral.

**Theorem 2.1** ***(The complex inversion formula for the exponential Laplace transform)***
*Let $F(\Phi^{-1}(p))$ be analytic function of $p$ (assuming that $F(\Phi^{-1}(p))$ has not the branch point) except at finite number of poles and each of poles lies to the left of the vertical line $\Re p = c$ . If $F(\Phi^{-1}(p)) \to 0$ as $p \to \infty$ through the left plane $\Re p \leq c$ , and*

$$\mathcal{L}_{\mathcal{E}}\{f(x);p\} = F(p) = \int_0^\infty \mathcal{E}'(x)e^{-\Phi(p)\mathcal{E}(x)}f(x)dx,$$

*then*

$$\mathcal{L}_{\mathcal{E}}^{-1}\{F(p)\} = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\Phi^{-1}(p))e^{p\mathcal{E}(x)}dp. \tag{2-1}$$

**Proof:** By definition of the exponential Laplace transform (1-1) and letting $\Phi(p) = r$, we have

$$F(\Phi^{-1}(r)) = \int_0^\infty \mathcal{E}'(x)e^{-r\mathcal{E}(x)}f(x)dx,$$

now, by setting $t = \mathcal{E}(x)$ in the above relation, we obtain

$$F(\Phi^{-1}(r)) = \int_0^\infty e^{-rt}f(\mathcal{E}^{-1}(t))dt = \mathcal{L}\{f(\mathcal{E}^{-1}(t));r\}.$$

At this point, by the complex inversion formula for the Laplace transform and setting back $\mathcal{E}^{-1}(t) = x, r = p$, we get finally

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\Phi^{-1}(p))e^{p\mathcal{E}(x)}dp.$$

**Theorem 2.2** ***(The exponential Laplace transform of ${}_{\mathcal{E}}\delta_x$-derivatives)***
*Let $f, f', ..., f^{(n-1)}$ are continuous functions with piecewise continuous derivative $f^{(n)}$ on the interval $x \geq 0$ and if all functions are of exponential order $e^{\Phi(c)A(x)}$ as $x \to \infty$ (i.e. $|f(x)| \leq Me^{\Phi(c)A(x)}$) for some constants $c, M$,then for $n = 1, 2, ...$*

$$\begin{aligned}\mathcal{L}_{\mathcal{E}}\{{}_{\mathcal{E}}\delta_x^n f(x);p\} &= \Phi^n(p)\mathcal{L}_{\mathcal{E}}\{f(x);p\} - \Phi^{n-1}(p)f(0^+) \\ &- \Phi^{n-2}(p)({}_{\mathcal{E}}\delta_x f)(0^+) - \cdots - ({}_{\mathcal{E}}\delta_x^{n-1} f)(0^+)\end{aligned} \tag{2-2}$$

*where the ${}_{\mathcal{E}}\delta_x$-derivative operator is defined as follows*

$${}_{\mathcal{E}}\delta_x = \frac{1}{\mathcal{E}'(x)}\frac{d}{dx} = e^{a(x)}\frac{d}{dx},$$

*and by notation*

$$ {}_{\mathcal{E}}\delta_x^2 = ({}_{\mathcal{E}}\delta_x)({}_{\mathcal{E}}\delta_x) = \frac{1}{\mathcal{E}'^2(x)}\frac{d^2}{dx^2} - \frac{\mathcal{E}''(x)}{\mathcal{E}'^3(x)}\frac{d}{dx} = e^{2a(x)}\frac{d^2}{dx^2} - a'(x)e^{2a(x)}\frac{d}{dx}, $$

*the ${}_{\mathcal{E}}\delta_x$-derivative for any positive integer power can be found.*

**Proof:** Using the definitions of the exponential Laplace transform (1-1) and the ${}_{\mathcal{E}}\delta_x$-derivative, by integration by parts, we obtain

$$ \mathcal{L}_{\mathcal{E}}\{{}_{\mathcal{E}}\delta_x f(x);p\} = \int_0^\infty e^{-\Phi(p)\mathcal{E}(x)} f'(x)dx = e^{-\Phi(p)\mathcal{E}(x)} f(x)\,|_0^\infty + \Phi(p)\int_0^\infty \mathcal{E}'(x)e^{-\Phi(p)\mathcal{E}(x)} f(x)dx. $$

Since $f$ is of exponential order $e^{\Phi(c)\mathcal{E}(x)}$ as $x \to \infty$, it follows that

$$ \lim_{x\to\infty} e^{-\Phi(p)\mathcal{E}(x)} f(x) = 0 $$

consequently

$$ \mathcal{L}_{\mathcal{E}}\{{}_{\mathcal{E}}\delta_x f(x);p\} = \Phi(p)\mathcal{L}_{\mathcal{E}}\{f(x);p\} - f(0^+). $$

Similarly by repeated application of the above relation once again, we get

$$ \begin{aligned} \mathcal{L}_{\mathcal{E}}\{{}_{\mathcal{E}}\delta_x^2 f(x);p\} &= \Phi(p)\mathcal{L}_{\mathcal{E}}\{{}_{\mathcal{E}}\delta_x f(x);p\} - ({}_{\mathcal{E}}\delta_x f)(0^+) \\ &= \Phi^2(p)\mathcal{L}_{\mathcal{E}}\{f(x);p\} - \Phi(p)f(0^+) - ({}_{\mathcal{E}}\delta_x f)(0^+) \end{aligned} $$

and by repeating the above scheme for${}_{\mathcal{E}}\delta_x^n f(x)$, we can readily arrive at (2-2).

**Theorem 2.3** ***(The convolution theorem for the exponential Laplace transform)*** *If $F(p), G(p)$ are the exponential Laplace transform of the functions $f(x), g(x)$ respectively, then*

$$ F(p)G(p) = \mathcal{L}_{\mathcal{E}}\{f*g\} = \mathcal{L}_{\mathcal{E}}\{\int_0^x \mathcal{E}'(t)g(t)f(\mathcal{E}^{-1}(\mathcal{E}(x) - \mathcal{E}(t)))dt\} \qquad (2\text{-}3) $$

**Proof:** Using the definition of the exponential Laplace transform for $F(p), G(p)$ , we have

$$ \begin{aligned} F(p)G(p) &= (\int_0^\infty \mathcal{E}'(y)e^{-\Phi(p)\mathcal{E}(y)} f(y)dy)(\int_0^\infty \mathcal{E}'(t)e^{-\Phi(p)\mathcal{E}(t)} g(t)dt) \\ &= \int_0^\infty\int_0^\infty \mathcal{E}'(y)\mathcal{E}'(t)e^{-\Phi(p)(\mathcal{E}(t)+\mathcal{E}(y))} f(y)g(t)dydt. \end{aligned} $$

Now, by substitution $\mathcal{E}(t)+\mathcal{E}(y) = \mathcal{E}(x)$ and changing the order of integration in the double integral ,we get

$$ \begin{aligned} F(p)G(p) &= \int_0^\infty \mathcal{E}'(x)e^{\Phi(p)\mathcal{E}(x)}dx \int_0^x \mathcal{E}'(t)g(t)f(\mathcal{E}^{-1}(\mathcal{E}(x) - \mathcal{E}(t)))dt \\ &= \mathcal{L}_{\mathcal{E}}\{\int_0^x \mathcal{E}'(t)g(t)f(\mathcal{E}^{-1}(\mathcal{E}(x) - \mathcal{E}(t)))dt\}. \end{aligned} $$

**Theorem 2.4** ***(Generalized product theorem)***

*Let* $\mathcal{L}_{\mathcal{E}}\{f(x);p\} = F(p)$ *and assuming that* $\Psi_1(p)$, $\Psi_2(p)$ *are analytic functions such that,* $\mathcal{L}_{\mathcal{E}}\{k(x,t);p\} = \Psi_1(p)\mathcal{E}'(t)e^{-\mathcal{E}(t)\Phi(\Psi_2(p))}$, *then the following relation holds true*

$$\mathcal{L}_{\mathcal{E}}\left\{\int_0^\infty k(x,t)f(t)dt\right\} = \Psi_1(p)F(\Psi_2(p)). \tag{2-4}$$

**Proof:** By applying the $\mathcal{L}_{\mathcal{E}}$-transform on the integral, we can easily get the result.

$$\begin{aligned}
\mathcal{L}_{\mathcal{E}}\left\{\int_0^\infty k(x,t)f(t)dt\right\} &= \int_0^\infty \mathcal{E}'(x)e^{-\Phi(p)\mathcal{E}(x)}\int_0^\infty k(x,t)f(t)dt \\
&= \int_0^\infty f(t)dt\int_0^\infty \mathcal{E}'(x)e^{-\Phi(p)\mathcal{E}(x)}k(x,t)dx \\
&= \Psi_1(p)\int_0^\infty \mathcal{E}'(t)e^{-\Phi(\Psi_2(p))\mathcal{E}(t)}f(t)dt = \Psi_1(p)F(\Psi_2(p)).
\end{aligned}$$

In view of the theorems of the exponential Laplace transform expressed in this section we may apply this transform to LPFDEs in the next sections.

# 3 The Time-Fractional Diffusion Equation of Single Order on Fractals

**Problem 3.1:** In connection with initial-value problems we consider diffusion-type partial fractional differential equation in the Riemann-Liouville sense in the form [8,19]

$$ {}_tD^{\alpha}_{0+}u(x,t) = -C\sqrt{(x^2+1)^3}\frac{\partial u(x,t)}{\partial x}, \qquad C > 0,\ 0 < \alpha \le 1, \tag{3-1}$$

with Cauchy type initial and boundary conditions as

$$ {}_tD^{\alpha-1}_{0+}u(x,0^+) = f(x), \quad u(0,t) = 0, \quad x,t \in \mathbb{R}^+, \tag{3-2}$$

since, the equation (3-1) is a LPFDE with non-constant coefficients we set

$$\mathcal{E}(x) = \int e^{-\frac{3}{2}\ln(x^2+1)}dx = \frac{x}{\sqrt{x^2+1}}, \Phi(p) = p \tag{3-3}$$

in the integral (1-1) and apply this new integral transform (the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform) in space and the Laplace transform in time as follows

$$\begin{aligned}
\mathcal{L}\{u(x,t);s\} &= \tilde{u}(x,s) = \int_0^\infty e^{-st}u(x,t)dt, \quad \Re s > 0 \\
\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}\{u(x,t);p\} &= \hat{u}(p,t) = \int_0^\infty \frac{1}{\sqrt{(x^2+1)^3}}e^{-p\frac{x}{\sqrt{x^2+1}}}u(x,t)dx, \quad \Re p > 0.
\end{aligned}$$

Then ,by using the Laplace transform of the Riemann-Liouville derivative [18] and the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform of the equation (3-1), we obtain

$$\begin{aligned}
\mathcal{L}\{{}_tD^{\alpha}_{0+}u(x,t);s\} &= s^\alpha\tilde{u}(x,s) - {}_tD^{\alpha-1}_{0+}u(x,0^+), \\
\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}\{\sqrt{(x^2+1)^3}u(x,t);p\} &= p\hat{u}(p,t) - u(0,t)
\end{aligned}$$

where by utilizing the Cauchy type initial conditions (3-2), we arrive at

$$\hat{\tilde{u}}(p,s) = \frac{1}{s^\alpha + Cp}F(p) \tag{3-4}$$

where $F(p)$ is the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform of the initial condition$f(x)$.

At this point, by considering the complex inversion formula for the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform (2-1) and the convolution theorem (2-3), we obtain

$$\tilde{u}(x,s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tilde{u}}(p,s)e^{p\frac{x}{\sqrt{x^2+1}}}dp = \frac{1}{C}\mathcal{L}^{-1}_{\frac{x}{\sqrt{x^2+1}}}\{\frac{1}{\frac{s^\alpha}{C}+p}\}*f(x) = \frac{1}{C}e^{-s^\alpha\frac{x}{C\sqrt{x^2+1}}}*f(x)$$

where the convolution of the two functions $f,g$ for the $\mathcal{L}_{\frac{x}{\sqrt{x^2+1}}}$-transform ,can be expressed by the relation (2-3) as follows

$$f*g = \int_0^x \frac{1}{\sqrt{(t^2+1)^3}}g(t)f\left(\frac{\frac{x}{\sqrt{x^2+1}}-\frac{t}{\sqrt{t^2+1}}}{\sqrt{1-(\frac{x}{\sqrt{x^2-1}}-\frac{t}{\sqrt{t^2+1}})^2}}\right)dt.$$

Now, in regard to the inverse Laplace transform of the functions $e^{-s^\alpha\frac{x}{C\sqrt{x^2+1}}}$ via the Wright functions [11]

$$\mathcal{L}^{-1}\{e^{-s^\alpha\frac{x}{C\sqrt{x^2+1}}}\} = \frac{1}{t}W(-\alpha,0;-\frac{x}{C\sqrt{x^2+1}}t^{-\alpha}),$$

we get the explicit solution of the Cauchy type problem (3-1)-(3-2) as follows

$$u(x,t) = \int_0^x \frac{1}{\sqrt{(\tau^2+1)^3}}G^\alpha(\frac{x}{\sqrt{x^2+1}}-\frac{\tau}{\sqrt{\tau^2+1}},t)f(\tau)d\tau, \tag{3-5}$$

where the Green function $G^\alpha$ is given by

$$G^\alpha(x,t) = \frac{1}{Ct}W(-\alpha,0;-\frac{x}{C}t^{-\alpha}), \tag{3-6}$$

provided that the integral on the right-hand side of (3-5) is convergent.

## 4 The Time-Fractional Difussion Equation of Distributed Order

The earlier idea of fractional derivative of distributed order was developed by Caputo [5-7] and later other researchers study some linear and non-linear fractional differential equations of distributed order by analyzing some interesting cases of the order-density function see [3], [20]. In this paper by the notion of fractional derivative of distributed order, we consider other form of Giona and Roman Equation .

**Problem 4.1:** The following equation

$$\int_0^1 b(\alpha)[{}_t^C D_{0^+}^\alpha u(x,t)]d\alpha = \sqrt{1+x^2}\frac{\partial u(x,t)}{\partial x}$$

$$x,t>0 \quad , \quad b(\alpha)\geq 0, \int_0^1 b(\alpha)d\alpha = 1 \tag{4-1}$$

is called the *fractional diffusion of distributed order* equation subject to initial and boundary conditions $u(x,0) = f(x), u(0,t) = 0$ and the order-density function $b(\alpha)$ which enables us to determine the intensity of diffusions.

In order to solve the equation (4-1), we extend the approach by Mainardi [13] and Naber [17] to find a general representation of the fundamental solution related to a generic order-density function $b(\alpha)$. In this respect, by applying the Laplace transform of fractional derivative in the Caputo sense with respect to $t$ [18]

$$\mathcal{L}\{{}_t^C D_{0+}^{\alpha} u(x,t); s\} = s^{\alpha}\tilde{u}(x,s) - s^{\alpha-1}u(x,0^+)$$

and the $\mathcal{L}_{\sinh^{-1}(\mathrm{x})}$-transform with respect to $x$ and setting $n = 1, \Phi(p) = p$ in (2-2)

$$\mathcal{L}_{\sinh^{-1}(\mathrm{x})}\{\sqrt{1+x^2}\frac{\partial u(x,t)}{\partial x}\}; p\} = \mathcal{L}_{\sinh^{-1}(\mathrm{x})}\{{}_{\sinh^{-1}(x)}\delta_x u(x,t); p\} = p\hat{u}(p,t) - u(0,t)$$

we obtain

$$(\int_0^1 b(\alpha)s^{\alpha}d\alpha)\hat{\tilde{u}}(p,s) + p\hat{\tilde{u}}(p,s) = \frac{1}{s}(\int_0^1 b(\alpha)s^{\alpha}d\alpha)F(p)$$

from which

$$\hat{\tilde{u}}(p,s) = \frac{B(s)}{s(B(s)-p)}F(p), \quad \Re s > 0 \tag{4-2}$$

where $F(p)$ is the $\mathcal{L}_{\sinh^{-1}(\mathrm{x})}$-transform of the function $f(x)$ and

$$B(s) = \int_0^1 b(\alpha)s^{\alpha}d\alpha.$$

By inverting the $\mathcal{L}_{\sinh^{-1}(\mathrm{x})}$-transform of (4-2) ,we get the remaining Laplace transform as the following expression

$$\tilde{u}(x,s) = f(x)*\frac{-B(s)}{2s}e^{\sinh^{-1}(x)B(s)} \tag{4-3}$$

where the convolution of the two functions $f, g$ for the $\mathcal{L}_{\sinh^{-1}(\mathrm{x})}$-transform by (2-3) ( by using the fact that $\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})$ ) can be written as

$$\begin{aligned} f*g &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{1+t^2}}g(t)f(\sinh(\sinh^{-1}(x) - \sinh^{-1}(t))dt, \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{1+t^2}}g(t)f(\frac{1}{2}(\frac{x+\sqrt{x^2+1}}{t+\sqrt{t^2+1}} - \frac{t+\sqrt{t^2+1}}{x+\sqrt{x^2+1}}))dt. \end{aligned} \tag{4-4}$$

By virtue of the Titchmarsh theorem [4] for the inverse Laplace transform of the function

$$\tilde{u}_1(x,s) = \frac{-B(s)}{2s}e^{\sinh^{-1}(x)B(s)}$$

which has branch cut on the real negative semiaxis we have the following

$$u_1(x,t) = -\frac{1}{\pi}\int_0^\infty e^{-rt}\Im\{\tilde{u}_1(x, re^{i\pi})\}dr. \tag{4-5}$$

In order to simplify the above relation (4-5), we need to evaluate the imaginary part of the function $-\tilde{u}_1(x, re^{i\pi})$ along the ray $s = re^{i\pi}$, $r > 0$ where the branch cut of the function $s^\alpha$ is defined . In this regard, by writing

$$B(re^{i\pi}) = \rho\cos\gamma\pi + i\rho\sin\gamma\pi, \qquad \begin{cases} \rho = \rho(r) = |B(re^{i\pi})| \\ \gamma = \gamma(r) = \frac{1}{\pi}\arg[B(re^{i\pi})] \end{cases}$$

and substituting in the above relation, (4-5) leads to the following

$$u_1(x,t) = -\frac{1}{2\pi}\int_0^\infty \frac{\rho}{r}e^{-rt+\sinh^{-1}(x)\rho\cos(\pi\gamma)}\sin(\pi\gamma + \sinh^{-1}(x)\rho\sin(\pi\gamma))dr.$$

Finally, by using the convolution product given by relation (4-4) ,$u(x,t)$ is expressed as an integral representation

$$\begin{aligned} u(x,t) &= -\frac{1}{2\pi}f(x) * \{\int_0^\infty \frac{\rho}{r}e^{-rt+\sinh^{-1}(x)\rho\cos(\pi\gamma)}\sin(\pi\gamma + \sinh^{-1}(x)\rho\sin(\pi\gamma))dr\} \\ &= -\frac{1}{2\pi}\int_0^\infty \frac{e^{-rt}}{r}\int_{-\infty}^\infty \frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\rho e^{\rho\cos(\pi\gamma)} \\ &\times \sin(\pi\gamma + \ln(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}})\rho\sin(\pi\gamma))f(\tau)\,d\tau dr \end{aligned} \tag{4-6}$$

provided that the integrals on the right-hand side of (4-6) are convergent.
The explicit solution (4-6) of the time-fractional diffusion equation of distributed order (4-1) can be simplified in particular cases. For example if we set the order density function with respect to Dirac delta function $b(\alpha) = \delta(\alpha - n)$, $0 < n < 1$ the time-fractional diffusion equation of distributed order (4-1) is converted to time-fractional disturbance equation of single order $n$, so that

$$B(s) = s^n,\ \rho = \rho(r) = r^n,\ \gamma = n.$$

In this case since the inverse Laplace transform of $s^{\frac{n}{2}-1}e^{\sinh^{-1}(x)s^{\frac{n}{2}}}$ in (4-3) can be easily obtained in terms of the Wright functions [11]

$$\mathcal{L}^{-1}\{s^{\frac{n}{2}-1}e^{\sinh^{-1}(x)s^{\frac{n}{2}}}\} = \frac{1}{t^{\frac{n}{2}}}W(-\frac{n}{2}, 1-\frac{n}{2}; \sinh^{-1}(x)t^{-\frac{n}{2}})$$

and the formal solution $u(x,t)$ (4-6), takes the form

$$u(x,t) = -\frac{1}{2\pi t^{\frac{n}{2}}}\int_{-\infty}^\infty W(-\frac{n}{2}, 1-\frac{n}{2}; \ln(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}})t^{-\frac{n}{2}})f(\tau)d\tau. \tag{4-7}$$

## 5 Evaluation of Integrals

In this section by using the generalized product theorem for $\mathcal{L}_{\mathcal{E}}$-transform (2-4) we may obtain the values of some improper integrals of elementary and special functions.The integrands of

these integrals are coincident to kernels of generalized product theorem integral (2-4).
**Problem 5.1:** By using generalized product theorem for the $\mathcal{L}_{x^\beta}$-transform,show that

$$\int_0^\infty \mathrm{Erfc}(\frac{\tau^\beta}{2x^{\frac{\beta}{2}}})d\tau = \frac{2^{\frac{1}{\beta}}}{\sqrt{\pi}}\Gamma(\frac{\beta+1}{2\beta})x^{\frac{1}{2}}, \qquad \frac{\beta+1}{2\beta} \neq -k, \quad k\in\mathbb{Z} \tag{5-1}$$

where Erfc is complementary error function.

By setting $\Phi(p) = p$ and applying the $\mathcal{L}_{x^\beta}$-transform on the right hand side of above relation and using the fact that $\mathcal{L}_{x^\beta}\{\mathrm{Erfc}(\frac{\tau^\beta}{2x^{\frac{\beta}{2}}})\} = \frac{e^{-\tau^\beta\sqrt{p}}}{p}$, we get

$$\begin{aligned}\mathcal{L}_{x^\beta}\{\int_0^\infty \mathrm{Erfc}(\frac{1}{2}\frac{\tau^\beta}{2x^{\frac{\beta}{2}}})d\tau\} &= \mathcal{L}_{x^\beta}\{\int_0^\infty \tau^{\beta-1}\mathrm{Erfc}(\frac{\tau^\beta}{2x^{\frac{\beta}{2}}})\frac{1}{\tau^{\beta-1}}d\tau\} \\ &= \frac{1}{p}\left(\mathcal{L}_{\tau^\beta}\{\frac{1}{\tau^{\beta-1}}\}\right)_{p\to\sqrt{p}} = \frac{\Gamma(\frac{1}{\beta})}{p^{\frac{1}{2\beta}+1}}\end{aligned}$$

which, by using complex inversion formula for $\frac{\Gamma(\frac{1}{\beta})}{p^{\frac{1}{2\beta}+1}}$ we obtain

$$\mathcal{L}^{-1}_{x^\beta}\{\frac{\Gamma(\frac{1}{\beta})}{p^{\frac{1}{2\beta}+1}}\} = 2\beta\frac{\Gamma(\frac{1}{\beta})}{\Gamma(\frac{1}{2\beta})}x^{\frac{1}{2}}.$$

Now, by recalling the Legendre's duplication formula for the Gamma function as follows

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

we finally get

$$\int_0^\infty \mathrm{Erfc}(\frac{\tau^\beta}{2x^{\frac{\beta}{2}}})d\tau = \frac{2^{\frac{1}{\beta}}}{\sqrt{\pi}}\Gamma(\frac{\beta+1}{2\beta})x^{\frac{1}{2}}.$$

## 6 Conclusions

This paper provides some new results in the area of fractional calculus and a new integral transform ($\mathcal{L}_\mathcal{E}$-transform ) which was implemented to solve two Cauchy type fractional diffusion equations of single and distributed order.
It may be concluded that the $\mathcal{L}_\mathcal{E}$-transform method is very powerful efficient technique in finding exact solution for LPFDEs with non-constant coefficients and this method could lead to a promising approach for many applications in applied sciences.

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