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Weak solutions of a fractional-order nonlocal boundary value problem in reflexive Banach spaces

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Abstract In this work, we establish an existence result (based on O'Regan fixed point theorem) for a nonlinear fractional-order nonlocal boundary value problem.

Keywords: *Fractional calculus; Nonlocal boundary value problem.*

1 Preliminaries and Introduction

Let $L_1(I)$ be the space of Lebesgue integrable functions on the interval $I = [0, 1]$. Unless otherwise stated, E is a reflexive Banach space with norm $||.||$ and dual E^*. We will denote by E_w the space E endowed with the weak topology $\sigma(E, E^*)$ and denote by $C[I, E]$ the Banach space of strongly continuous functions $u : I \to E$ with sup-norm $||.||_0$.
We recall that the fractional integral operator of order $\beta > 0$ with left-hand point a is defined by (see [4], [9], [10] and [15])

$$I_a^\beta\, u(t) \;=\; \int_a^t \frac{(t\;-\;s)^{\beta\;-\;1}}{\Gamma(\beta)}\, u(s)\; ds.$$

We recall the following definitions. Let E be a Banach space and let $u : I \to E$. Then

(1) $u(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\varphi \in E^*$ we have $\varphi(u(.))$ continuous (measurable) at t_0.

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if h takes weakly convergent sequences in E to weakly convergent sequences in E.

If u is weakly continuous on I, then u is strongly measurable (see [3]), hence weakly measurable.
Note that in reflexive Banach spaces weakly measurable functions are Pettis integrable (see [1], [3] and [8] for the definition) if and only if $\varphi(u(.))$ is Lebesgue integrable on I for every $\varphi \in E^*$ (see [3]).
Now, we present some auxiliary results that will be needed in this paper. Firstly, we state O'Regan fixed point theorem ([7]).

Theorem 1.1 *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C[I, E]$. Suppose $T : Q \to Q$ is weakly sequentially continuous and assume $TQ(t)$ is weakly relatively compact in E for each $t \in I$, holds. Then the operator T has a fixed point in Q.*

The following theorems can be found in [2], [16] and [5] respectively.

Theorem 1.2 *(Dominated convergence theorem for Pettis integral)*
Let $u : I \to E$. Suppose there is a sequence (u_n) of Pettis integrable functions from I into E such that $\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$ a.e. for $\varphi \in E^*$. If there is a scalar function $\psi \in L_1(I)$ with $||u_n(\cdot)|| < \psi(\cdot)$ a.e. for all n, then u is Pettis integrable and

$$\int_J u_n(s)\, ds \;\to\; \int_J u(s)\, ds \quad \text{weakly} \;\; \forall\, t \in I.$$

Theorem 1.3 *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Theorem 1.4 *Let Q be a weakly compact subset of $C[I, E]$. Then $Q(t)$ is weakly compact subset of E for each $t \in I$.*

Finally, we state some results which is an immediate consequence of the Hahn-Banach theorem.

Theorem 1.5 *Let E be a normed space with $u_0 \neq 0$. then there exists a $\varphi \in E^*$ with $||\varphi|| = 1$ and $\varphi(u_0) = ||u_0||$.*

Theorem 1.6 *If $u_0 \in E$ is such that $\varphi(u_0) = 0$ for every $\varphi \in E^*$, then $u_0 = 0$.*

In this work we study the existence of solutions, in the Banach space $C[I, E]$, of the nonlocal boundary value problem

$$\left\{ \begin{array}{rcl} D^\beta \; u(t) & + f(t, u(t)) & = \; 0, \\ & \beta & \in \; (1, 2), \\ & t & \in \; (0, 1), \\ I^\gamma u(t)|_{t=0} \; = & & 0, \\ & \gamma & \in \; (0, 1], \\ & \alpha \; u(\eta) & = \; u(1), \\ & 0 \; < \; \eta & < 1, \\ & 0 \; \leq \; \alpha \; \eta^{\beta-1} & < \; 1. \end{array} \right. \tag{1}$$

Now consider the fractional-order integral equation

$$\begin{aligned} u(t) \; = \; & - \; I^\beta \; f(t, u(t)) - \tfrac{\alpha \; t^{\beta-1}}{1-\alpha \; \eta^{\beta-1}} \int_0^\eta \tfrac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \; f(s, u(s)) \; ds \\ + \; & \tfrac{t^{\beta-1}}{1-\alpha \; \eta^{\beta-1}} \int_0^1 \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)} \; f(s, u(s)) \; ds, \end{aligned}$$
$$\beta \in (1, 2), t \in (0, 1).$$

In [7] the author studied the integral equation

$$y(t) \; = \; x_0 \; + \; \int_0^t f(s, y(s)) \; ds, \;\; t \; \in \; [0, T], \; x_0 \; \in \; E$$

where $E = (E, |.|)$ is a real Banach space, under the assumptions that $f(t, .)$ is weakly sequentially continuous for each $t \in [0, T]$ and $f(., y(.))$ is Pettis integrable on $[0, T]$ for each continuous function $y : [0, T] \to E$ and $|f(t, y)| \leq h_r(t)$ for a.e. $t \in [0, T]$ and all $y \in E$ with $|y| \leq r, r > 0, h_r \in L_1[0, T]$.
Also, in [6] the author studied the Volterra-Hammerstein integral equation

$$y(t) \; = \; h(t) \; + \; \int_0^t k(t, s) \; f(s, y(s)) \; ds, \;\; t \; \in \; [0, T], \; T \; > \; 0,$$

under the assumptions that $f : [0, T] \times B \to B$ is weakly-weakly continuous and $h : [0, T] \to B$ is weakly continuous, where B is a reflexive Banach space. Here we study the existence of weak solution of the fractional-order integral equation (2) such that the function $f : I \times B_r \to E$ satisfies the following conditions:

(1) For each $t \in I, f_t = f(t, .)$ is weakly sequentially continuous.

(2) For each $u \in E_r$, $f(., u(.))$ is weakly measurable on I.

(3) for any $r > 0$, the weak closure of the range of $f(I \times B_r)$ is weakly compact in E (or equivalently; there exists an M_r such that $||f(t,u)|| \leq M_r$ for all $(t,u) \in I \times B_r$).

Definition 1.1 *by a weak solution of (2) we mean a function $u \in C[I, E]$ such that for all $\varphi \in E^*$*

$$\varphi(u(t)) = - I^\beta \phi(f(t,u(t))) - \frac{\alpha t^{\beta-1}}{1-\alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \varphi(f(s,u(s))) ds + \frac{t^{\beta-1}}{1-\alpha \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi(f(s,u(s))) ds, \beta \in (1,2), t \in (0,1).$$

2 Fractional-order integrals in reflexive Banach spaces

Here, we define the fractional-order integral operator in reflexive Banach spaces. Definition given below is an extension of such a notion for real-valued functions.

Definition 2.1 *Let $u : I \to E$ be a weakly measurable function, such that $\varphi(u(.)) \in L_1(I)$, and let $\alpha > 0$. Then the fractional (arbitrary) order Pettis integral (shortly FPI) $I^\alpha u(t)$ is defined by*

$$I^\alpha u(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} u(s) ds.$$

In the above definition the sign " $\int$ " denotes the Pettis integral.
Such an integral is well defined (see [11]):

Lemma 2.1 *Let $u : I \to E$ be a weakly measurable function, such that $\varphi(u(.)) \in L_1(I)$, and let $\alpha > 0$. The fractional (arbitrary) order Pettis integral*

$$I^\alpha u(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} u(s) ds$$

exists for almost every $t \in I$ as a function from I into E and $\varphi(I^\alpha u(t)) = I^\alpha \varphi(u(t))$.

The following lemma can be found in [12]

Lemma 2.2 *Let $u : I \to E$ be weakly continuous function on $[0,1]$. Then, FPI of u exists for almost every $t \in [0,1]$ as a weakly continuous function from $[0,1]$ to E. Moreover,*

$$\varphi(I^\alpha u(t)) \;=\; I^\alpha \varphi(u(t)), \;\; \text{for all} \;\; \varphi \;\in\; E^*.$$

Definition 2.2 *Let $u : I \to E$. We define the fractional-Pseudo derivative (shortly FPD) of u of order $\alpha \in (n-1, n), n \in N$ by*

$$\frac{d^\alpha}{dt^\alpha}\, u(t) \;=\; D^n I^{n-\alpha} u(t).$$

In the above definition the sign "D" denotes the Pseudo differential operator (see [8]).

The following lemma can be found in [13]

Lemma 2.3 *Let $u : [0,1] \to E$ be weakly continuous function on $[0,1]$ such that the real-valued function $I^{n-\alpha}\varphi u$ is n-times differentiable. Then, the FPD of u of order $\alpha \in (n-1, n)$, exists.*

Definition 2.3 *A function $u : I \to E$ is called Pseudo solution of (1) if $u \in C[I,E]$ has FPD of order $\beta \in (1,2), I^\gamma u(t)|_{t=0} = 0, \gamma \in (0,1], \alpha u(\eta) = u(1), 0 < \eta < 1, 0 \le \alpha\eta^{\beta-1} < 1$ and satisfies*

$$\frac{d^2}{dt^2}\varphi(I^{2-\beta}\; u(t)) \;+\; \varphi(f(t,u(t))) \;=\; 0, \;\; \text{a.e. on } (0,1), \;\; \text{for each } \varphi \;\in\; E^*.$$

Now, for the properties of the integrals of fractional-orders in reflexive spaces we have the following lemma [11]:

Lemma 2.4 *Let $u : I \to E$ be weakly measurable and $\varphi(u(.)) \in L_1(I)$. If $\alpha,\ \beta \in (0,1)$, we have:*

(1) $I^\alpha I^\beta u(t) = I^{\alpha+\beta} u(t)$ *for a.e.* $t \in I$.

(2) $\lim_{\alpha\to 1} I^\alpha u(t) = I^1 u(t)$ *weakly uniformly on I if only these integrals exist on I.*

(3) $\lim_{\alpha\to 0} I^\alpha u(t) = u(t)$ *weakly in E for a.e.* $t \in I$.

(4) If, for a fixed $t \in I$, $\varphi(u(t))$ is bounded for each $\varphi \in E^*$, then $lim_{t\to 0} I^\alpha u(t) = 0$.

3 Main result

In this section we present our main result by proving the existence of solutions of the equation (2) in $C[I,E]$.
Let E be a reflexive Banach space. And let

$$E_r = \{u \in C[I,E]: ||u||_0 < \frac{M_r}{\Gamma(1 + \beta)} + r\} \quad (r > 0),$$

where $||.||_0$ is the sup-norm. We will consider the set

$$B_r = \{u(t) \in E: u \in E_r, t \in I\}.$$

Now, we are in a position to formulate and prove our main result.

Theorem 3.1 *Let the assumptions (1) - (3) are satisfied.*

$$\text{If} \quad \frac{(\alpha + 1) M_r}{(1 - \alpha \eta^{\beta-1}) \Gamma(1 + \beta)} < r$$

Then equation (2) has at least one weak solution $u \in C[I,E]$.

Proof: Let us define the operator T as

$$Tu(t) = - I^\beta f(t,u(t)) - \frac{\alpha t^{\beta-1}}{1-\alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)) ds$$
$$+ \frac{t^{\beta-1}}{1-\alpha \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)) ds, \ \beta \in (1,2), \ t \in I.$$

We will solve equation (2) by finding a fixed point of the operator T.
We claim

$$T: C[I,E] \to C[I,E].$$

To prove our claim, first note that assumption (2) implies that for each $u \in C[I,E]$, $f(.,u(.))$ is weakly measurable on I. The fact that f has weakly compact range means that $\varphi(f(.,u(.)))$ is Lebesgue integrable on I for every $\varphi \in E^*$ and thus the operator T is well defined. Now, we show that if $u \in C[I,E]$, then $Tu \in C[I,E]$. Note that there exists $r > 0$ with $||u||_0 = \sup_{t \in I} ||u(t)|| < \frac{M_r}{\Gamma(1 + \beta)} + r$.
Now assumption (3) implies that

$$||f(t,u(t))|| \le M_r \quad \text{for} \quad t \in [0,1].$$

Let $t, \tau \in [0,1]$ with $t > \tau$. Without loss of generality, assume $Tu(t) - Tu(\tau) \neq 0$. Then there exists (a consequence of Theorem 1.5) $\varphi \in E^*$ with $||\varphi|| = 1$ and

$$||Tu(t) \ - \ Tu(\tau)|| \ = \ \varphi(Tu(t) \ - \ Tu(\tau)).$$

Thus

$$\begin{aligned}
&||Tu(t) \ - \ Tu(\tau)|| \leq \\
&\leq \ | \ \textstyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds \\
&- \ \textstyle\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds \ | \\
&+ \ \tfrac{\alpha}{1-\alpha \ \eta^{\beta-1}} \ \textstyle\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds \ |t^{\beta-1} \ - \ \tau^{\beta-1}| \\
&+ \ \tfrac{1}{1-\alpha \ \eta^{\beta-1}} \ \textstyle\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds \ |t^{\beta-1} \ - \ \tau^{\beta-1}| \\
&\leq \ | \textstyle\int_0^\tau \frac{(t-s)^{\beta-1} \ - \ (\tau-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds| \\
&+ \ | \textstyle\int_\tau^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ \varphi(f(s,u(s))) \ ds| \\
&+ \ \tfrac{\alpha \ M_r}{1-\alpha \ \eta^{\beta-1}} \textstyle\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \ ds \ |t^{\beta-1} - \tau^{\beta-1}| \\
&+ \ \tfrac{M_r}{1-\alpha \ \eta^{\beta-1}} \textstyle\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \ ds \ |t^{\beta-1} - \tau^{\beta-1}| \\
&\leq \ \tfrac{M_r}{\Gamma(\beta)} \ \left(\textstyle\int_0^\tau |(t-s)^{\beta-1} \ - \ (\tau - s)^{\beta-1}| \ ds \ + \ \int_\tau^t (t-s)^{\beta-1} \ ds \right) \\
&+ \ \tfrac{\alpha \ M_r \ \eta^\beta}{(1-\alpha \ \eta^{\beta-1}) \ \Gamma(1+\beta)} \ |t^{\beta-1} \ - \ \tau^{\beta-1}| \\
&+ \ \tfrac{M_r}{(1-\alpha \ \eta^{\beta-1}) \ \Gamma(1+\beta)} \ |t^{\beta-1} \ - \ \tau^{\beta-1}| \\
&\leq \ \tfrac{M_r}{\Gamma(1+\beta)} \left(2 \ (t-\tau)^\beta \ + \ |t^\beta - \tau^\beta|\right) \\
&+ \ \tfrac{M_r \ (\alpha \ \eta^\beta \ + \ 1)}{(1-\alpha \ \eta^{\beta-1}) \ \Gamma(1+\beta)} \ |t^{\beta-1} \ - \ \tau^{\beta-1}|.
\end{aligned}$$

which proves that $Tu \in C[I,E]$.
Now, let

$$\begin{aligned}
Q = \{u \ \in \ E_r : \ (\forall \ t, \tau \in I) \\
(||u(t) - u(\tau)|| \ &\leq \ \frac{M_r}{\Gamma(1+\beta)} \left(2 \ (t-\tau)^\beta \ + \ |t^\beta - \tau^\beta|\right) \\
&+ \ \frac{M_r(\alpha \ \eta^\beta \ + \ 1)}{(1-\alpha \ \eta^{\beta-1}) \ \Gamma(1+\beta)} |t^{\beta-1} - \tau^{\beta-1}|\},
\end{aligned}$$

Note that Q is nonempty, closed, bounded, convex and equicontinuous subset of $C[I,E]$. Now, we claim that $T : Q \to Q$ and is weakly sequentially continuous. If this is true then according to Theorem 1.3, TQ is bounded in $C[I,E]$ (hence, Theorem 1.4, implies $TQ(t)$ is weakly relatively compact in E for each $t \in I$) and the result follows immediately from Theorem 1.1. It remains to prove our claim. First we show that T maps Q into Q. To see this, note that the inequality (2) shows that TQ is norm continuous. Now, take $u \in Q$; without loss of generality, we may assume that $I^\alpha f(t,u(t)) \neq 0$, then, by Theorem 1.5, there exists $\varphi \in E^*$ with $||\varphi|| = 1$ and $||I^\alpha f(t,u(t))|| = \varphi(I^\alpha f(t,u(t)))$. Thus

$$\begin{aligned}
||Tu(t)|| \le& \\
\le\ & ||I^\beta\ f(t,u(t))||\ +\ ||\tfrac{\alpha\ t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^\eta\ \tfrac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds|| \\
+\ & ||\tfrac{t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^1\ \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds|| \\
=\ & \varphi(I^\beta\ f(t,u(t)))\ +\ \varphi\left(\tfrac{\alpha\ t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^\eta\ \tfrac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds\right) \\
+\ & \varphi\left(\tfrac{t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^1\ \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds\right) \\
=\ & I^\beta\ \varphi(f(t,u(t)))\ +\ \tfrac{\alpha\ t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^\eta\ \tfrac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\ \varphi(f(s,u(s)))\ ds \\
+\ & \tfrac{t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^1\ \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ \varphi(f(s,u(s)))\ ds \\
\le\ & M_r\ \int_0^t\ \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ ds\ +\ \tfrac{\alpha\ M_r\ t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^\eta\ \tfrac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\ ds \\
+\ & \tfrac{M_r\ t^{\beta-1}}{1-\alpha\ \eta^{\beta-1}}\ \int_0^1\ \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ ds \\
\le\ & \tfrac{M_r\ t^\beta}{\Gamma(1+\beta)}\ +\ \tfrac{M_r\ t^{\beta-1}}{(1-\alpha\ \eta^{\beta-1})\ \Gamma(1+\beta)}\ (\alpha\ \eta^\beta\ +\ 1) \\
\le\ & \tfrac{M_r}{\Gamma(1+\beta)}\ +\ \tfrac{M_r\ (\alpha\ +\ 1)}{(1-\alpha\ \eta^{\beta-1})\ \Gamma(1+\beta)} \\
<\ & \tfrac{M_r}{\Gamma(1+\beta)}\ +\ r,
\end{aligned}$$

therefore

$$||Tu||_0\ <\ \frac{M_r}{\Gamma(1+\beta)}\ +\ r.$$

Thus $T : Q \to Q$. Finally, we will show that T is weakly sequentially continuous. To see this, let $\{u_n\}_{n=1}^\infty$ be a sequence in Q and let $u_n(t) \to u(t)$ in E_w for each $t \in [0,1]$. Recall [5] that a sequence $\{u_n\}_{n=1}^\infty$ is weakly convergent in $C[I,E]$ if and only if it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t,.)$, the Lebsegue dominated convergence theorem (see assumption (3)) for the Pettis integral [2] implies for each $\varphi \in E^*$ that $\varphi(Tu_n(t)) \to \varphi(Tu(t))$ a.e. on I, $Tu_n(t) \to Tu(t)$ in E_w. So $T : Q \to Q$ is weakly sequentially continuous. The proof is complete. ■

Now, we are looking for sufficient conditions to ensure the existence of Pseudo solution to the boundary value problem (1).

Theorem 3.2 *If $f : I \times B_r \to E$ satisfies the assumptions of Theorem 3.1, then the boundary value problem (1) has at least one solution $u \in C[I,E]$.*

Proof: Let us remark, that by assumptions (2), (3) the FPI of f of order $\beta > 1$ exists and

$$\varphi(I^{\beta} f(t, u(t))) \ = \ I^{\beta}\ \varphi(f(t, u(t))), \ \ \text{for all} \ \ \varphi \ \in \ E^{*}.$$

Let u be a solution of equation (2), then

$$u(t) \ = \ - \ I^{\beta}\ f(t, u(t)) \ - \ \frac{\alpha\ t^{\beta-1}}{1 - \alpha\ \eta^{\beta-1}} \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\ f(s, u(s))\ ds$$
$$+ \ \frac{t^{\beta-1}}{1 - \alpha\ \eta^{\beta-1}} \int_0^{1} \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)}\ f(s, u(s))\ ds, \ \beta \ \in \ (1, 2), \ t \ \in \ (0, 1).$$

It is clear that

$$I^{\gamma} u(t)|_{t=0} \ = \ 0, \ \gamma \ \in \ (0, 1], \ \alpha\ u(\eta) \ = \ u(1).$$

Furthermore, we have

$$u(t) \ = \ - \ I^{\beta}\ f(t, u(t)) \ + \ K\ t^{\beta-1}, \tag{2}$$

where

$$K \ = \ \frac{-\alpha}{1-\alpha\ \eta^{\beta-1}} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\ f(s, u(s))\ ds$$
$$+ \ \frac{1}{1-\alpha\ \eta^{\beta-1}} \int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ f(s, u(s))\ ds.$$

since $u \in C[I, E]$, then $\varphi(I^{2-\beta} u(t)) = I^{2-\beta} \varphi(u(t))$, for all $\varphi \in E^{*}$ (see Lemma 2.2). From equation (2), we deduce that

$$\begin{aligned} \varphi u(t) \ &= \ - \ \varphi(I^{\beta}\ f(t, u(t))) \ + \ \varphi\ K\ t^{\beta-1} \\ &= \ - \ I^{\beta}\ \varphi(f(t, u(t))) \ + \ \varphi\ K\ t^{\beta-1}. \end{aligned} \tag{3}$$

Operating by $I^{2-\beta}$ on both sides of the equation (3) and using the properties of fractional calculus in the space $L_1[0, 1]$ (see [14] and [15]) result in

$$I^{2-\beta}\ \varphi u(t) \ = \ -I^{2}\ \varphi(f(t, u(t))) \ + \ \varphi\ K\ \Gamma(\beta)\ t.$$

Therefore,

$$\varphi(I^{2-\beta}\ u(t)) \ = \ -I^{2}\ \varphi(f(t, u(t))) \ + \ \varphi\ K\ \Gamma(\beta)\ t.$$

Thus

$$\frac{d^2}{dt^2} \varphi(I^{2-\beta}\ u(t)) \ = \ - \ \varphi(f(t, u(t))) \ \ \text{a.e. on} \ \ (0, 1).$$

That is u has the FPD of order $\beta \in (1, 2)$ and u is a solution of the differential equation (1) which complete the proof. ■

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