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Symbolic dynamics

On application of balance method to approximation of invariant measures of a dynamical system

N.B. Ampilova, ampilova@math.spbu.ru

E.I.Petrenko, evgeni.petrenko@mail.ru

Math.and Mech. Faculty, St.Petersburg State University, Universitetskii pr. 28, Starii Petergof,195504 St.Petersburg, Russia

Abstract

Invariant measures of a dynamical system describe the asymptotic behaviour of its orbits, hence the elaboration of numerical methods of their approximation is of considerable importance in applications. We use the concept of symbolic image which is a finite approximation of a dynamical system. Symbolic image is constructed as an oriented graph for a mapping f and a fixed covering of its phase space. Vertices of the graph correspond to the cells of the covering and edges mark the existence of nonempty intersections of the covering cells with their images. We construct an invariant measure (stationary process) on the graph of symbolic image, which is the approximation of an invariant measure of the initial system. Application of balance method allows us to construct an invariant measure in such a way to assign the measure to all edges of the graph.

key words:dynamical system, symbolic image, entropy, stationary process on a graph, linear programming

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1 Introduction

This paper is devoted to the elaboration of numerical methods of the construction of an invariant measure of a dynamical system using the notion of symbolic image [6] — a directed graph constructed by the system and its finite covering. The advantage of such a method is that many problems (localization of periodic orbits and invariant sets, estimation of Lyapunov exponents, estimation of topological entropy) may be solved by using well known algorithms for directed graphs.

We consider a method of the construction of an invariant measure on a directed graph. It was shown in [8] that such a measure is an approximation of an invariant measure of the initial system. The algorithm of the construction of such a measure using prime cycles was designed and implemented in [1]. In a prime cycle with l edges the value $1/l$ is assigned to every edge. A coefficient (weight) is designated to every prime cycle, being the sum of weights equals to one. The measure of the edge belonging to more than one cycle is defined as the sum of the measures which the edge has in every cycle. If an edge does not belong to any cycle, its measure is zero. The measure of a vertex is the sum of measures of outcoming (or incoming) edges. This method, while clear, has an evident disadvantage: the number of prime cycles may be very significant and the algorithm becomes time-consuming. An optimization may lead to cycles missing. Hence, the measure is not assigned to all edges of the graph.

The method proposed in this work is aimed at the construction of an invariant measure, such that to assign a value to every edge. To solve the problem we formulate it as a task of linear programming. It allows us to construct a stationary process on the graph (with a given accuracy), using a method of the sequential balance of the vertices measures. L.M. Bregman proved the convergence of the method in [2].

The paper is organized as follows: next section is dedicated to the notion of symbolic image. In section 3 definitions of Markov chain, stationary process on a graph and its entropy are given. Section 4 describes the algorithm of the construction of the invariant measure. In the next section halting criteria of the algorithm are fiven. Finally, in sections 6 and 7 we give examples of entropy estimation and summarize our results.

2 Symbolic image of a dynamical system

Let ϕ be a discrete dynamical system generated by a homeomorphism f on a compact $M \in R^n$. Symbolic image of a dynamical system f [6] is an oriented graph G, constructed in accordance with a covering $\{M_i\}, i = 1, \dots, k$ of M by closed sets, being vertices correspond to the covering cells and the existence of the edge (i,j) means that $f(M_i) \cap M_j \neq \emptyset$. The symbolic image is a finite approximation of the system f.

It depends on the covering and may be specified by the following parameters: d — diameter of the covering (the largest of diameters of M_i); q — upper bound of the symbolic image (the largest of diameters of $f(M_i)$); r — lower bound of the symbolic image, which is the minimum of the distances between $f(M_i)$ and M_j, if $f(M_i) \cap M_j = \emptyset$. Being a relationship between the parameters and an value ε is given, there is a correspondence between the ε-orbits of the system and paths on G [9]. The construction of a sequence of symbolic images corresponding to a sequential subdivision of the set M results in obtaining sequential approximations of the system dynamics.

3 Markov chain on a graph

Consider an oriented graph $G = (V, E)$.

Definition 1 *[5] A Markov chain μ on a graph $G = (V, E)$ is an assignment of probabilities $\mu(I) \geq 0, I \in V$ and conditional probabilities $\mu(e|b(e)) \geq 0$ such that $\sum_{I \in V} \mu(I) = 1; \sum_{e \in E_I} \mu(e|I) = 1 \, \forall I \in V$, where E_I denotes the set of edges outcoming from I, $b(e)$ denotes the beginning of the edge e and $\mu(e) = \mu(b(e))\mu(e|b(e))$.*

Let a row vector $\mathbf{p}$ be the initial state distribution, such that $p_I = \mu(I)$ and the conditional probabilities $\mu(e|I)$ form a stohastic square matrix defined by $P_{IJ} = \sum_{e \in E_I^J} \mu(e|I)$, where E_I^J denotes the set of edges outcoming from I and incoming to J.

Definition 2 *[5] The Markov chain is said to be stationary if the equality*

$$\mathbf{p}P = \mathbf{p} \tag{1}$$

holds.

Taking into account the definition of Markov chain, it easy to understand that stationarity condition means the following: for any vertex the sum of measures of incoming edges equals to the sum of measures of outcoming ones.

It is well known [10] that the entropy of Markov process may be computed by the formula

$$h(\mu) = -\sum_{e \in E(G)} \mu(b(e))\mu(e|b(e)) \log(\mu(e|b(e))). \tag{2}$$

For practical computation we use the equivalent form

$$h(\mu) \;=\; \sum_{b(e)\in V} \mu(b(e)) \log(\mu(b(e))) - \sum_{e\in E} \mu(e) \log(\mu(e)). \qquad (3)$$

Let G be an irreducible graph. Then the entropy of every stationary process μ on G satisfies the inequality [5]

$$h(\mu) \le \log \lambda, \qquad (4)$$

where λ is the maximal eigenvalue of the adjacency matrix of G. If G is an arbitrary graph then in (4) λ is the maximal value of eigenvalues of all irreducible components of G.

4 Construction of an invariant measure μ

Assign probabilities to all edges of the graph G arbitrary. Denote by $P = \{p_{ij}\}, i, j = 1, \dots, n$, the matrix formed by these values. Our goal is to transform P in such a way to obtain a stationary process on G.

This problem may be formulated as a part of the following linear programming task.

Maximize the function $\sum_{i,j} x_{ij} \ln \frac{p_{ij}}{x_{ij}}$ on conditions

$$\begin{array}{l} \sum_{j=1}^{n} x_{ij} = a_i, \sum_{i=1}^{n} x_{ij} = b_j, x_{ij} \ge 0; \\ \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j; a_i, b_j > 0; p_{ij} \ge 0; \sum_{i,j} x_{ij} = 1. \end{array} \qquad (5)$$

Our problem may be considered as a particular case when $a_i = b_i, i = 1, \dots, n$.

A method of approximative solution of (5) based on the successive balance of rows and columns of P was proposed by G.V.Sheleihovsky. L.M.Bregman [2] proved its convergence. In [3] L.M.Bregman supposed and proved the relaxation method to solve both convex and linear programming tasks, which coincided with Sheleihovsky method for the tasks with linear restrictions.

Order elements x_{ij} on rows and form vector $x = \{x_k\}, k = 1, \dots, m$, where $m \le n^2$, (all x_k are different). Consider $S = \{x \in R^m, x_k > 0, k = 1, \dots, m\}$. Let f be a strictly convex function, $f \in C^1(S)$ and $f \in C^0(\overline{S})$. Then (5) may be formulated in the following way:

Minimize the function $\sum_k x_k \ln \frac{x_k}{p_k}$ on linear restrictions

$$Ax = b, \qquad (6)$$

where $x \in \overline{S}, b \in R^n$, A is $[n \times m]$ matrix. In such a form our problem is formulated with A as (1,-1) matrix and $b = 0$.

It was proved in [3] that a sequence $\{x^l\}$ converges to a solution of (6) if the following conditions are fulfilled

$$\begin{gathered} grad\, f(x^{l+1}) = grad\, f(x^l) + \lambda A_i, \\ (A_i, x^{l+1}) = b_i, \end{gathered} \tag{7}$$

where λ is unknown parameter, A_i is i^{th} row of A and b_i is the right part of i^{th} equation in the linear restrictions (6). The solving of (7) leads to obtaining both λ and the formula of transformation for x_{ij} :

$$x_{ij}^{l+1} = x_{ij}^l \sqrt{\frac{in(i)}{out(i)}},$$

for i^{th} row and

$$x_{ki}^{l+1} = x_{ki}^l \sqrt{\frac{out(i)}{in(i)}}$$

for i^{th} column, being $in(i)$ and $out(i)$ are sums of elements in column i and row i respectively. It should be noted that diagonal elements are not changed.

Algorithm. Denote the given accuracy of computation by ε. Let $i \in V$ and

$$\begin{aligned} beg(i) &= \{e \in E, e = (i,j), j \in V\}, \\ end(i) &= \{e \in E, e = (j,i), j \in V\}. \end{aligned}$$

- Assign measures to all edges of G. As the normalization step may be fulfilled at the and of operating period, we assume $\mu(e) = 1$, $\forall e \in E$.

- For each vertex i calculate its balans

  $$q(i) = |\sum_{e \in beg(i)} \mu(e) - \sum_{e \in end(i)} \mu(e)|.$$

  Construct the queue Q of the vertices of G, being a vertex i with the maximum $q(i)$ has the maximum priority. So, we assign the greatest priority to the most unbalanced vertex.

- In the cycle: select the next vertex i from Q.

  - If $q(i) < \varepsilon$, then complete the processing of i and go out from the cycle. (In view of the structure of Q such an inequality holds for all remaining elements.)
  - Else calculate
    * $out(i) = \sum_{e \in beg(i)} \mu(e)$
    * $in(i) = \sum_{e \in end(i)} \mu(e)$
    * $\forall e \in end(i)\, \mu(e) := \mu(e) * \sqrt{\frac{out(i)}{in(i)}}$.
    * $\forall e \in beg(i)\, \mu(e) := \mu(e) * \sqrt{\frac{in(i)}{out(i)}}$.
    * If some of values $out(i), in(i), \sqrt{\frac{out(i)}{in(i)}}$ is too large (or small), we fulfill the normalization.

- Fulfill the normalization. The algorithm is completed.

To provide the efficiency of the algorithm we have to save both forth and back directions of the edges, which results in the representation of the graph with using two hash-tables. Priority queue has been implemented using Fibonacci trees [4].

5 Halting problem

Let C_v be the volume of a cell C corresponding to a vertex v, $|V|$ be the number of vertices and ε be a given accuracy. We consider the following criteria for the algorithm halting:

1. Maximal discrepancy is not greater than ε.
$$\max_{v \in V} q_k(v) < \varepsilon. \tag{8}$$
2. Maximal discrepancy divided by the cell volume is not greater than ε.
$$\max_{v \in V} q_k(v) < \varepsilon C_v. \tag{9}$$
3. The sum of discrepances over all vertices at the step k is not greater than ε.
$$\sum_{v \in V} \left| \sum_{e \in V} m_{ev}^k - \sum_{t \in V} m_{vt}^k \right| < \varepsilon. \tag{10}$$

4. The average sum of discrepances over all vertices at the step k is not greater than ε.

$$\sum_{v \in V} \left| \sum_{e \in V} m_{ev}^k - \sum_{t \in V} m_{vt}^k \right| < \varepsilon |V|. \tag{11}$$

5. The sum of discrepances over all vertices at the step k divided by the cell volume is not greater than ε.

$$\sum_{v \in V} \left| \sum_{e \in V} m_{ev}^k - \sum_{t \in V} m_{vt}^k \right| < \varepsilon C_v. \tag{12}$$

6. The sum of variation of vertices weights for 1 step is not greater than ε.

$$\sum_{v \in V} \left| m_v^k - m_v^{k+1} \right| < \varepsilon. \tag{13}$$

7. The sum of variation of vertices weights for 1 step divided by the cell volume is not greater than ε.

$$\sum_{v \in V} \left| m_v^k - m_v^{k+1} \right| < \varepsilon C_v. \tag{14}$$

8. The sum of variation of edges weights for 1 step is not greater than ε.

$$\sum_{(i \to j) \in E} \left| m_{ij}^k - m_{ij}^{k+1} \right| < \varepsilon. \tag{15}$$

9. The sum of variation of edges weights for 1 step divided by the cell volume is not greater than ε.

$$\sum_{(i \to j) \in E} \left| m_{ij}^k - m_{ij}^{k+1} \right| < \varepsilon C_v. \tag{16}$$

The rate of convergence was estimated experimentally for

- Henon map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix},$$

$a = 1.4$, $b = 0.3$;

- Ikeda map

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} d & - & C_2(x\cos\tau(x,y) - y\sin\tau(x,y)) \\ & & C_2(x\sin\tau(x,y) + y\cos\tau(x,y)) \end{pmatrix},$$

where $\tau(x,y) = C_1 - \frac{C_3}{1+x^2+y^2}, d = 2, C_1 = 0.4, C_2 = 0.9, C_3 = 6;$

- Duffing system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - 0.1y. \end{aligned}$$

The number of steps equals 100 000 and $\varepsilon = 10^{-5}$. The dependence of the number of steps from a halting criteria is shown in table 1. and run time — in table 2.

System	Step	Vert.	Edge	1	2	3	4	5	6	7	8	9
Henon	6	138	433	1 194	2 394	2 337	1 034	1 034	1 194	2 394	1 194	2 394
	8	866	2 802	9 494	∞	∞	7 627	7 627	9 494	∞	9 495	∞
	10	4 692	15 350	30 543	∞	∞	18 622	18 622	30 543	∞	30 543	∞
	12	25 000	81 458	29 649	∞	∞	11 063	11 063	29 649	∞	29 649	∞
	14	143 345	467 347	8 167	∞	∞	1	1	8 176	∞	8 167	∞
Ikeda	6	1 145	5 348	12 219	77 103	∞	9 664	9 664	12 219	77 103	12 219	77 104
	8	10 750	49 779	40 421	∞	∞	18 859	18 859	40 421	∞	40 421	∞
	10	104 799	482 427	3 522	∞	∞	1	1	3 526	∞	3 522	∞
Duffing	6	6 252	25 116	94 434	∞	∞	16 892	16 892	94 434	∞	89 096	∞
	8	55 528	227 147	2 153	∞	∞	1	1	2 153	∞	1 683	∞
	10	757 622	3 119 866	2	∞	∞	1	1	2	∞	1	∞

Table 1: Convergence of balance method

System	Step	Vert.	Edge	1	2	3	4	5	6	7	8	9
Henon	6	138	433	46ms	125ms	125ms	31ms	31ms	46ms	125ms	78ms	140ms
	8	866	2 802	828ms	∞	∞	656ms	656ms	828ms	∞	828ms	∞
	10	4 692	15 350	11s	∞	∞	6s	6s	11s	∞	11s	∞
	12	25 000	81 458	48s	∞	∞	18s	18s	48s	∞	48s	∞
	14	143 345	467 347	77s	∞	∞	3s	3s	77s	∞	77s	∞
Ikeda	6	1 145	5 348	2s	11s	∞	1s	1s	2s	11s	2s	11s
	8	10 750	49 779	41s	∞	∞	19s	19s	41s	∞	41s	∞
	10	104 799	482 427	36s	∞	∞	2s	2s	36s	∞	36s	∞
Duffing	6	6 252	25 116	48s	∞	∞	9s	9s	48s	∞	46s	∞
	8	55 528	227 147	11s	∞	∞	921ms	921ms	11s	∞	8s	∞
	10	757 622	3119 866	33s	∞	∞	33s	33s	33s	∞	33s	∞

Table 2: Run time for balance method

Subdivision	Nodes	Entropy	$\log\lambda$
$9 \times 2^{2\times 6}$	231	1.384079	1.386763

Table 3: Entropy estimation for Henon map

Subdivision	Nodes	Entropy	$\log\lambda$
$9 \times 2^{2\times 6}$	162	0.916999	0.940517
$9 \times 2^{2\times 12}$	1 032	0.729919	0.731549
$9 \times 2^{2\times 11}$	672	0.756997	0.779899

Table 4: Entropy estimation for logistic map

6 Entropy estimation

To obtain the value of the entropy according to the constructed measure we use formula (3).

Example 1 *For Henon map consider area* $D = [-10, 10] \times [-10, 10]$ *and use both linear and punctual methods [11] to construct a symbolic image. The initial partition consists from 9 cells. On each step every cell is subdivided in 4 cells. Construct an invariant measure on the symbolic image and estimate the entropy for this measure.*

In table 3 the number of nodes, the entropy and the estimation of the entropy by (4) are given.

Example 2 *Consider logistic map* $f(x) = ax(1-x), x \in [0, 1]$, *for* $a = 4$ *and* $a = 3.569$. *The results are given in table 4.*

7 Conlusion

In this paper a numerical method of the construction of a approximation to an invariant measure of a dynamical system is considered. Such an approximation may be obtained as a stationary process (an invariant measure) on the graph of a symbolic image of the initial system. A measure is assigned to all edges of the graph using a linear programming technique. Experimental data about the rate of convergence are given. The entropy of the stationary process on a graph with regard to the obtained invariant measure is computed. Numerical experiments show that this value less than the entropy of corresponding topological Markov chain.

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