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Boundary Control For Cooperative Parabolic Systems Governed By Schrödinger Operator

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Abstract

In this paper, we study the existence of solutions for a cooperative parabolic systems governed by Schrödinger operator defined on $\mathbb{R}^n$*, then we discuss the optimal control of boundary type for this systems.*

Keywords: Cooperative parabolic systems in $\mathbb{R}^n$, Schrödinger operator, Existence of solution, Boundary control, Optimality conditions.

1 INTRODUCTION.

The linear quadratic optimal control problem described by a distributed parameter system has a variety of mechanical and technical sources and applications. The fundamental class of optimal controls and its mathematical approaches can be found in Lions (1971). The necessary and sufficient condition of optimality for systems ($n \times n$ systems) governed by different types of partial differential operators defined on spaces of functions of infinitely many variables and spaces of infinite order are discussed in El-Saify & Bahaa (2001,2002a,b,2003,), Kotarski (1989,1997),Kotarski & Bahaa(2005), and Kotarski & El-saify & Bahaa(2002a,b). Interest in the study of this class of operator is stimulated by problems in quantum field theory. Various optimization problems associated with the optimal control of distributed parameter cooperative systems have been studied by Gali & Serag (1994,1995), Fleckinger (1981,1994) and Fleckinger & Serag (1995).

We consider the following cooperative parabolic systems :

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + (-\Delta + q) y_1 = a y_1 + b y_2 + f_1 \quad \text{in} \quad \mathbb{R}^n \\ \dfrac{\partial y_2}{\partial t} + (-\Delta + q) y_2 = c y_1 + d y_2 + f_2 \quad \text{in} \quad \mathbb{R}^n \\ y_1 = g_1 \quad \text{as} \quad |x| \to \infty \\ y_2 = g_2 \quad \text{as} \quad |x| \to \infty, \\ y_1(x,0) = y_{1,0}(x) \quad \text{in} \quad \mathbb{R}^n \\ y_2(x,0) = y_{2,0}(x) \quad \text{in} \quad \mathbb{R}^n \end{cases} \tag{1}$$

where :

$$\begin{cases} a, b, c \& d \text{ are given numbers such that } b, c > 0 \\ \text{in this case, we say that the system(1)is cooperative} \end{cases} \tag{2}$$

$$q(x) \text{ is a positive function and tending to} \infty \text{at infinity.} \tag{3}$$

In [14], Gali et al. proved the existence of optimal control for system like (1) with $q(x) = 0$ and with positive weight function. Also they found the set of inequalities which described the distributed control for systems (1) with $q(x) = 0$ and defined on bounded domain [13]. The case of semilinear cooperative system with $q(x) = 0$ is discussed in [12].

In [11] Fleckinger, obtained the necessary and sufficient conditions for having the maximum principle and the existence of positive solutions for cooperative system (1) which are:

$$\begin{cases} a < \lambda(q), \quad d < \lambda(q) \\ (\lambda(q) - a)(\lambda(q) - d) > bc, \end{cases} \tag{4}$$

where $\lambda(q)$ is defined later.

Here, we shall use the same conditions (4) to prove the existence of the state of our system (1); then using the theory of Lions [19], we study the existence of boundary control for system (1). Our model in this problem is the Schrödinger operator.

2 Operator equation.

To prove the existence of the state $y = \{y_1, y_2\}$ of system (1), we state briefly some results introduced in [10] concerning the eigenvalue problem:

$$\begin{cases} (-\Delta + q)\phi = \lambda(q)\phi \quad \text{in} \quad \mathbb{R}^n \\ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad \phi > 0. \end{cases} \tag{5}$$

The associated variational space is $V_q(\mathbb{R}^n)$, the completion of $D(\mathbb{R}^n)$, with respect to the norm :

$$||y||_q = \left(\int_{\mathbb{R}^n} |\Delta y|^2 + q|y|^2 dx \right)^{\frac{1}{2}}.$$

Since the imbedding of $V_q(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is compact. Then the operator $(-\Delta + q)$ considered as an operator in $L^2(\mathbb{R}^n)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalue tending to infinity; moreover the smallest one which is called the principle eigenvalue denoted by $\lambda(q)$ is simple and is associated with an eigenfunction which does not change sign in $\mathbb{R}^n$. It is characterized by:

$$\lambda(q) \int_{\mathbb{R}^n} |y|^2 dx \le \int_{\mathbb{R}^n} |\Delta y|^2 + q|y|^2 dx \quad \forall y \in V_q(\mathbb{R}^n). \tag{6}$$

Now, to study our system (1) we have the embedding

$$V_q(\mathbb{R}^n) \times V_q(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$$

is continuous and compact then, we define a bilinear form

$$a : (V_q(\mathbb{R}^n))^2 \times (V_q(\mathbb{R}^n))^2 \to \mathbb{R}$$

by

$$\pi((y_1,y_2),(\phi_1,\phi_2)) = \frac{1}{b}\int_{\mathbb{R}^n}[\Delta y_1\Delta\phi_1 + qy_1\phi_1]dx + \frac{1}{c}\int_{\mathbb{R}^n}[\Delta y_2\Delta\phi_2 + qy_2\phi_2]dx$$
$$- \int_{\mathbb{R}^n} y_1\phi_2 dx - \frac{d}{c}\int_{\mathbb{R}^n} y_2\phi_2 dx - \frac{a}{b}\int_{\mathbb{R}^n} y_1\phi_1 dx - \int_{\mathbb{R}^n} y_2\phi_1 dx. \tag{7}$$

It is easy to check that π is a continuous bilinear form; and then by lax Milgram Lemma, we have the following theorem:

Theorem 2.1 *For* $f_1, f_2 \in L^2(\partial\mathbb{R}^n)$*, there exists a unique solution* $y = \{y_1, y_2\} \in (V_q(\mathbb{R}^n))^2$ *of system* (1) *if conditions (4) are satisfied.*

Proof We choose m large enough such that $a+m>0$ and $d+m>0$ and define on $V_q(\mathbb{R}^n)$ the equivalent norm

$$||y||^2_{q,m} = \int_{\mathbb{R}^n}[|\Delta y|^2 + (m+q)|y|^2]dx$$

and we write (6) as:

$$\pi((y_1,y_2),(\phi_1,\phi_2)) = \frac{1}{b}\int_{\mathbb{R}^n}[\Delta y_1\Delta\phi_1 + (q+m)y_1\phi_1]dx - \frac{a+m}{b}\int_{\mathbb{R}^n} y_1\phi_1 dx$$
$$- \int_{\mathbb{R}^n} y_2\phi_1 dx + \frac{1}{c}\int_{\mathbb{R}^n}[\Delta y_2\Delta\phi_2 + (q+m)y_2\phi_2]dx$$
$$- \frac{d+m}{c}\int_{\mathbb{R}^n} y_2\phi_2 dx - \int_{\mathbb{R}^n} y_1\phi_2 dx.$$

Then

$$\pi((y_1,y_2),(y_1,y_2)) = \frac{1}{b}\int_{\mathbb{R}^n}[|\Delta y_1|^2 + (q+m)|y_1|^2]dx - \frac{a+m}{b}\int_{\mathbb{R}^n} |y_1|^2 dx$$
$$- \int_{\mathbb{R}^n} y_1y_2 dx + \frac{1}{c}\int_{\mathbb{R}^n}[|\Delta y_2|^2 + (q+m)|y_2|^2]dx$$
$$- \frac{d+m}{c}\int_{\mathbb{R}^n} |y_2|^2 dx - \int_{\mathbb{R}^n} y_1y_2 dx.$$

By Cauchy Schwarz inequality, we have

$$
\begin{aligned}
\pi((y_1, y_2), (y_1, y_2)) \geq & \frac{1}{b} \int_{\mathbb{R}^n} [|\Delta y_1|^2 + (q+m)|y_1|^2] dx - \frac{a+m}{b} \int_{\mathbb{R}^n} |y_1|^2 dx \\
& + \frac{1}{c} \int_{\mathbb{R}^n} [|\Delta y_2|^2 + (q+m)|y_2|^2] dx - \frac{d+m}{c} \int_{\mathbb{R}^n} |y_2|^2 dx \\
& - 2 \left(\int_{\mathbb{R}^n} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |y_2|^2 dx \right)^{\frac{1}{2}},
\end{aligned}
$$

from (5), we deduce

$$
\pi((y_1, y_2), (y_1, y_2)) \geq \frac{1}{b}\left(1 - \frac{a+m}{\lambda(q)+m}\right)||y_1||^2_{q,m} + \frac{1}{c}\left(1 - \frac{d+m}{\lambda(q)+m}\right)||y_2||^2_{q,m}
$$
$$
\frac{2}{\lambda+m}||y_1||_{q,m}||y_2||_{q,m}.
$$

If (4) holds, then

$$
\pi((y_1, y_2), (y_1, y_2)) \geq C(||y_1||^2_{q,m} + ||y_2||^2_{q,m})
$$

which prove the coerciveness of the bilinear form π. Then for $f_1, f_2 \in L^2(\mathbb{R}^n)$, system (1) has a unique solution by Lax Milgram lemma.

3 Formulation of the control problem

The space $L^2(\Gamma) \times L^2(\Gamma)$ is the space of controls. For a control $u = \{u_1, u_2\} \in (L^2(\Gamma))^2$, the state $y(u) = \{y_1(u), y_2(u)\}$ of the system is given by the solution of

$$
\begin{cases}
\dfrac{\partial y_1(u)}{\partial t} + (-\Delta + q) y_1(u) = a y_1(u) + b y_2(u) + f_1 \quad \text{in} \quad \mathbb{R}^n \\
\dfrac{\partial y_2(u)}{\partial t} + (-\Delta + q) y_2(u) = c y_1(u) + d y_2(u) + f_2 \quad \text{in} \quad \mathbb{R}^n \\
y_1 = u_1 \quad \text{as} \quad |x| \to \infty \\
y_2 = u_2 \quad \text{as} \quad |x| \to \infty,
\end{cases}
\tag{8}
$$

The observation equation is given by $z(u) = \{z_1(u), z_2(u)\} = y(u) = \{y_1(u), y_2(u)\}$. For given $z_d = \{z_{d1}, z_{d2}\}$ in $(L^2(\mathbb{R}^n))^2$; the cost function is given by:

$$
J(v) = \int_{\mathbb{R}^n} (y_1(v) - z_{d1})^2 + (y_2(v) - z_{d2})^2 dx + (Nv, v)_{(L^2(\Gamma))^2} \tag{9}
$$

where $N \in L((L^2(\Gamma))^2, (L^2(\Gamma))^2)$ is hermitian positive definite operator:

$$(Nu, u) \geq \eta ||u||^2_{(L^2(\mathbb{R}^n))^2}. \tag{10}$$

The control problem then is to find

$$\begin{cases} u = \{u_1, u_2\} \in U_{ad} \quad \text{such that} \\ J(u) \leq J(v) \end{cases} \tag{11}$$

where U_{ad} is a closed convex subset of$(L^2(\Gamma))^2$.

Under the given consideration, we may apply the theorem of Lions [19] to obtain the following result:

Theorem 3.1 *Assume that (7) and (10) hold. If the cost function is given by (9), then there exists an optimal control* $u = \{u_1, u_2\}$*; Moreover it is characterized by the following equations and inequalities:*

$$\begin{cases} -\frac{\partial p_1(u)}{\partial t} + (-\Delta + q)p_1(u) - ap_2(u) - cp_2(u) = y_1(u) - z_{1d} \quad in \quad \mathbb{R}^n \\ -\frac{\partial p_2(u)}{\partial t} + (-\Delta + q)p_2(u) - bp_1(u) - dp_2(u) = y_2(u) - z_{2d} \quad in \quad \mathbb{R}^n \\ \qquad p_1(u) = 0 \quad p_2(u) = 0 \quad on \quad \Gamma \end{cases}$$

$$\int_\Gamma \frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2) d\Gamma + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0 \quad \forall v \in U_{ad}$$

together with (8), where $p(u) = \{p_1(u), p_2(u)\}$ *is the adjoint state.*

Proof

The control u is characterized by

$$J'(u)(v - u) \geq 0 \quad \forall u \in U_{ad}$$

which is equivalent to

$$(y(u) - z_d, y(v) - y(u))_{(L^2(\mathbb{R}^n))^2} + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0$$

i.e.,

$$(y_1(u) - z_{1d}, y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} + (y_2(u) - z_{2d}, y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0 \tag{12}$$

Since $(A^*P, Y) = (P, AY)$, where

$$A(\phi = \{\phi_1, \phi_2\}) \rightarrow A\phi = \{(-\Delta + q)\phi_1 - a\phi_1 - b\phi_2, (-\Delta + q)\phi_2 - c\phi_1 - d\phi_2\}$$

$$\text{for} \quad \phi \in (V'_q(\mathbb{R}^n))^2.$$

Then

$$\begin{aligned}(P, AY) &= (p_1, (-\Delta + q)y_1 - ay_1 - by_2) + (p_2, (-\Delta + q)y_2 - cy_1 - dy_2)\\ &= (p_1, (-\Delta + q)y_1) - a(p_1, y_1) - b(p_1, y_2) + (p_2, (-\Delta + q)y_2) - c(p_2, y_1)\\ &- d(p_2, y_2)\\ &= ((-\Delta + q)p_1, y_1) - a(p_1, y_1) - c(p_2, y_1) + ((-\Delta + q)p_2, y_2) - d(p_2, y_2)\\ &- b(p_1, y_2)\\ &= ((-\Delta + q)p_1 - ap_1 - cp_2, y_1) + ((-\Delta + q)p_2 - bp_1 - dp_2, y_2)\\ &= (A^*P, Y)\end{aligned}$$

where

$$A^*(P = \{p_1, p_2\}) \to \{(-\Delta + q)p_1 - ap_1 - cp_2, (-\Delta + q)p_2 - bp_1 - dp_2\}$$

where A^* is the adjoint for A, P is the adjoint state. Then $A^*P = Y(u) - Z_d$ can be written as

$$-\frac{\partial p_1(u)}{\partial t} + (-\Delta + q)p_1 - ap_1 - cp_2 = y_1(u) - z_{1d}$$

$$-\frac{\partial p_2(u)}{\partial t} + (-\Delta + q)p_2 - bp_1 - dp_2 = y_2(u) - z_{2d}$$

$$p_1(u) = p_2(u) = 0.$$

So (12) is equivalent to

$$(-\frac{\partial p_1(u)}{\partial t} + (-\Delta + q)p_1 - ap_1 - cp_2, y_1(v) - y_1(u)) + (-\frac{\partial p_2(u)}{\partial t} + (-\Delta + q)p_2 - bp_1 - dp_2,$$

$$y_2(v) - y_2(u)) + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0$$

$$(p_1(u), \frac{\partial}{\partial t}(y_1(v) - y_1(u)) + (-\Delta + q)(y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} - (\frac{\partial p_1(u)}{\partial \nu_A}, y_1(v) - y_1(u))_{L^2(\Gamma)} +$$

$$(p_1(u), \frac{\partial}{\partial \nu_A}(y_1(v) - y_1(u))_{L^2(\Gamma)} - a(p_1(u), y_1(v) - y_1(u)) - b(p_1(u), y_2(v) - y_2(u)) +$$

$$(p_2(u), \frac{\partial}{\partial t}(y_2(v) - y_2(u)) + (-\Delta + q)(y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} - (\frac{\partial p_2(u)}{\partial \nu_A}, y_2(v) - y_2(u))_{L^2(\Gamma)}$$

$$+(p_2(u), \frac{\partial}{\partial \nu_A}(y_2(v) - y_2(u))_{L^2(\Gamma)} - c(p_2(u), y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} - d(p_2(u),$$

$$y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0$$

From (8), we obtain

$$(p_1(u), a(y_1(v) - y_1(u)) + b(y_2(v) - y_2(u)) + f_1 - f_1 - a(y_1(v) - y_1(u)))_{L^2(\mathbb{R}^n)}$$

$$+(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1)_{L^2(\Gamma)} + (0, \frac{\partial}{\partial \nu_A}(y_1(v) - y_1(u))_{L^2(\Gamma)} - c(p_2(u), y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)}$$

$$(p_2(u), c(y_1(v) - y_1(u)) + d(y_2(v) - y_2(u)) + f_2 - f_2 - c(y_1 - y_1(u)))_{L^2(\mathbb{R}^n)}$$

$$+(\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2)_{L^2(\Gamma)} + (0, \frac{\partial}{\partial \nu_A}(y_2(v) - y_2(u))_{L^2(\Gamma)} - d(p_2(u), y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)}$$

$$+(Nu, v - u)_{(L^2(\Gamma))^2} \geq 0.$$

Then we have

$$(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1)_{L^2(\Gamma)} + (\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2)_{L^2(\Gamma)} + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0.$$

i.e.,

$$\int_\Gamma (\frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2))d\Gamma + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0 \quad \forall u \in U_{ad}, v \in U_{ad}.$$

Which completes the proof of the theorem.

Remark 3.2 *To study the optimal control for the scalar case*

$$\begin{cases} \dfrac{\partial y}{\partial t} + (-\Delta + q)y = ay + f \quad in \quad \mathbb{R}^n \\ y(x) = g \quad in \quad \Gamma, \end{cases} \tag{13}$$

we define a bilinear form $\pi : V_q(\mathbb{R}^n) \times V_q(\mathbb{R}^n) \to \mathbb{R}$ *by*

$$\pi(y, \phi) = \int_{\mathbb{R}^n} (\nabla y \nabla \phi + qy\phi)dx - a \int_{\mathbb{R}^n} y\phi dx$$

As in theorem (1), we can prove π *is coercive if* $a < \lambda(q)$ *and then there exists a unique solution of (13) for* $f \in L^2(\mathbb{R}^n)$*. Therefore, the state of the system is given by the solution of:*

$$\begin{cases} \dfrac{\partial y}{\partial t} + (-\Delta + q)y(u) = ay(u) + f + u \quad in \quad \mathbb{R}^n \\ y(u) = u \quad in \quad \Gamma, \end{cases} \tag{14}$$

where u *is given in the space* $L^2(\Gamma)$ *of controls. For given* z_d *in* $L^2(\mathbb{R}^n)$*, the cost function is given by*

$$J(v) = \int_{\mathbb{R}^n} |y(v) - z_d|^2 dx + \int_\Gamma (Nv)v d\Gamma$$

where N is a given hermitian positive definite operator. Then we have the following characterization of optimal control for this system :

$$\begin{cases} \dfrac{\partial p(u)}{\partial t} + (-\Delta + q)p(u) - ap(u) = y_1(u) - z_d \quad in \quad \mathbb{R}^n \\ p(u) = 0 \quad in \quad \Gamma, \end{cases}$$

$$\int_\Gamma \frac{\partial p(u)}{\partial \nu_A}(v-u)d\Gamma + (Nu, v-u)_{L^2(\Gamma)} \geq 0, \quad \forall \quad v \in U_{ad},$$

together with (14), where $p(u)$ is the adjoint state.

Remark 3.3 *Also it is evident that by modifying:*

- the boundary conditions,

- the nature of the control (distributed, boundary),

- the nature of the observation,

- the initial differential system,

an infinity of variations on the above problem are possible to study.

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