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Ordinary differential equations

PERIODIC SOLUTIONS FOR THIRD AND FOURTH ORDER DELAY DIFFERENTIAL EQUATION IMPULSES WITH FREDHOLM OPERATOR OF INDEX ZERO

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Abstract

The purpose of this paper is to study the periodic solution of a certain class of third and fourth order delay differential equation impulses with Fredholm operator of index zero. We obtain the existence of periodic solution and Mawhin's continuation theorem. The delay conditions for the Schwarz inequality of the periodic solutions are also obtained. An example is also furnished which demonstrates validity of main result. We establish some new sufficient conditions which ensure that every solution of this equation impulses to at least one periodic solution.

Keywords and phrases: third and fourth order delay differential equations; Impulses; Periodic solutions; Mawhin's continuation theorem; Fredholm operator of index zero.

1 Introduction

The theory of impulsive delay differential equations is promising as an important role of investigation, since it is better than the corresponding theory of delay differential equation without impulse effects. Furthermore, such equations may demonstrate several real-world phenomena in physics,chemistry, biology, engineering, etc. In the last few years the theory of periodic solutions and delay differential equations with impulses has been studied by many authors, respectively [3, 5, 7, 8]. There are several books and a lot of papers dealing with the periodic solution of delay differential equations [1, 2, 4, 6, 9]. Periodic solutions of impulsive delay differential equations is a new research area and there are many publications in this field. The paper deals with impulsive equations with constant delay and Fredholm operator of index zero. We obtain the theorems of existence of periodic solution based on the Mawhin's continuation theorem.

In [11, 22, 23], the periodic solution of delay differential equations was considered. Also, boundedness of solutions was investigated in [22]. Afterward, many books and papers dealt with the delay differential equations and given many results, for example, [10, 12, 13, 14, 18, 19], etc.In recent years, the periodic solutions for some types of second and third-order delay differential equation with deviating argument were investigated; see [15, 16, 17, 21]. In [19], Sadek obtained stability and boundedness of a kind of third-order delay differential equation system. By using the continuation theorem of Mawhin's coincidence degree theory [14], we obtain some new results which complement and extend the corresponding works already known; see[15, 16, 17, 20, 21].

2 Preliminaries

Let $PC(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ be continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$,
$PC^1(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)\}$, as the space of continuous everywhere and continuously differentiable everywhere functions excluding t_k points.
$PC^2(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x''(t_k^+)$ and $x''(t_k^-)$ exist and $x''(t_k^-) = x''(t_k)\}$, as the space of continuous everywhere and continuously differentiable everywhere functions excluding t_k points.
Let $X = \{x(t) \in PC^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \sup_{t\in[0,T]} |x(t)|$,
$Y = PC(\mathbb{R},\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^n$, with norm $\|y\| = \max\{|u|_\infty, |c|\}$, where $u \in PC(\mathbb{R},\mathbb{R}), c = (c_1, \dots c_{2n}) \in R^n \times \mathbb{R}^n$, $|c| = \max_{1\le k\le 2n}\{|c_k|\}$.
$Z = PC(\mathbb{R},\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^n$, with norm $\|z\| = \max\{|v|_\infty, |d|\}$, where $v \in PC(\mathbb{R},\mathbb{R}), d = (d_1, \dots d_{2n}) \in R^n \times \mathbb{R}^n$, $|d| = \max_{1\le k\le 2n}\{|d_k|\}$.

Then X, Y and Z are Banach spaces. $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ are a Fredholm operator of index zero, where $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y, R : Z \to Z$ are projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad \ker R = \operatorname{Im} L,$$

$$X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} R.$$

It continues that

$$L|_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Similarly it follows that

$$L|_{D(L)\cap \ker Q} : D(L) \cap \ker Q \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_q. Let Ω be an open bounded subset of Y, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact in $\overline{\Omega}$, if $RN(\overline{\Omega})$ is bounded and $K_q(I - R)N : \overline{\Omega} \to Y$ is compact.

This paper obtains the existence of periodic solutions for the third-order delay differential equations with impulses

$$\begin{aligned} x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \geq 0, t \neq t_k,\\ \Delta x(t_k) = I_k,\\ \Delta x'(t_k) = J_k,\\ \Delta x''(t_k) = K_k. \end{aligned} \tag{1}$$

where $f(t+T, x) = f(t, x)$, $g(t+T, x) = g(t, x)$, $h(t+T) = h(t)$, $\tau(t+T) = \tau(t)$, $p(t + T) = p(t)$, $\tau(t) \geq 0$;
$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$, $x(t_k^-) = \lim_{t\to t_k^-} x(t)$, $x(t_k^-) = x(t_k)$;
$\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, $x'(t_k^+) = \lim_{t\to t_k^+} x'(t)$, $x'(t_k^-) = \lim_{t\to t_k^-} x'(t)$, $x'(t_k^-) = x'(t_k)$;
$\Delta x''(t_k) = x''(t_k^+) - x''(t_k^-)$, $x''(t_k^+) = \lim_{t\to t_k^+} x''(t)$, $x''(t_k^-) = \lim_{t\to t_k^-} x''(t)$, $x''(t_k^-) = x''(t_k)$.

The results is related to not only f, g, and h parameters with the impulses I_k, J_k, K_k and the delay τ. We assume that the following conditions:

(H1) $f(t+T,x)=f(t,x)$, $f\in C(\mathbb{R}^2,\mathbb{R})$ and $g(t+T,x)=g(t,x)$, $h(t+T)=h(t)$, $h,g\in C(\mathbb{R},\mathbb{R})$, with $\tau(t+T)=\tau(t)$, $\tau(t)\geq 0$, $p(t+T)=p(t)$, $p,\tau\in C(\mathbb{R},\mathbb{R})$;

(H2) $\{t_k\}$ satisfies $t_k<t_{k+1}$ and $\lim_{k\to\pm\infty} t_k=\pm\infty$, $k\in Z$,
$I_k(x,y), J_k(x,y), K_k(x,y)\in C(\mathbb{R}^2,\mathbb{R})$, and there is a positive n such that $\{t_k\}\cap[0,T]=\{t_1,t_2,\dots,t_n\}, t_{k+n}=t_k+T$,
$I_{k+n}(x,y)=I_k(x,y), J_{k+n}(x,y)=J_k(x,y), K_{k+n}(x,y)=K_k(x,y)$.

(H3) There are constants $\sigma,\beta\geq 0$ such that

$$|f(t,x)|\leq\sigma|x|,\quad \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{2}$$
$$xf(t,x)\geq\beta|x|^2,\quad \forall(t,x)\in[0,T]\times\mathbb{R}; \tag{3}$$

(H4) There are constants $\sigma,\beta\geq 0$ such that

$$|g(t,x)|\leq\sigma|x|,\quad \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{4}$$
$$x^2g(t,x)\geq\beta|x|^2,\quad \forall(t,x)\in[0,T]\times\mathbb{R}; \tag{5}$$

(H5) there are constants $\beta_i\geq 0$ $(i=1,2,3)$ such that

$$|h(x)|\geq\beta_1+\beta_2|x|, \tag{6}$$
$$|h(x)-h(y)|\leq\beta_3|x-y|; \tag{7}$$

(H6) there are constants $\gamma_i>0$ $(i=1,2,3)$, such that $|\int_x^{x+\lambda J_k(x,y)}h(s)ds|\leq |J_k(x,y)|(\gamma_1+\gamma_2|x|+\gamma_3|J_k(x,y)|)$, $\quad\forall\lambda\in(0,1)$;

(H7) there are constants $a_k, a_k', a_k''\geq 0$ such that $|K_k(x,y)|\leq a_k|x|^2+a_k'|x|+a_k''$;

(H8) $zK_k(x,y)\leq 0$ and there are constants $b_k\geq 0$ such that $|K_k(x,y)|\leq b_k$.

Lemma 1 *[[4]] Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. We assume that the following conditions are satisfied:*

(i) $Lx\neq\lambda Nx, \forall x\in\partial\Omega\cap D(L), \lambda\in(0,1)$;

(ii) $RNx\neq 0$, *for all* $x\in\partial\Omega\cap\ker L$;

(iii) $\deg\{KRNx, \Omega\bigcap\ker L, 0\}\neq 0$, *where* $K:\operatorname{Im} R\to\ker L$ *is an isomorphism.*

Then the abstract equation $Lx=Nx$ has at least one solution in $\overline{\Omega}\bigcap D(L)$.

We assume the operators $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ by

$$Lx = (x''', \Delta x(t_1), \ldots, \Delta x(t_n), \Delta x'(t_1), \ldots, \Delta x'(t_n), \Delta x''(t_1), \ldots, \Delta x''(t_n)), \tag{8}$$

and $N : X \to Y$, $N : Y \to Z$ by

$$\begin{aligned} Nx = (&-f(t, x''(t)) - g(t, x'(t)) - h(x(t - \tau(t))) + p(t), \\ &I_1(x(t_1)), \ldots, I_n(x(t_n)), J_1(x'(t_1)), \ldots, J_n(x'(t_n)), K_1(x''(t_1)), \ldots, K_n(x''(t_n))). \end{aligned} \tag{9}$$

Lemma 2 *[[4]] L is a Fredholm operator of index zero with*

$$\ker L = \{x(t) = c, t \in \mathbb{R}\}, \tag{10}$$

and

$$\begin{aligned} &\operatorname{Im} L(y, z, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ &= \int_0^T (y(s) + z(s))ds + \sum_{k=1}^{n} b_k(T - t_k) + \sum_{k=1}^{n} a_k + x'(0)T = 0. \end{aligned} \tag{11}$$

Let the linear operators $P : X \to X$, $Q : Y \to Y$ and $R : Z \to Z$ be defined by

$$Px = x(0), \tag{12}$$

$$\begin{aligned} &Q(y, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ &= \frac{2}{T^2}[\int_0^T (T - s)y(s)ds + \sum_{k=1}^{n} b_k(T - t_k) + \sum_{k=1}^{n} a_k + x'(0)T], 0, \ldots, 0), \end{aligned} \tag{13}$$

and

$$\begin{aligned} &R(z, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ &= \frac{2}{T^2}[\int_0^T (T - s)z(s)ds + \sum_{k=1}^{n} b_k(T - t_k) + \sum_{k=1}^{n} a_k + x'(0)T], 0, \ldots, 0). \end{aligned} \tag{14}$$

Lemma 3 *[[8]] If $\alpha > 0$, $x(t) \in PC^2(\mathbb{R}, \mathbb{R})$ with $x(t + T) = x(t)$, then*

$$\int_0^T \int_{t-\alpha}^{t} |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt \tag{15}$$

and

$$\int_0^T \int_{t}^{t+\alpha} |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt. \tag{16}$$

Let

$$
\begin{aligned}
A_1(t,\alpha) &= \sum_{t-\alpha\le t_k\le t} a_k, \quad A_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a_k,\\
B_1(t,\alpha) &= \sum_{t-\alpha\le t_k\le t} a'_k, \quad B_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a'_k,\\
C_1(t,\alpha) &= \sum_{t-\alpha\le t_k\le t} a''_k, \quad C_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a''_k,\\
I_1 &= \Big(\int_0^T A_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T A_2^2(t,\alpha)dt\Big)^{1/2},\\
I_2 &= \Big(\int_0^T B_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T B_2^2(t,\alpha)dt\Big)^{1/2},\\
I_3 &= \int_0^T A_1^2(t,\alpha)dt + \int_0^T A_2^2(t,\alpha)dt,\\
I_4 &= \int_0^T A_1(t,\alpha)B_1(t)dt + \int_0^T A_2(t,\alpha)B_2(t)dt,\\
I_5 &= \int_0^T B_1^2(t,\alpha)dt + \int_0^T B_2^2(t,\alpha)dt
\end{aligned}
$$

The following Lemma is important for us to the delay $\tau(t)$.

Lemma 4 *Suppose $\tau(t)\in C(\mathbb{R},\mathbb{R})$ with $\tau(t+T)=\tau(t)$ and $\tau(t)\in[-\alpha,\alpha]$ for all $t\in[0,T]$, $x(t)\in PC^1(\mathbb{R},\mathbb{R})$ with $x(t+T)=x(t)$ and there is a positive n such that $\{t_k\}\cap[0,T]=\{t_1,t_2,\dots,t_n\}$, $\Delta x(t_k)=\lambda I_k(x(t_k),x'(t_k))$ for all $\lambda\in(0,1)$ and $t_{k+n}=t_k+T, I_{k+n}(x,y)=I_k(x,y)$. Furthermore there exist nonnegative constants a_k,a_k such that $|I_k(x,y)|\le a_k|x|+a'_k$. Then*

$$
\begin{aligned}
&\int_0^T |x(t)-x(t-\tau(t))|^2dt\\
&\le 2\alpha^2\int_0^T |x'(t)|^2dt + 2\alpha I_1|x(t)|_\infty\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2} \qquad (17)\\
&\quad + 2\alpha I_2\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2} + I_3|x(t)|_\infty^2 + I_4|x(t)|_\infty + I_5.
\end{aligned}
$$

3 Third-order delay differential equation

We establish the theorems of existence of periodic solution based on the following two conditions.

Theorem 1 *We assume that* (H1)–(H8) *hold. Then* (1) *has at least one* T*-periodic solution and*

$$\sum_{k=1}^{n} a_k < 1, \tag{18}$$

$$\begin{aligned}&\Big[\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)\Big]M^2 + \beta_3\Big[2|\tau(t)|_\infty^2 \\ &+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M + I_3(|\tau(t)|_\infty)M^2\Big]^{1/2} < \beta,\end{aligned} \tag{19}$$

where

$$M = \frac{1}{1-\sum_{k=1}^{n} a_k}(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}).$$

proof: Consider the abstract equation $Lx = \lambda Nx$, with $\lambda \in (0,1)$, where L and N are given by (8) and (9). Let

$$\Omega_1 = \{x \in D(L) : \ker L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

For $x \in \Omega_1$, (1) Integrating the interval on $[0,T]$, using Schwarz inequality, we get

$$\begin{aligned}&|\int_0^T h(x(t-\tau(t))dt| \\ &= |\int_0^T p(t)dt - \int_0^T f(t,x''(t))dt - \int_0^T g(t,x'(t))dt + \sum_{k=1}^{n} K_k(x(t_k),x''(t_k))| \\ &\le T|p(t)|_\infty + \sigma\int_0^T |x''(t)|dt + \sum_{k=1}^{n} b_k \\ &\le \sigma T^{1/2}\Big(\int_0^T |x''(t)|^2 dt\Big)^{1/2} + T|p(t)|_\infty + \sum_{k=1}^{n} b_k.\end{aligned}$$

From the above formula, there is a interval on $t_0 \in [0,T]$ such that

$$|h(x(t_0-\tau(t_0))| \le \frac{\sigma}{T^{1/2}}(\int_0^T |x''(t)|^2 dt)^{1/2} + |p(t)|_\infty + \frac{1}{T}\sum_{k=1}^{n} b_k.$$

From (6),we get

$$\beta_1 + \beta_2|x(t_0-\tau(t_0))| \le \frac{\sigma}{T^{1/2}}(\int_0^T |x''(t)|^2 dt)^{1/2} + |p(t)|_\infty + \frac{1}{T}\sum_{k=1}^{n} b_k.$$

Then
$$|x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2} + d,$$
where $d = \big(||p(t)|_\infty + \frac{1}{T}\sum_{k=1}^n b_k - \beta_1|\big)/\beta_2$. So there is an integer m and an interval $t_1 \in [0,T]$ such that $t_0 - \tau(t_0) = mT + t_1$. Therefore
$$|x(t_1)| = |x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2} + d,$$
$$x(t) = x(t_1) + \int_{t_1}^t x''(s)ds + \sum_{t_1\le t_k<t} K_k(x(t_k), x''(t_k)).$$
Thus
$$\begin{aligned}|x(t)|_\infty &\le |x(t_1)| + \int_{t_1}^t |x''(s)|ds + \sum_{t_1\le t_k<t} |K_k(x(t_k))|\\ &\le \frac{\sigma}{\beta_2 T^{1/2}}(\int_0^T |x''(t)|^2dt)^{1/2} + d + \int_0^T |x''(t)|dt + \sum_{k=1}^n a_k|x|_\infty + \sum_{k=1}^n a'_k + \sum_{k=1}^n a''_k\\ &\le |x|_\infty\sum_{k=1}^n a_k + (\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2})\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2} + d + \sum_{k=1}^n a'_k + \sum_{k=1}^n a''_k.\end{aligned}$$
It continues that
$$\begin{aligned}|x(t)|_\infty &\le \frac{d+\sum_{k=1}^n a''_k}{1-\sum_{k=1}^n a_k} + \frac{1}{1-\sum_{k=1}^n a_k}(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2})(\int_0^T |x''(t)|^2dt)^{1/2}\\ &= c_1 + M(\int_0^T |x''(t)|^2dt)^{1/2},\end{aligned} \tag{20}$$
where c_1 is a positive constant. On the other hand, multiplying both side of (1) by $x'(t)$, we have
$$\begin{aligned}&\int_0^T x'''(t)x''(t)dt + \lambda\int_0^T f(t,x''(t))x'(t)dt\\ &+\lambda\int_0^T g(t,x'(t))x'(t)dt + \lambda\int_0^T h(t,x(t-\tau(t))x'(t)dt\\ &= \lambda\int_0^T p(t)x'(t)dt.\end{aligned}$$
Since
$$\int_0^T x'''(t)x''(t)dt = -\frac{1}{2}\sum_{i=1}^n[(x''(t_k^+))^2 - (x''(t_k))^2],$$

Our assumption (H7) that

$$\begin{aligned}
&(x'(t_k^+))^2 - (x'(t_k))^2 \\
&= (x'(t_k^+) + x'(t_k))(x'(t_k^+) - (x'(t_k)) \\
&= \Delta x'(t_k)(2x'(t_k) + \Delta x'(t_k)) \\
&= \lambda K_k(x(t_k), x'(t_k))(2x'(t_k) + \lambda K_k(x(t_k), x'(t_k)) \\
&= 2\lambda K_k(x(t_k), x'(t_k))x'(t_k) + [\lambda K_k(x(t_k), x'(t_k))]^2 \le b_k^2.
\end{aligned}$$

In (5), by use Schwarz inequality

$$\begin{aligned}
&\beta \int_0^T |x''(t)|^2 dt \\
&\le - \int_0^T h(x(t-\tau(t))x'(t)dt + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{k=1}^{n} b_k^2 \\
&= \int_0^T [h(x(t) - h(x(t-\tau(t))]x'(t)dt - \int_0^T h(x(t))x'(t)dt \\
&\quad + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{i=1}^{n} b_k^2 \\
&\le \int_0^T |h(x(t)) - h(x(t-\tau(t))||x'(t)|dt + |p(t)|_\infty \int_0^T |x'(t)|dt \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^{n} b_k^2 \\
&\le \Big[\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2} + |p(t)|_\infty T^{1/2}\Big]\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^{n} b_k^2.
\end{aligned} \tag{21}$$

From (H5) and (H6), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt| \\
&= |\int_{x(0)}^{x(t_1)} h(s)ds + \int_{x(t_1^+)}^{x(t_2)} h(s)ds + \cdots + \int_{x(t_n^+)}^{x(T)} h(s)ds| \\
&= |\int_{x(0)}^{x(T)} h(s)ds - \sum_{k=1}^{n} \int_{x(t_k)}^{x(t_k^+)} h(s)ds| \\
&\le \sum_{k=1}^{n} |\int_{x(t_k)}^{x(t_k)+\lambda K_k(x(t_k),x'(t_k))} h(s)ds| \\
&\le \sum_{k=1}^{n} [|K_k(x(t_k),x'(t_k))|(\gamma_1 + \gamma_2|x(t_k)| + \gamma_3|K_k(x(t_k),x'(t_k))|)] \\
&\le [\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)]|x(t)|_\infty^2 + c_2|x(t)|_\infty + c_3,
\end{aligned}
$$

where c_2, c_3 are constants. From (20), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt| \\
&\le [\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)]M^2 \int_0^T |x'(t)|^2 dt + c_4(\int_0^T |x'(t)|^2 dt)^{1/2} + c_5,
\end{aligned}
\tag{22}
$$

where c_4, c_5 are constants. From Lemma 4, we get

$$
\begin{aligned}
&\int_0^T |h(x(t) - h(x(t-\tau(t)))|^2 dt \\
&\le \beta_3^2 \int_0^T |x(t) - x(t-\tau(t))|^2 dt \\
&\le \beta_3^2 [2|\tau(t)|_\infty^2 \int_0^T |x'(t)|^2 dt + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)|x(t)|_\infty \Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + 2|\tau(t)|_\infty I_2(|\tau(t)|_\infty)\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + I_3(|\tau(t)|_\infty)|x(t)|_\infty^2 \\
&\quad + I_4(|\tau(t)|_\infty)|x(t)|_\infty + I_5(|\tau(t)|_\infty)].
\end{aligned}
$$

Substituting (20) into the above inequality, we get

$$\begin{aligned}
&\int_0^T |h(x(t) - h(x(t-\tau(t)))|^2 dt \\
&\leq \beta_3^2[2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M \\
&\quad + I_3(|\tau(t)|_\infty)M^2] \int_0^T |x'(t)|^2 dt + c_6\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + c_7,
\end{aligned}$$

where c_6, c_7 are constants. From above inequality

$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2} \quad for \quad a \geq 0, b \geq 0, \tag{23}$$

we get

$$\begin{aligned}
&\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2} \\
&\leq \beta_3[2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M \\
&\quad + I_3(|\tau(t)|_\infty)M^2]^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + c_6^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/4} + c_7^{1/2}.
\end{aligned}$$

Substituting the above formula and (22) in (21), we get

$$\begin{aligned}
&\{\beta - [\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)]M^2 - \beta_3[2|\tau(t)|_\infty^2 \\
&+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M + I_3(|\tau(t)|_\infty)M^2]^{1/2}\} \int_0^T |x'(t)|^2 dt \\
&\leq c_8(\int_0^T |x'(t)|^2 dt)^{\frac{3}{4}} + c_9(\int_0^T |x'(t)|^2 dt)^{1/2} + c_{10},
\end{aligned}$$

where c_8, c_9, c_{10} are constants. There is a constant $M_1 > 0$ such that

$$\int_0^T |x'(t)|^2 dt \leq M_1. \tag{24}$$

From (20), we get

$$|x(t)|_\infty \leq d + M(\int_0^T |x'(t)|^2 dt)^{1/2} \leq d + M(M_1)^{1/2}.$$

Then there is a constant $M_2 > 0$ such that $|x(t)|_\infty \le M_2$. Therefore, integrating (1) on the interval $[0, T]$, using Schwarz inequality, we get

$$\begin{aligned}
\int_0^T |x'''(t)|dt &= \int_0^T |-f(t, x''(t)) - g(t, x'(t)) - h(x(t-\tau(t))) + p(t)|dt \\
&\le \int_0^T |f(t, x''(t))|dt + \int_0^T |g(t, x''(t))|dt + \int_0^T |h(x(t-\tau(t)))|dt + \int_0^T |p \\
&\le \sigma \int_0^T |x''(t)|dt + h_\delta T + T|p(t)|_\infty \\
&\le \sigma T^{1/2}(\int_0^T |x''(t)|^2 dt)^{1/2} + h_\delta T + T|p(t)|_\infty \\
&\le \sigma T^{1/2}(M_1)^{1/2} + h_\delta T + T|p(t)|_\infty,
\end{aligned}$$

where $h_\delta = \max_{|x|\le\delta} |g(x)|$. Then there is a constant $M_3 > 0$ such that

$$\int_0^T |x''(t)|dt \le M_3. \tag{25}$$

From (24),then there are $t_2 \in [0, T]$ and $c > 0$ such that $|x'(t_2)| \le c$ for $t \in [0, T]$

$$|x'(t)|_\infty \le |x'(t_2)| + \int_0^T |x''(t)|dt + \sum_{k=1}^n b_k. \tag{26}$$

Then there is a constant $M_4 > 0$ such that

$$|x'(t)|_\infty \le M_4. \tag{27}$$

It follows that there is a constant $I_2 > \max\{M_2, M_4\}$ such that $\|x\| \le I_2$, Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \ker L, RNx = 0\}$. If $x \in \Omega_2$, then $x(t) = c \in R$ and satisfies

$$RN(x, 0) = (-\frac{2}{T^2} \int_0^T [f(t, 0) + g(t, 0) + h(c) - p(t)]dt, 0, \ldots, 0) = 0. \tag{28}$$

we get

$$\int_0^T [f(t, 0) + g(t, 0) + h(c) - p(t)]dt = 0. \tag{29}$$

In (29),there must be a interval $t_0 \in [0, T]$ such that

$$h(c) = -f(t_0, 0) - g(t_0, 0) + p(t_0). \tag{30}$$

From (30) and assumption (H3), (H4), we get

$$\beta_1 + \beta_2|c| \le |h(c)| \le |f(t_0, 0)| + |g(t_0, 0)| + |p(t_0)| \le \sigma \times 0 + |p(t)|_\infty. \quad (31)$$

Then

$$|c| \le \frac{||p(t)|_\infty - \beta_1|}{\beta_2} \quad (32)$$

which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where $\Omega_3 = \{x \in X : |x| < ||p(t)|_\infty - \beta_1|/\beta_2 + 1\}$. By Lemmas 2, we can see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument,

(i) $Lx \ne \lambda Nx$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(ii) $RNx \ne 0$ for all $x \in \partial\Omega \cap \ker L$.

Finally we prove that (iii) of Lemma 1 is satisfied. We take $H(x, \mu) : \Omega \times [0, 1] \to X$,

$$H(x, \mu) = \mu x + \frac{2(1-\mu)}{T^2} \int_0^T [-f(t, x''(t)) - g(t, x'(t)) + h(x(t - \tau(t)) + p(t)]dt.$$

From assumptions (H3) and (H4), we can easily verify $H(x, \mu) \ne 0$, for all $(x, \mu) \in \partial\Omega \cap \ker L \times [0, 1]$, which results in

$$\begin{aligned} \deg\{KRNx, \Omega \cap \ker L, 0\} &= \deg\{H(x, 0), \Omega \cap \ker L, 0\} \\ &= \deg\{H(x, 1), \Omega \cap \ker L, 0\} \ne 0, \end{aligned}$$

where $K(x, 0, \ldots, 0) = x$. Therefore, by Lemma 1, Equation (1) has at least one T-periodic solution.

4 Fourth-order delay differential equation

We establish criteria for the existence of positive periodic solutions to the following fourth-order delay differential equation. The simplified model takes the form

$$\ddot{\ddot{x}}(t) + a\dddot{x}(t) + f_1(\ddot{x}(t - \tau(t))) + g_1(\dot{x}(t - \tau(t))) + h_1(x(t - \tau(t))) = p_1(t). \quad (33)$$

where $f_1(t + T, x) = f_1(t, x)$, $g_1(t + T, x) = g_1(t, x)$, $h_1(t + T) = h_1(t)$, $\tau(t + T) = \tau(t)$, $p_1(t + T) = p_1(t)$, $\tau(t) \ge 0$.

We assume that the following conditions:

(H9) $|f_1(x)| \leq K + \delta_1|x|$ for $x \in \mathbb{R}$

(H10) $xg_1(x) > 0$ and $|g_1(x)| > K + |p_1|_0 + \delta_1|x|$ for $|x| \geq D$

(H11) $x^2h_1(x) > 0$ and $|h_1(x)| > K + |p_1|_0 + \delta_2|x|$ for $|x| \geq D$

(H12) $\lim_{x\to-\infty} \frac{h_1(x)}{x^2} \leq \delta_3$.

The main purpose of this paper is to establish the existence of positive periodic solutions to (33). An example to compute the main result is given.

Lemma 5 *[[4]] Let X and Z be two Banach space. Consider a Fredholm operator equation*

$$Lx = \lambda N(x, \lambda), \tag{34}$$

where $L : \mathrm{Dom}\, L \cap X \to Z$ is a operator of index zero, $\lambda \in (0, 1)$ is a parameter. Let P and Q denote two projectors such that

$$P : X \to \ker L, \quad \text{and} \quad Q : Z \to Z/ImL.$$

Assume that $N : \bar{\Omega} \times (0, 1) \to Z$ is L-compact on $\bar{\Omega} \times (0, 1)$, where Ω is open bounded in X. In addition, suppose that

(a) For each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda N(x, \lambda)$

(b) For each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$,

(c) $\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$.

Then $Lx = N(x, 1)$ has at least one solution in $\bar{\Omega}$.

Theorem 2 *Suppose that exist positive constants $\delta_1, \delta_2, \delta_3 \leq 0$, $K > 0$ and $D > 0$, such that (H9–H12).Then (33) has at least one ω-periodic solution for $a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1 + \omega)\delta_3 < 1$.*

Proof: To use Lemma 5 for (33), we take $X = \{x \in C^3(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t)$ for all $t \in \mathbb{R}\}$ and $Z = \{z \in C(\mathbb{R}, \mathbb{R}) : z(t + \omega) = z(t)$ for all $t \in \mathbb{R}\}$ and denote $|x|_0 = \max_{t\in[0,\omega]} |x(t)|$ and
$\|x\| = \max\{|x|_0, |\dot{x}|_0, |\ddot{x}|_0\, |\dddot{x}|_0\}$. Then X and Z are Banach spaces, for $x \in X$ and $z \in Z$, able with the norm forms $\|\cdot\|$ and $|\cdot|_0$, respectively. Let

$$Lx(t) = \ddddot{x}, \quad x \in X,\, t \in \mathbb{R};$$
$$N(x(t), \lambda) = -a\dddot{x}(t) - \lambda f_1(\ddot{x}(t - \tau(t))) - \lambda g_1(\dot{x}(t - \tau(t))) - h_1(x(t - \tau(t)))$$
$$+\lambda p_1(t), \quad x \in X,\, t \in \mathbb{R};$$
$$Px(t) = \frac{1}{\omega}\int_0^\omega x(t)dt, \quad Qz(t) = \frac{1}{\omega}\int_0^\omega z(t)dt, \quad x \in X,\, t \in \mathbb{R};$$

where $x \in X$, $z \in Z$, $t \in \mathbb{R}$, $\lambda \in (0,1)$.

We prove that L is a Fredholm mapping of index 0, that $P : X \to \ker L$ and $Q \to Z/\operatorname{Im} L$ are projectors, and that N is L-compact on $\bar{\Omega}$ for any given open and bounded subset Ω in X.

The equivalent differential equation for the operator $Lx = \lambda N(x,\lambda)$, $\lambda \in (0,1)$, takes the form

$$\dddot{x}(t)+\lambda a\ddot{x}(t)+\lambda^2 f_1(\ddot{x}(t-\tau(t)))+\lambda^2 g_1(\dot{x}(t-\tau(t)))+\lambda h_1(x(t-\tau(t))) = \lambda^2 p_1(t). \tag{35}$$

Let $x \in X$ be a solution of (35) for a certain $\lambda \in (0,1)$. Integrating (35) over $[0,\omega]$, we obtain

$$\int_0^\omega \left[\lambda^2 f_1(\ddot{x}(t-\tau(t))) + \lambda^2 g_1(\dot{x}(t-\tau(t))) + \lambda h_1(x(t-\tau(t))) - \lambda^2 p_1(t)\right] dt = 0. \tag{36}$$

Thus, there is a point $\xi \in [0,\omega]$, such that

$$\lambda^2 f_1(\ddot{x}(\xi-\tau(\xi))) + \lambda^2 g_1(\dot{x}(\xi-\tau(\xi))) + \lambda h_1(x(\xi-\tau(\xi))) - \lambda^2 p_1(\xi) = 0$$

Thus using the condition (H9),

$$\begin{aligned} |h_1(x(\xi-\tau(\xi)))| &\le |f_1(\ddot{x}(\xi-\tau(\xi)))| + |g_1(\dot{x}(\xi-\tau(\xi)))| + |p_1(\xi)| \\ &\le K + \delta_1|\ddot{x}(\xi-\tau(\xi))| + \delta_2|\dot{x}(\xi-\tau(\xi))| + |p_1|_0 \\ &\le K + |p_1|_0 + \delta_2|\ddot{x}|_0 + \delta_1|\dot{x}|_0\,. \end{aligned} \tag{37}$$

We will prove that there is a point $t_0 \in [0,\omega]$ such that

$$|x(t_0)| < |\ddot{x}|_0 + |\dot{x}|_0 + D. \tag{38}$$

Case 1: $\delta_1, \delta_2 = 0$. If $|x(\xi-\tau(\xi))| > D$, (H9)–(H12) and (37) ensure $K+|p_1|_0 < |h_1(x(\xi-\tau(\xi)))| \le K + |p_1|_0$, which is a contradiction. So

$$|x(\xi-\tau(\xi))| \le D. \tag{39}$$

Case 2: $\delta_1, \delta_2 > 0$. If $|x(\xi-\tau(\xi))| > D$, then $K + |p_1|_0 + \delta_1|\dot{x}(\xi-\tau(\xi))| + \delta_2|x(\xi-\tau(\xi))| < |h_1(x(\xi-\tau(\xi)))| \le K + |p_1|_0 + \delta_1|\ddot{x}|_0 + \delta_2|\dot{x}|_0$. So that

$$|x(\xi-\tau(\xi))| \le |\ddot{x}|_0. \tag{40}$$

Hence from (39) and (40), we see in either case 1 or case 2 that

$$|x(\xi-\tau(\xi))| \le |\ddot{x}|_0 + D.$$

Let $\xi - \tau(\xi) = 2k\pi + t_0$, where k is an integer and $t_0 \in [0, \omega]$. Then

$$|x(t_0)| = |x(\xi - \tau(\xi))| < |\ddot{x}|_0 + D.$$

So (38) holds, and then

$$|x|_0 \leq |\dot{x}(t_0)| + \int_0^\omega |\ddot{x}(s)|ds < (\omega + 1)|\ddot{x}|_0 + D. \tag{41}$$

Let $G(\theta) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1+\omega)(\delta_3+\theta), \theta \in [0, \infty)$. From the assumption $G(0) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1+\omega)\delta_3 < 1$ and $G(\theta)$ is continuous on $[0, \infty)$, we know that there must be a small constant $\theta_0 > 0$ such that $G(\theta) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1+\omega)(\delta_3+\theta) < 1,\ \theta \in (0, \theta_0]$. Let $\varepsilon = \theta_0/2$, once we can obtain that $a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1+\omega)(\delta_3+\varepsilon) < 1$ For such a small $\varepsilon > 0$, in view of assumption (H_4), we find that there must be a constant $\rho > D$, which is independent of λ and x, such that

$$\frac{h_1(x)}{x^2} < (\delta_3 + \varepsilon), \quad \text{for } x < -\rho. \tag{42}$$

Thus putting $\Delta_1 = \{t : t \in [0, \omega], x(t - \tau(t)) > \rho\}$, $\Delta_2 = \{t : t \in [0, \omega], x(t - \tau(t)) < -\rho\}$, $\Delta_3 = \{t : t \in [0, \omega], |x(t - \tau(t))| \leq \rho\}$, $\Delta_4 = \{t : t \in [0, \omega], |x(t - \tau(t))| \geq \rho\}$ and $h_\rho = \sup_{|x| \leq \rho} h_1(x)$, we have

$$\int_{\Delta_1} |h_1(t - \tau(t))|dt < \omega(\delta_1 + \varepsilon)|x|_0, \int_{\Delta_2} |h_1(t - \tau(t))|dt < \omega(\delta_2 + \varepsilon)|x|_0,$$
$$\int_{\Delta_3} |h_1(t - \tau(t))|dt < \omega(\delta_3 + \varepsilon)|x|_0, \int_{\Delta_4} |h_1(t - \tau(t))|dt \leq \omega h_\rho.$$

From (36), we have

$$\begin{aligned} \int_0^\omega h_1(x(t - \tau(t)))dt = \Big(\int_{E_1} + \int_{E_2} + \int_{E_3} + \int_{E_4}\Big) h_1(x(t - \tau(t)))dt \\ \leq \int_0^\omega |f_1(\ddot{x}(t - \tau(t)))|dt \qquad (43) \\ + \int_0^\omega |g_1(\dot{x}(t - \tau(t)))|dt + \int_0^\omega |h_1(x(t - \tau(t)))|dt + \int_0^\omega |p_1(t)|dt. \end{aligned}$$

That is

$$\begin{aligned} \int_{E_1} |h_1(x(t - \tau(t)))|dt \leq \int_{E_2} |h_1(x(t - \tau(t)))|dt + \int_{E_3} |h_1(x(t - \tau(t)))|dt \\ + \int_{E_4} |h_1(x(t - \tau(t)))|dt \\ + \int_0^\omega |f_1(\ddot{x}(t - \tau(t)))|dt + \int_0^\omega |g_1(\dot{x}(t - \tau(t)))|dt + \int_0^\omega |h_1(x(t - \tau(t)))|dt + \omega|p_1|_0. \end{aligned} \tag{44}$$

Using the condition (H9), we have

$$\begin{aligned}\int_0^{\omega}|f_1(\ddot{x}(t-\tau(t)))|dt &= \int_{-\tau(0)}^{\omega-\tau(\omega)}\frac{1}{1-\ddot{\tau}(\nu(s))}|f_1(\ddot{x}(s))|ds\\ &= \int_0^{\omega}\frac{1}{1-\ddot{\tau}(\nu(s))}|f_1(\ddot{x}(s)|ds\\ &\le \int_0^{\omega}\frac{\delta_1}{1-\ddot{\tau}(\nu(s))}|\ddot{x}(s)|ds+\int_0^{\omega}\frac{K}{1-\ddot{\tau}(\nu(s))}ds\\ &\le \delta_1|b|_2\Big(\int_0^{\omega}|\ddot{x}(s)|ds\Big)^{1/2}+|b|_2K\sqrt{\omega}.\end{aligned} \tag{45}$$

Thus, by (44) and (45), we have

$$\begin{aligned}\int_0^{\omega}|\dddot{x}(s)|ds &\le a\int_0^{\omega}|\dddot{x}(s)|ds+\int_0^{\omega}|f_1(\ddot{x}(t-\tau(t)))|dt+\int_0^{\omega}|g_1(\dot{x}(t-\tau(t)))|dt\\ &\quad+\int_0^{\omega}|h_1(x(t-\tau(t)))|dt+\omega|p_1|_0\\ &= a\int_0^{\omega}|\dddot{x}(s)|ds+\int_0^{\omega}|f_1(\ddot{x}(t-\tau(t)))|dt+\int_0^{\omega}|g_1(\dot{x}(t-\tau(t)))|dt\\ &\quad+\Big(\int_{\Delta_1}+\int_{\Delta_2}+\int_{\Delta_3}+\int_{\Delta_4}\Big)|h_1(x(t-\tau(t)))|dt+\omega|p_1|_0\\ &\le a\sqrt{\omega}\Big(\int_0^{\omega}|\dddot{x}(s)|^2ds\Big)^{1/2}+2\delta_1|b|_2\Big(\int_0^{\omega}|\ddot{x}(s)|^2ds\Big)^{1/2}+2\delta_2|b|_2\Big(\int_0^{\omega}|\dot{x}(s)|^2ds\Big)^{1/2}\\ &\quad+2\omega(\delta_3+\varepsilon)|x|_0+2K\sqrt{\omega}|b|_2+2\omega f_\rho+2|p_1|_0.\end{aligned} \tag{46}$$

Since $x(0)=x(\omega)$, there exists $t_1\in[0,\omega]$, such that $\ddot{x}(t_1)=0$, Hence for $t\in[0,\omega]$,

$$|\ddot{x}|_0\le\int_0^{\omega}|\dddot{x}(t)|dt\le\sqrt{\omega}\Big(\int_0^{\omega}|\dddot{x}(s)|^2ds\Big)^{1/2}, \tag{47}$$

$$\Big(\int_0^{\omega}|\ddot{x}(s)|^2ds\Big)^{1/2}\le\sqrt{\omega}\max_{t\in[0,\omega]}|\ddot{x}(t)|\le\omega\Big(\int_0^{\omega}|\dddot{x}(s)|^2ds\Big)^{1/2}. \tag{48}$$

Since $x(t)$ is periodic function, for $t\in[0,\omega]$, we have

$$|\dddot{x}(t)|\le\int_0^{\omega}|\ddddot{x}(t)|dt, \tag{49}$$

$$\Big(\int_0^{\omega}|\ddot{x}(s)|^2ds\Big)^{1/2}\le\sqrt{\omega}\max_{t\in[0,\omega]}|\dddot{x}(t)|\le\sqrt{\omega}\int_0^{\omega}|\ddddot{x}(t)|dt. \tag{50}$$

Substituting (50) in (47), we have

$$|\ddot{x}|_0 \leq \omega \int_0^\omega |\dddot{x}(t)|dt. \tag{51}$$

Substituting (51) in (41),

$$|x|_0 \leq D + \omega(1+\omega) \int_0^\omega |\dddot{x}(t)|dt. \tag{52}$$

Substituting (48),(50) and (52) in (46), and using inequality (49), we have

$$|\dddot{x}|_0 \leq \int_0^\omega |\dddot{x}(t)|dt \leq \frac{2K\sqrt{\omega}|b|_2 + 2\omega h_\rho + 2\omega|p_1|_0 + 2\omega(\delta_2+\varepsilon)D}{1 - a\omega - 2\delta_1|b|_2\omega^{\frac{3}{2}} - 2\delta_2|b|_2\omega^{\frac{5}{2}} - 2\omega^2(1+\omega)(\delta_3+\varepsilon)} \equiv A_3. \tag{53}$$

Substituting (53) in (51) and (52), we have

$$|x|_0 \leq D + \omega(1+\omega)A_3 \equiv A_1, \quad |\dot{x}|_0 \leq \omega A_3 \equiv A_2. \tag{54}$$

Let $A_0 = \max\{A_1, A_2, A_3, A_4\}$ and take $\Omega = \{x \in X : \|x\| \leq A_0\}$. The priori bounds show that condition (a) of Lemma 5 is satisfied. If $x \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}$, then x is a constant with $x(t) = A_0$ or $x(t) = -A_0$. Then

$$\begin{aligned} QN(x,0) &= \frac{1}{\omega}\int_0^\omega \big[- a\dddot{x}(t) - h_1(x(t-\tau(t))\big] dt \\ &= \frac{1}{\omega}\int_0^\omega -f_1(x)dt = \frac{1}{\omega}\int_0^\omega -f_1 A_0 dt \neq 0 \end{aligned}$$

Finally, consider the homotopy mapping

$$H(x,\mu) = \mu x + \frac{1-\mu}{\omega}\int_0^\omega h_1(x)dt, \quad \mu \in [0,1].$$

Since for every $\mu \in [0,1]$ and x in the intersection of $\ker L$ and $\partial\Omega$, we have

$$xH(x,\mu) = \mu x^2 + \frac{1-\mu}{\omega}\int_0^\omega x h_1(x)dt > 0,$$

This continues that

$$\begin{aligned} \deg\{QN(x,0), \Omega\cap\ker L, 0\} &= \deg\{-h_1(x), \Omega\cap\ker L, 0\} \\ &= \deg\{-x, \Omega\cap\ker L, 0\} \\ &= \deg\{-x, \Omega\cap R, 0\} \neq 0. \end{aligned}$$

All conditions in Lemma 5 are satisfied; therefore, (33) has at least one solution in Ω. Our results complement and extend known results and are given with examples.

Example 1

Consider the third order delay differential equation with impulses

$$\begin{aligned} x'''(t) + \frac{1}{3}x''(t) + \frac{1}{6}x'(t) + \frac{1}{21}x(t - \frac{1}{10}\cos t) &= \sin t, \quad t \neq k, \\ I_k(x, y) &= \frac{\sin\frac{k\pi}{3}}{120}x + \frac{y}{1+y^2}, \\ J_k(x, y) &= -\frac{2x^2y}{1+x^4y^2}, \\ K_k(x, y) &= -\frac{4x^4y}{1+x^8y^2}, \end{aligned} \tag{55}$$

where $t_k = k$, $f(t,x) = \frac{1}{3}x^2$,$g(t,x) = \frac{1}{6}x$, $h(y) = \frac{1}{21}y$, $p(t) = \sin t$, $\tau(t) = \frac{1}{10}\cos t$, it is easy to see that $|\tau(t)|_\infty = \frac{1}{10}$, $T = 2\pi$, $\{k\} \cap [0, 2\pi] = \{1,2,3,4,5,6,7,8\}$, $\sigma = \beta = \frac{1}{3}$, $\beta_1 = 0$, $\beta_2 = \beta_3 = \frac{1}{21}$. Since $|I_k(x,y)| \leq \frac{1}{120}|x| + \frac{1}{2}$,
$|J_k(x,y)| \leq 1$,$|\int_x^{x+I_k(x,y)} h(s)ds| \leq |I_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|I_k(x,y)|)$,
$|K_k(x,y)| \leq 1$,$|\int_x^{x+J_k(x,y)} h(s)ds| \leq |J_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|J_k(x,y)|)$,
then we take $a_k = \frac{1}{120}$, $a'_k = \frac{1}{2}$, $b'_k = 1$ $(k = 1,2,3,4,5,6,7,8)$, $\gamma_1 = 0$, $\gamma_2 = 1/21$, $\gamma_3 = 1/42$.

$$\sum_{k=1}^{8} a_k = \frac{1}{20} < 1,$$

$$M = \frac{1}{1 - \sum_{k=1}^{n} a_k}\left(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}\right) = \frac{1}{1 - \frac{1}{20}}\left(\frac{\frac{1}{3}}{\frac{1}{21}(2\pi)^{1/2}} + (2\pi)^{1/2}\right) < 8.$$

By Theorem 1, Equation (55) has at least one 2π-periodic solution.

Example 2

Consider the fourth order delay differential equation with impulses

$$\begin{aligned} \ddot{\ddot{x}}(t) + \frac{1}{2\pi}\dddot{x}(t) + \frac{7}{3\pi^2}\ddot{x}(t - \cos 2t)) + \frac{7}{2\pi^2}\dot{x}(t - \cos 2t)) \\ + \frac{3}{2}e^{-(\dot{x}(t-\cos 2t))^2} + h_1(x(t - \cos 2t)) = \frac{1+\sin 2t}{4} \end{aligned}$$

where $p_1(t) = (1 + \sin 2t)/4$, $\tau(t) = \cos 2t$, $f_1(u) = \frac{7}{3\pi^2}u + \frac{3}{2}e^{-u^2}$, $g_1(u) =$

$\frac{7}{2\pi^2}u + \frac{3}{2}e^{-u^2}$ and

$$h_1(u) = \begin{cases} \frac{7}{3\pi^2}u + \frac{3}{2} + tan^{-1}u, & \text{for } u > D, \\ \left(\frac{7}{2\pi^2} + \frac{3}{2} + \frac{\pi}{2}\right), & \text{for } |u| \le D, \\ \frac{7}{3\pi^2}u - \frac{3}{2} + tan^{-1}u, & \text{for } u < -D. \end{cases}$$

So we can chose $\delta_1 = \delta_2 = \delta_3 = 7/(3\pi^2)$, $D = 1$, $K = 1$, $|p_1|_0 = 1/2$, $|b|_2 < \sqrt{\omega}$, $\omega = \pi/4$. Therefore, fourth order delay differential equation has at least one periodic solution.

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