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Optimal control

Time-Optimal Control Problem For Infinite Order Parabolic Equations With Control Constraints

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Abstract

A time optimal control problem for infinite order parabolic equations is considered. A time optimal control problem is replaced by an equivalent one with a performance index in the form of integral form. Constraints on controls are assumed. To obtain the optimality conditions for the Neumann problem, the generalization of the Dubovitskii-Milyutin Theorem given by Walczak in Refs.[41,42] was applied.

1 Introduction

In recent years, significant emphasis has been given to the study of optimal control for systems governed by parabolic and hyperbolic partial differential equations with first boundary conditions or with Cauchy conditions. In these studies, the differential equations are either in general form or in divergence form. Questions concerning necessary conditions for optimality and existence of optimal controls for these problems have been investigated for example in [8-19,22-43].

In (Refs. [15,18,19,29,32]), the optimal control problems for systems described by parabolic and hyperbolic operators with infinite order and consist of one equation have been discussed. Also we extended the discussion in [9-12] to $n \times n$ coupled systems of elliptic, parabolic and hyperbolic types involving different types of operators. To obtain optimality conditions the arguments of (Ref.[38]) have been applied.

Making use of the Dubovitskii-Milyutin theorem from [20]), following (Refs. [22-35]) Kotarski have obtained necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables and with Dirichlet and Neumann boundary conditions. The interest in the study of this class of operators is stimulated by problems in quantum field theory.

In [1] a distributed Pareto optimal control problem for the parabolic operator with an infinite number of variables and with Neumann boundary conditions is considered. In [2] a time optimal control problem for parabolic equations with an infinite number of variables is considered. In [29] a distributed control problem for a hyperbolic system with mixed control state constraints involving operator of infinite order is considered. In [31] a distributed control problem for Neumann parabolic problem with time delay is considered. Also in [32], a distributed control problem for a hyperbolic system involving operator of infinite order with Dirichlet conditions is considered.

In this paper, the application of the generalized Dubovitskii- Milyutin Theorem from [41,42] will be demonstrated on an optimization Neumann problem for system described by parabolic operator with infinite order. A time optimal control problem for infinite order parabolic equations is considered. A time optimal control problem is replaced by an equivalent one with a performance index in the form of integral form. Constraints on controls are assumed.

This paper is organized as follows. In section 2, we introduce some functional spaces with an infinite order. In section 3, we formulate the optimal control problem and we introduce the main results of this paper. In section 4, we introduce a real example for this problem.

2 Some Functional Spaces (Refs.[3-5]).

The object of this section is to give the definition of some function spaces of infinite order, and the chains of the constructed spaces which will be used later. We define the Sobolev space $W^{\infty}\{a_\alpha, 2\}(\mathbb{R}^n)$ (which we shall denoted by $W^{\infty}\{a_\alpha, 2\}$) of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of $\mathbb{R}^n, n \geq 1$, as follows,

$$W^{\infty}\{a_\alpha, 2\} = \left\{ \phi(x) \in C^{\infty}(\mathbb{R}^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha \phi||_2^2 < \infty \right\}$$

where $a_\alpha \geq 0$ is a numerical sequence and $||.||_2$ is the canonical norm in the space $L^2(\mathbb{R}^n)$(all functions are assumed to be real valued), and

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}....(\partial x_n)^{\alpha_n}},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $W^{-\infty}\{a_\alpha, 2\}$ is defined as the formal conjugate space to the space $W^{\infty}\{a_\alpha, 2\}$, namely:

$$W^{-\infty}\{a_\alpha, 2\} = \{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \psi_\alpha(x)\},$$

where $\psi_\alpha \in L^2(\mathbb{R}^n)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha ||\psi_\alpha||_2^2 < \infty$..

The duality pairing of the spaces $W^{\infty}\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is postulated by the formula

$$(\psi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{\mathbb{R}^n} \psi_\alpha(x) D^\alpha \phi dx,$$

where

$$\phi \in W^{\infty}\{a_\alpha, 2\}, \quad \psi \in W^{-\infty}\{a_\alpha, 2\}.$$

From above, $W^{\infty}\{a_\alpha, 2\}$ is everywhere dense in $L^2(\mathbb{R}^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ denotes the topological dual space with respect to $L^2(\mathbb{R}^n)$, so we have the following chain:

$$W^{\infty}\{a_\alpha, 2\} \subseteq L^2(\mathbb{R}^n) \subseteq W^{-\infty}\{a_\alpha, 2\}.$$

We now introduce $L^2(0,T;L^2(\mathbb{R}^n))$ which we shall denoted by $L^2(Q)$, where $Q=\mathbb{R}^n\times]0,T[$, denotes the space of measurable functions $t\to\phi(t)$ such that

$$||\phi||_{L^2(Q)}=(\int_0^T||\phi(t)||_2^2dt)^{\frac{1}{2}}<\infty,$$

endowed with the scalar product $(f,g)=\displaystyle\int_0^T(f(t),g(t))_{L^2(\mathbb{R}^n)}dt$, $L^2(Q)$ is a Hilbert space. In the same manner we define the spaces $L^2(0,T;W^\infty\{a_\alpha,2\})$, and $L^2(0,T;W^{-\infty}\{a_\alpha,2\})$, as its formal conjugate resp.

Finally we have the following chains:

$$L^2(0,T;W^\infty\{a_\alpha,2\})\subseteq L^2(Q)\subseteq L^2(0,T;W^{-\infty}\{a_\alpha,2\}),$$

Finally, let us introduce the space

$$W(0,T):=\left\{y;\quad y\in L^2(0,T;W^\infty\{a_\alpha,2\}),\frac{\partial y}{\partial t}\in L^2(0,T;W^{-\infty}\{a_\alpha,2\})\right\},$$

in which a solution of a parabolic equation with an infinite order will be contained.The spaces considered in this paper are assumed to be real.

3 Time-Optimal Control Problem For Parabolic Equations

Let us take into account the following optimization problem:

$$\frac{\partial y}{\partial t}+A(t)y=u,\qquad x\in R^n,\quad t\in(0,T), \tag{1}$$

$$y(x,0)=y_p(x),\quad x\in R^n, \tag{2}$$

$$\frac{\partial^\omega y(x,t)}{\partial\nu_A^\omega}=0,\qquad x\in\Gamma,\quad t\in(0,T), \tag{3}$$

$$y(x,T)\in K,\qquad\qquad x\in R^n, \tag{4}$$

$$u\in U_{ad}, \tag{5}$$

$$T\to\min, \tag{6}$$

where y_p is a given element in $L^2(\mathbb{R}^n)$, $A(t)$ is a bounded infinite order self-adjoint elliptic partial differential operator mapping $W^{\infty}(\mathbb{R}^n)$ onto $W^{-\infty}(\mathbb{R}^n)$, which takes the form [18-19]

$$(A\Phi)(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha}\Phi(x,t),$$

and $\frac{\partial^\omega}{\partial\nu_A^\omega}$ is the co-normal derivatives with respect to A, for $|\omega| = 0, 1, 2, .., |\omega| \leq \alpha - 1$.

For each $t \in]0, T[$, we define the following bilinear form on $W^{\infty}(\mathbb{R}^n)$:

$$\pi(t;\phi,\psi) = (A(t)\phi,\psi)_{L^2(\mathbb{R}^n)}, \quad \phi,\psi \in W^{\infty}(\mathbb{R}^n).$$

Then

$$\begin{aligned}\pi(t;\phi,\psi) &= \big(A(t)\phi,\psi\big)_{L^2(\mathbb{R}^n)} \\ &= \big(A(t)\phi(x),\psi(x)\big)_{L^2(\mathbb{R}^n)} \\ &= \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}\phi(x,t),\psi(x)\Big)_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n}\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^{\alpha}\phi(x)D^{\alpha}\psi(x)dx\end{aligned} \tag{7}$$

The bilinear form (3.7) is coercive on $W^{\infty}\{a_\alpha, 2\}$ that is, there exists $\eta \in \mathbb{R}$, such that:

$$\pi(t;\phi,\phi) = \eta||\phi||^2_{W^{\infty}(\mathbb{R}^n)}, \quad \eta > 0. \tag{8}$$

It is well known that the ellipticity of $A(t)$ is sufficient for the coercitivness of $\pi(t;\phi,\psi)$ on $W^{\infty}(\mathbb{R}^n)$ [38]. Then

$$\begin{aligned}\pi(t;\phi,\phi) &= \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}\phi(x,t),\phi(x,t)\Big) \\ &\geq \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha||D^{\alpha}\phi(x)||^2_{L^2(\mathbb{R}^n)}\Big) \\ &= \eta||\phi(x)||^2_{W^{\infty}(\mathbb{R}^n)}\ .\end{aligned}$$

$$\begin{cases} \forall \phi, \psi \in W^\infty(\mathbb{R}^n) \text{ the functiont} \to \pi(t;\phi,\psi) \text{ is continuously differentiable in} \\ ]0,T[; and \end{cases}$$

$$\pi(t;\phi,\psi) = \pi(t;\psi,\phi) \tag{9}$$

Note. The operator $\frac{\partial}{\partial t} + A(t)$ is an infinite order parabolic operator which maps $L^2(0,T;W^\infty\{a_\alpha,2\})$ onto $L^2(0,T;W^{-\infty}\{a_\alpha,2\})$.

Remark 3.1 *The equations (3.1)-(3.3) have the unique generalized solution such as*

$$y \in W(0,T) := \left\{ y |\ y \in L^2(0,T;W^\infty(R^n)), \quad \frac{\partial y}{\partial t} \in L^2(0,T;W^{-\infty}(\mathbb{R}^n)) \right\}$$

continuously dependent on the right-hand side of (3.1) and the initial condition (3.2). Moreover, $y \in W(0,T)$ is a continuous function $[0,T] \to L^2(R^n)$ (compare with Theorems 1.1 and 1.2 Chapt. 3 [38])

Let us denote by $U := L^2(0,T;L^2(R^n)) = L^2(Q)$, the space of controls and by $Y := L^2(0,T;W^\infty(R^n))$ the space of states.

We assume that U_{ad} is a closed, convex subset of U and K is a closed, convex subset of $L^2(R^n)$ with non-empty interior.

3.1 An equivalent optimization problem

The optimization problem (3.1)-(3.6) can be replaced by another equivalent one with a fixed time T and a performance index in a form of integral (compare with [40]). To show that we need two auxiliary theorems.

Theorem 3.2 *Let $T^0 > 0$ be the optimal time for the problem (3.1)-(3.6). If $int\,K \neq \emptyset$, then*

$$y(x,T^0) \in \partial K \qquad (boundary\ of K) \tag{10}$$

for any y satisfying (3.1)-(3.4).

Proof Any solution of (3.1)-(3.3) is continuous with respect to t. If (3.10) is not true, then there exists an admissible state y such as $y(x,T^0) \in \text{int}K$. Thus a $\hat{T} < T^0$ exists so that $y(x,\hat{T}) \in K$. This contradicts the optimality of T^0 and hence (3.10) must be fulfilled.

Theorem 3.3 *Let $T^0 > 0$ be the optimal time for the problem (3.1)-(3.6), let u^0 and y^0 be an optimal control and the corresponding state, respectively. Then with the assumptions given above there exists a non-trivial element $g \in L^2(R^n)$ so that the pair (y^0, u^0) is optimal for the following control problem with the fixed time T^0:*

$$I(y,u) := \int_{R^n} g(x)y(x,T^0)dx \to min \tag{11}$$

subject to the constraints (3.1)-(3.3).

Proof The linearity of the equations (3.1)-(3.3) implies that the endpoints $y(x,T^0)$ of all admissible states y form a convex set Y_{T^0}. From Theorem (3.1) we have $Y_{T^0} \cap \text{int}\, K = \emptyset$ and $y^0(0,T^0) \in \partial K$. Since $\text{int}K \neq \emptyset$ thus there exists a closed hyperplane separating Y_{T^0} and K containing $y^0(x,T^0)$, i.e. there is a nonzero $g \in [L^2(R^n)]^* = L^2(R^n)$

$$\begin{aligned}\sup_{y \in Y_{T^0}} \int_{R^n} g(x)y(x,T^0)dx &\leq \int_{R^n} g(x)y^0(x,T^0)dx \\ &\leq \inf_{y\in K} \int_{R^n} g(x)y(x,T^0)dx.\end{aligned}$$

This completes the proof.

Remark 3.4 *If the set K has a special form i.e.*

$$K = \{y\}\,(x,T); \quad ||y - z||_{L^2(R^n)} \leq \epsilon,$$

where $\epsilon > 0$ and $z \in L^2(R^n)$ are given, then g is known explicitly and is expressed by $g(x) = 2(y^0(x,T) - z(x))$.

Remark 3.5 *The method fails if $intK = \emptyset$, e.g. in the case when K consists of a single point.*

3.2 Optimality conditions

Now based on Theorem (3.2) we can be ready to formulate the necessary condition of optimality for problem (3.1)-(3.6).

Theorem 3.6 *Assuming that $T^0 > 0$ is the optimal time for the problem (3.1)-(3.6), u^0 and y^0 are the optimal control and the corresponding state, respectively. Then with the assumptions given above at the beginning of section 3, there exist* *an element $g \in L^2(R^n)$ and the adjoint state $p \in W(0,T)$ so that the following system of partial differential equations and inequalities must be satisfied:*

State equations:

$$\frac{\partial y^0}{\partial t} + A(t)y^0 = u^0, \qquad x \in R^n, \quad t \in (0, T^0) \tag{12}$$

$$\frac{\partial^\omega y^0(x,t)}{\partial \nu_A^\omega} = 0, \qquad x \in \Gamma, \quad t \in (0, T^0), \tag{13}$$

$$y^0(x, 0) = y_p(x), \qquad x \in R^n, \tag{14}$$

$$y^0(x, T^0) \in K, \qquad x \in R^n. \tag{15}$$

Adjoint equations:

$$-\frac{\partial p}{\partial t} + A(t)p = 0, \qquad x \in R^n, \quad t \in (0, T^0), \tag{16}$$

$$\frac{\partial^\omega p(x,t)}{\partial \nu_A^\omega} = 0, \qquad x \in \Gamma, \quad t \in (0, T^0), \tag{17}$$

$$p(x, T^0) = g(x), \qquad x \in R^n. \tag{18}$$

Maximum conditions:

$$\int_Q p(u - u^0)dxdt \geq 0 \quad \forall u \in U_{ad}, \tag{19}$$

$$\int_{R^n} g(x)(y - y^0)dx \geq 0 \quad \forall y \in K. \tag{20}$$

Proof According to Theorem (3.2) our problem is equivalent to the one with the fixed time T^0 and the performance index in the integral form (3.11). For such a new problem we formulate the necessary conditions of optimality by applying the generalized Dubovitskii-Milyutin Theorem (Theorem 1.8.1 in [26]). Let us denote by Q_1, Q_2, Q_3 the sets in the space $E := Y \times U$ as follows

$$Q_1 := \left\{ (y,u) \in E; \quad \begin{array}{r} \dfrac{\partial y}{\partial t} + A(t)y = u \\ y(x,0) = y_p(x) \\ \dfrac{\partial^\omega y(x,t)}{\partial \nu_A^\omega} = 0, \end{array} \right\}$$

$$Q_2 := \left\{ (y,u) \in E; \quad y \in Y, u \in U_{ad} \right\},$$

$$Q_3 := \left\{ (y,u) \in E; \quad y(x,T^0) \in K, u \in U_{ad} \right\}.$$

Thus the optimization problem may be formulated in such a form

$$I(y,u) \to min \qquad \text{subject to} \qquad (y,u) \in Q_1 \cap Q_2 \cap Q_3.$$

We approximate the sets Q_1, and Q_2 by the regular tangent cones (RTC), Q_3 by the regular admissible cone (RAC) and the performance functional by the regular cone of decrease (RFC).

The tangent cone to the set Q_1 at (y^0,u^0) has the form

$$RTC(Q_1,(y^0,u^0)) = \left\{ (\tilde{y},\tilde{u}) \in E;\ P'(y^0,u^0)(\tilde{y},\tilde{u}) = 0 \right\}$$

$$= \left\{ (\tilde{y},\tilde{u}) \in E; \qquad \begin{array}{r} \dfrac{\partial \tilde{y}}{\partial t} + A(t)\tilde{y} = \tilde{u} \\ \tilde{y}(x,0) = 0 \\ \dfrac{\partial^\omega \tilde{y}(x,t)}{\partial \nu_A^\omega} = 0, \end{array} \right\}$$

where $P'(y^0,u^0)(\tilde{y},\tilde{u})$ is the Frèchet differential of the operator

$$P(y,u) := (\frac{\partial y}{\partial t} + A(t)y - u, y(x,0) - y_p(x))$$

mapping from the space

$$W := L^2(0,T;W^\infty(R^n)) \times L^2(Q)$$

into the space

$$Z := L^2(0,T;W^{-\infty}(R^n)) \times L^2(R^n).$$

Applying theorem on the existence of the solution to the equation (3.1)-(3.3) Remark (3.1) it is easy to prove that $P'(y^0,u^0)$ is the mapping from the space W onto Z as required in the Lusternik Theorem (Theorem 9.1 in [20]).

The tangent cone $RTC(Q_2,(y^0,u^0))$ to the set Q_2 at (y^0,u^0) has the form $Y \times RTC(U_{ad},u^0)$, where $RTC(U_{ad},u^0)$ is the tangent cone to the set U_{ad} at the point u^0. It is known that the tangent cones are closed [36].

Then we can show that:-

$$RTC(Q_1 \cap Q_2, (y^0, u^0)) = RTC(Q_1, (y^0, u^0)) \cap RTC(Q_2, (y^0, u^0)).$$

Further taking into account Theorem 3.3 in [42] it is easy to see that the adjoint cones $[RTC(Q_1, (y^0, u^0))]^*$ and $[RTC(Q_2, (y^0, u^0))]^*$ are of the same sense.

The admissible cone $RAC(Q_3, (y^0, u^0))$ to the set Q_3 at (y^0, u^0) has the form $RAC(K, y^0(x, T^0)) \times U$, where $RAC(K, y^0(x, T^{0s}))$ is the admissible cone to the set K at the point $y^0(x, T^0)$.

According to Theorem 7.5 [20] the regular cone of decrease for the performance functional is given by

$$RFC(I, (y^0, u^0)) = \left\{ (\tilde{y}, \tilde{u}) \in E; \int_{R^n} g(x)\tilde{y}(x, T^0)dx < 0 \right\}.$$

If $RFC(I, (y^0, u^0)) \neq \emptyset$, then the adjoint cone consists of the elements of the form (Theorem 10.2 [20])

$$f_4(\tilde{y}, \tilde{u}) = -\lambda_0 \int_{R^n} g(x)\tilde{y}(x, T^0)dx, \quad \text{where} \quad \lambda_0 \geq 0.$$

The functionals belonging to $[RTC(Q_1, (y^0, u^0))]^*$ have the form (Theorem 10.1 [20])

$$f_1(\tilde{y}, \tilde{u}) = 0 \qquad \forall (\tilde{y}, \tilde{u}) \in RTC(Q_1, (y^0, u^0)).$$

The functionals

$$f_2(\tilde{y}, \tilde{u}) \in [RTC(Q_2, (y^0, u^0))]^* \quad \text{and } f_3(\tilde{y}, \tilde{u}) \in [RAC(Q_3, (y^0, u^0))]^*$$

can be expressed as follows

$$f_2(\tilde{y}, \tilde{u}) = f_2^1(\tilde{y}) + f_2^2(\tilde{u}),$$
$$f_3(\tilde{y}, \tilde{u}) = f_3^1(\tilde{y}) + f_3^2(\tilde{u})$$

where $f_2^1(\tilde{y}) = 0 \, \forall \tilde{y}$ and $f_3^2(\tilde{y}) = 0 \, \forall \tilde{u} \in U$ (Theorem 10.1 in [20]), $f_2^2(\tilde{u})$ is the support functional to the set U_{ad} at the point u^0 and $f_3^1(\tilde{y})$ is the support functional to the set K at the point $y^0(x, T^0)$ (Theorem 10.5 in [20]).

Since all assumptions of the Dubovitskii-Milyutin Theorem are satisfied and we know suitable adjoint cones then we are ready to write down the Euler-Lagrange Equation in the following form

$$f_2^2(\tilde{u}) + f_3^1(\tilde{y}) = \lambda_0 \int_{R^n} g(x)\tilde{y}(x, T^0)dx, \forall (\tilde{y}, \tilde{u}) \in RTC(Q_1, (y^0, u^0)). \qquad (21)$$

Introducting the adjoint variable p by the equation (3.16)-(3.18) and taking into account that $\tilde{y}$ is the solution of $P'(y^0, u^0)(\tilde{y}, \tilde{u}) = 0$ for any fixed $\tilde{u}$, we obtain

$$0 = \int_Q \left(\frac{\partial p}{\partial t} + A(t)p \right) \tilde{y} dx dt = \int_{R^n} \left(-p(x, T^0)\tilde{y}(x, T^0) \right) dx$$
$$+ \int_{R^n} p(x,0)\tilde{y}(x,0)dx + \int_Q p \left(\frac{\partial \tilde{y}}{\partial t} + A(t)\tilde{y} \right) dx dt$$
$$= - \int_{R^n} p(x, T^0)\tilde{y}(x, T^0)dx + \int_Q p\tilde{u} dx dt.$$

Hence

$$\int_{R^n} g(x)\tilde{y}(x, T^0)dx = \int_Q p\tilde{u} dx dt. \tag{22}$$

From (3.21) and (3.22) we get

$$f_2^2(\tilde{u}) + f_3^1(\tilde{y}) = \frac{1}{2}\lambda_0 \int_Q p\tilde{u} dx dt + \frac{1}{2}\lambda_0 \int_{R^n} g(x)\tilde{y}(x, T^0)dx \tag{23}$$

A number λ_0 cannot be equal to 0 because in such a case all functionals in the Euler-Lagrange Equation would be zero which is impossible according to the Dubovitskii-Milyutin Theorem. Using the definition of the support functional and dividing both members of the obtained inequalities by λ_0 from (3.23) we obtain the maximum conditions (3.19)-(3.20).

If $RFC(I, (y^0, u^0)) = \emptyset$ then the optimality conditions (3.12)-(3.20) are fulfilled with equality in the maximum conditions (3.19)-(3.20). This last remark completes the proof.

4 Real example

We shall us the following notation:

$$\begin{aligned} Q =& Q_T = \Omega \times]0, T[, \quad \Omega \text{ an open subset of } R^n; \\ \Sigma =& \Sigma_T = \Gamma \times]0, T[, \\ \Gamma =& \text{boundary of } \Omega, \quad \Sigma = \text{lateral boundary of } Q, \\ & V \subset H \subset V'. \end{aligned}$$

Let us consider the system whose state is given by

$$\frac{d}{dt} y(t; v) + A(t)y(t; v) = f + Bv \tag{24}$$

$$y(0,v)=y_0 \tag{25}$$

where $A(t)y=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y}{\partial x_j})$, a_{ij} be given functions in $\Omega\times]0,T[=Q$ with

$$\begin{cases} a_{ij}\in L^{\infty}(Q), \\ \sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\geq\alpha(\xi_1^2+.....+\xi_n^2) \quad \alpha>0, \quad \xi_i\in R, \\ \text{almost everywhere in } \Omega, \end{cases}$$

$$B\in L(U,L^2(0,T,V')) \tag{26}$$

Let U_{ad} be a given closed, convex subset of U and let y_1 be a given element in H.

We assume that

$$\begin{cases} \text{there exists a } v\in U_{ad} \text{ such that} \\ y(\tau;v)=y_1 \quad \text{for an appropriate } \tau\in[0,T] \text{ and } \tau\leq T \end{cases} \tag{27}$$

The optimal time is defined by

$$\tau_0=\inf\tau, \quad \tau \text{ such that (4.4) holds} \tag{28}$$

The Problem which we shall study are :

existence of an optimal control, that is, existence of $u\in U_{ad}$ such that

$$y(\tau_0;u)=y_1; \tag{29}$$

Lemma 4.1 *We assume that*

$$\begin{cases} \forall\phi,\psi\in W^1(\mathbb{R}^n) \text{ the function } t\to\pi(t;\phi,\psi) \text{ is continuously differentiable in} \\ ]0,T[; \text{and} \end{cases}$$

$$\pi(t;\phi,\psi)=\pi(t;\psi,\phi) \tag{30}$$

and there exists a λ *such that*

$$\pi(t;\phi,\phi)+\lambda|\phi|^2\geq\alpha||\phi||^2, \quad \alpha>0, \quad \forall\phi\in V, t\in]0,T[, \tag{31}$$

(4.4) and (4.3) hold and that U_{ad} *is bounded. Then there exists an optimal control, that is* $u\in U_{ad}$ *such that (4.6) is satisfied.*

Proof See Theorem 17.1 in [38] Chapter III, section 17.2.

Example Take $U = L^2(\Sigma)$. Let the state $y(v)$ be given by

$$\frac{\partial}{\partial t}y(v) + A(t)y(v) = f,$$
$$\frac{\partial y}{\partial \nu_A}(v) = v,$$
$$y(x, 0, v) = y_0(x).$$

Theorem 17.1 [38] may be applied to this example. Hence the theorem covers the case of boundary control.

Remark 4.2 *Theorem 17.1 [38] may be modified without any difficulty to cover the case of systems with Dirichlet boundary conditions (cf [38] section 9) and where the control is exercised through the boundary.*

Comments

The main result of the paper contains necessary and sufficient conditions of optimality (of Pontryagin's type) for infinite order parabolic system that give characterization of optimal control. But it is easily seen that obtaining analytical formulas for optimal control is very difficult. This results from the fact that state equations (3.12)-(3.15), adjoint equations (3.16)-(3.18) and maximum conditions (3.19)-(3.20) are mutually connected that cause that the usage of derived conditions is difficult. Therefore we must resign from the exact determining of the optimal control and therefore we are forced to use approximations methods. Those problems need further investigations and form tasks for future research.

Also it is evident that by modifying:

- the boundary conditions,
- the nature of the control (distributed, boundary),
- the nature of the observation,
- the initial differential system,

an infinity of variations on the above problem are possible to study with the help of Dubovitskii-Milyutin formalism.

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