*DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 4, 2010
Electronic Journal,
reg. N ФC77-39410 at 15.04.2010
ISSN 1817-2172*

*http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru*

Normal Forms of Hamiltonian Systems [1]

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Abstract

Equivalence of Hamiltonian systems in the neighborhood of a critical point relatively to the group of formal canonical transformations is considered.

Definitions of metanormal and normal forms of a Hamiltonian system that do not require restrictions on the unperturbed part of the Hamiltonian and a method of their finding are introduced.

Relationships between the introduced Hamiltonian normal forms and the normal forms of Hamiltonian systems defined by A.D. Bruno and K.R. Meyer are studied.

Hamiltonian normal forms for real single degree of freedom Hamiltonian systems are obtained in the case when the unperturbed part of the Hamiltonian is monomial and in the case when the unperturbed part of the Hamiltonian is an irreducible binomial with coprime indices.

[1]This research was financially supported by the Russian Foundation for Basic Research (project no. 09–01–00734-a)

Part I

Hamiltonian Systems

1 Hamiltonian Systems and Canonical Transformations

1.1 Formal Hamiltonian Systems of Differential Equations

In this section we give some basic facts from the theory of Hamiltonian systems. Detailed exposition is given for instance in [9].

Let f be a formal power series in variables $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ with coefficients in the field $\mathbb{K}$, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Further the words "formal power" will be omitted.

Henceforth, $f(u,v) = \sum_{k=0}^{\infty} f_k(u,v)$ denotes the decomposition of a series f into the sum of homogeneous polynomials f_k of degree k.

Definition 1 *The least $r \geq 0$ for which $f_r \not\equiv 0$ is called the order of f. The order of f is denoted by* $\operatorname{ord} f$. *If $f \equiv 0$, then* $\operatorname{ord} f = +\infty$.

Definition 2 *For any series $f = f(u,v)$, $\varphi = \varphi(u,v)$, define a series called the Poisson bracket of $f = f(u,v)$ and $\varphi = \varphi(u,v)$ by*

$$\widehat{f}(\varphi) = \{f, \varphi\} = \sum_{j=1}^{n} \frac{\partial f}{\partial u_j}\frac{\partial \varphi}{\partial v_j} - \frac{\partial f}{\partial v_j}\frac{\partial \varphi}{\partial u_j}. \tag{1}$$

In particular, the variables u_j, v_k themselves satisfy the identities:

$$\{u_j, v_k\} \equiv \delta_k^j,\ \{u_j, u_k\} \equiv 0,\ \{v_j, v_k\} \equiv 0 \quad (j, k = \overline{1, n}\,),$$

where δ_k^j denotes the Kronecker delta.

The Poisson bracket has the following properties:

1) $\widehat{f}(\varphi) = -\widehat{\varphi}(f)$ (skew-symmetry);
2) $\widehat{f}(\alpha\,\varphi + \beta\,\psi) = \alpha\,\widehat{f}(\varphi) + \beta\,\widehat{f}(\psi) \quad \forall \alpha, \beta \in \mathbb{K}$ (linearity);
3) $\widehat{f}\,(\varphi\,\psi) = \widehat{f}(\varphi)\,\psi + \varphi\,\widehat{f}\,(\psi)$ (Leibniz's law);
4) $\widehat{f}(\{\varphi, \psi\}) = \{\widehat{f}\,(\varphi), \psi\} + \{\varphi, \widehat{f}\,(\psi)\}$ (Jacobi's identity).

Consider a Hamiltonian represented by series

$$H(u,v) = H_r(u,v) + \sum_{k=r+1}^{\infty} H_k(u,v) \quad (r \geq 2). \tag{2}$$

The homogeneous polynomial H_r of the least degree in decomposition (2) is called the unperturbed Hamiltonian, and the series $H - H_r$ is called perturbation.

A Hamiltonian system with the Hamiltonian H is defined as follows.

$$\dot{u}_j = \frac{\partial H}{\partial v_j}, \quad \dot{v}_j = -\frac{\partial H}{\partial u_j} \qquad (j = \overline{1, n}\,). \tag{3}$$

The origin O is an equilibrium point of (3) because $\operatorname{ord} H \geq 2$.

Definition 3 *A series $\varphi = \varphi(u, v)$ is called an integral of the unperturbed Hamiltonian H_r if*

$$\widehat{H}_r(\varphi) = \{H_r, \varphi\} = 0.$$

Let $\varphi = \sum_{k=0}^{\infty} \varphi_k$, then $\{H_r, \varphi\} = \sum_{k=0}^{\infty}\{H_r, \varphi_k\}$ is the decomposition of $\{H_r, \varphi\}$ into the sum of homogeneous polynomials $\{H_r, \varphi_k\}$ of degree $k + r - 2$.

Hence, if φ is an integral of the unperturbed Hamiltonian H_r, then the homogeneous polynomials φ_k are integrals of H_r as well.

1.2 Formal Canonical Transformations

Consider arbitrary series $u_j(x, y)$, $v_k(x, y)$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $\operatorname{ord} u_j$, $\operatorname{ord} v_k \geq 1$ $(j, k = \overline{1, n}\,)$.

For any series $f(u, v)$, let

$$\widetilde{f}(x, y) = f(u(x, y), v(x, y)).$$

Clearly, the order of $\widetilde{f}$ is greater than or equal to the order of f.

Definition 4 *Series $u_j = \widetilde{u}_j(x, y)$, $v_k = \widetilde{v}_k(x, y)$ with $\operatorname{ord} \widetilde{u}_j$, $\operatorname{ord} \widetilde{v}_k = 1$ are said to define a formal canonical transformation if*

$$\{\widetilde{u}_j, \widetilde{v}_k\} \equiv \delta_k^j, \quad \{\widetilde{u}_j, \widetilde{u}_k\} \equiv 0, \quad \{\widetilde{v}_j, \widetilde{v}_k\} \equiv 0 \quad (j, k = \overline{1, n}\,). \tag{4}$$

This definition is equivalent to the matrix equality

$$(D(\widetilde{u}, \widetilde{v})/D(x, y))\, I\, (D(\widetilde{u}, \widetilde{v})/D(x, y))^T = I, \tag{5}$$

where $I = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. Hence, $\det(D(\widetilde{u}, \widetilde{v})/D(x, y)) = \pm 1$.

Proposition 1 *The Poisson bracket is invariant under formal canonical transformations $u_j = \widetilde{u}_j(x, y)$, $v_k = \widetilde{v}_k(x, y)$, i. e.*

$$\{f, \varphi\} = \{\widetilde{f}, \widetilde{\varphi}\}. \tag{6}$$

Proof By Definition 2,

$$\{\widetilde{f}, \widetilde{\varphi}\} = \sum_{j=1}^{n} \frac{\partial \widetilde{f}}{\partial x_j}\frac{\partial \widetilde{\varphi}}{\partial y_j} - \frac{\partial \widetilde{f}}{\partial y_j}\frac{\partial \widetilde{\varphi}}{\partial x_j}. \tag{7}$$

Substituting $\dfrac{\partial}{\partial x_j} = \sum\limits_{k=1}^{n} \dfrac{\partial \widetilde{u}_k}{\partial x_j}\dfrac{\partial}{\partial u_k} + \dfrac{\partial \widetilde{v}_k}{\partial x_j}\dfrac{\partial}{\partial v_k}$, $\dfrac{\partial}{\partial y_j} = \sum\limits_{k=1}^{n} \dfrac{\partial \widetilde{u}_k}{\partial y_j}\dfrac{\partial}{\partial u_k} + \dfrac{\partial \widetilde{v}_k}{\partial y_j}\dfrac{\partial}{\partial v_k}$ into (7) yields $\{\widetilde{f}, \widetilde{\varphi}\} =$

$$= \left(\frac{\partial \widetilde{u}_k}{\partial x_j}\frac{\partial f}{\partial u_k} + \frac{\partial \widetilde{v}_k}{\partial x_j}\frac{\partial f}{\partial v_k}\right)\left(\frac{\partial \widetilde{u}_l}{\partial y_j}\frac{\partial \varphi}{\partial u_l} + \frac{\partial \widetilde{v}_l}{\partial y_j}\frac{\partial \varphi}{\partial v_l}\right) - \left(\frac{\partial \widetilde{u}_k}{\partial y_j}\frac{\partial f}{\partial u_k} + \frac{\partial \widetilde{v}_k}{\partial y_j}\frac{\partial f}{\partial v_k}\right)\left(\frac{\partial \widetilde{u}_l}{\partial x_j}\frac{\partial \varphi}{\partial u_l} + \frac{\partial \widetilde{v}_l}{\partial x_j}\frac{\partial \varphi}{\partial v_l}\right) =$$

$$= \frac{\partial f}{\partial u_k}\frac{\partial \varphi}{\partial u_l}\{\widetilde{u}_k, \widetilde{u}_l\} + \frac{\partial f}{\partial u_k}\frac{\partial \varphi}{\partial v_l}\{\widetilde{u}_k, \widetilde{v}_l\} + \frac{\partial f}{\partial v_k}\frac{\partial \varphi}{\partial u_l}\{\widetilde{v}_k, \widetilde{u}_l\} + \frac{\partial f}{\partial v_k}\frac{\partial \varphi}{\partial v_l}\{\widetilde{v}_k, \widetilde{v}_l\} \overset{(4)}{=}$$

$$= \frac{\partial f}{\partial u_j}\frac{\partial \varphi}{\partial v_j} - \frac{\partial f}{\partial v_j}\frac{\partial \varphi}{\partial u_j} = \{f, \varphi\};$$

it was understood that a repeated index was to be summed over from 1 to n. □

Proposition 2 *Formal canonical transformation $u_j = \widetilde{u}_j(x, y)$, $v_k = \widetilde{v}_k(x, y)$ transforms the system* (3) *to the system*

$$\dot{x}_j = \frac{\partial \widetilde{H}}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial \widetilde{H}}{\partial x_j} \quad (j = \overline{1, n}\,), \tag{8}$$

where according to the notation introduced, $\widetilde{H}(x, y) = H(\widetilde{u}, \widetilde{v})$.

Proof According to (3), differentiating u_j, v_k with respect to t gives

$$\dot{u}_j = \{u_j, H\} \overset{(6)}{=} \{\widetilde{u}_j, \widetilde{H}\}, \quad \dot{v}_k = \{v_k, H\} \overset{(6)}{=} \{\widetilde{v}_k, \widetilde{H}\}.$$

On the other hand,

$$\dot{u}_j = \frac{\partial \widetilde{u}_j}{\partial x_l}\dot{x}_l + \frac{\partial \widetilde{u}_j}{\partial y_l}\dot{y}_l, \quad \dot{v}_k = \frac{\partial \widetilde{v}_j}{\partial x_l}\dot{x}_l + \frac{\partial \widetilde{v}_j}{\partial y_l}\dot{y}_l.$$

Thus, we obtain the equalities:

$$\frac{\partial \widetilde{u}_j}{\partial x_l}\left(\dot{x}_l - \frac{\partial \widetilde{H}}{\partial y_l}\right) + \frac{\partial \widetilde{u}_j}{\partial y_l}\left(\dot{y}_l + \frac{\partial \widetilde{H}}{\partial x_l}\right) = 0, \quad \frac{\partial \widetilde{v}_k}{\partial x_l}\left(\dot{x}_l - \frac{\partial \widetilde{H}}{\partial y_l}\right) + \frac{\partial \widetilde{v}_k}{\partial y_l}\left(\dot{y}_l + \frac{\partial \widetilde{H}}{\partial x_l}\right) = 0.$$

The Jacobian of a formal canonical transformation is nonzero; consequently, these equalities imply (8). □

1.3 Lie Transforms

For any series $f = f(x, y)$ with $\operatorname{ord} f \geq 3$, an operator $exp(\widehat{f}\,)$ is defined by

$$exp(\widehat{f}\,)(\varphi) = \sum_{k=0}^{\infty} \frac{\widehat{f}^{\,k}(\varphi)}{k!},$$

where $\varphi = \varphi(x, y)$ is a series, $\widehat{f}^{\,k} = \widehat{f} \cdots \widehat{f}$, k times, is the k^{th} composition of $\widehat{f}$ from (1) with itself, and $\widehat{f}^{\,0}(\varphi) = \varphi$. Meanwhile, if $\widehat{f}^{\,k}(\varphi) \not\equiv 0$, then $\operatorname{ord} \widehat{f}^{\,k}(\varphi) = \operatorname{ord} \varphi + k(\operatorname{ord} f - 2) \geq k$; otherwise, $\operatorname{ord} \widehat{f}^{\,k}(\varphi) = +\infty$.

Proposition 3 *The operator $exp(\widehat{f}\,)$ has the following properties:*

1) $exp(\widehat{f}\,)(\alpha\varphi + \beta\psi) = \alpha\, exp(\widehat{f}\,)(\varphi) + \beta\, exp(\widehat{f}\,)(\psi) \quad \forall \alpha, \beta \in \mathbb{K}$;

2) $exp(\widehat{f}\,)(\varphi\,\psi) = exp(\widehat{f}\,)(\varphi)\, exp(\widehat{f}\,)(\psi)$;

3) $exp(\widehat{f}\,)(\{\varphi, \psi\}) = \{exp(\widehat{f}\,)(\varphi), exp(\widehat{f}\,)(\psi)\}$;

4) $exp(-\widehat{f}\,)(exp(\widehat{f}\,)(\varphi)) = \varphi$,

i. e. $exp(\widehat{f}\,)$ is an automorphism of the algebra of series supplied by operation $\{\,.\,,\,.\,\}$.

Proof 1) is an implication of the linearity of $\widehat{f}$;

2) is verified by using Leibniz's law:

$$exp(\widehat{f}\,)(\varphi\,\psi) = \sum_{k=0}^{\infty} \frac{\widehat{f}^{\,k}(\varphi\,\psi)}{k!} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{C_k^l\, \widehat{f}^{\,l}(\varphi)\, \widehat{f}^{\,k-l}(\psi)}{k!} =$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{\widehat{f}^{\,l}(\varphi)\, \widehat{f}^{\,k-l}(\psi)}{l!\,(k-l)!} = exp(\widehat{f}\,)(\varphi)\, exp(\widehat{f}\,)(\psi);$$

3) is verified in a similar way by using Jacobi's identity;

4) follows by the identities

$$exp(-\widehat{f}\,)(exp(\widehat{f}\,)(\varphi)) = \sum_{k=0}^{\infty} \frac{1}{k!}\,(-\widehat{f}\,)^k \left(\sum_{l=0}^{\infty} \frac{1}{l!}\,\widehat{f}^{\,l}(\varphi)\right) = \sum_{k,l=0}^{\infty} \frac{(-\widehat{f}\,)^k(\widehat{f}^{\,l}(\varphi))}{k!\,l!} =$$

$$= \sum_{k,l=0}^{\infty} \frac{(-1)^k\, \widehat{f}^{\,k+l}(\varphi)}{k!\,l!} = \sum_{m=0}^{\infty} \frac{\widehat{f}^{\,m}(\varphi)}{m!} \sum_{k=0}^{m} \frac{(-1)^k m!}{k!\,(m-k)!} = \widehat{f}^{\,0}(\varphi) = \varphi. \;\Box$$

Proposition 4 *Series*

$$u_j = exp(\widehat{f}\,)(x_j), \quad v_k = exp(\widehat{f}\,)(y_k) \qquad (\mathrm{ord}\, f \geq 3,\;\; j,k = \overline{1,n}\,) \tag{9}$$

define a formal canonical transformation.

Proof By the 3^{rd} property of the operator $exp(\widehat{f}\,)$, we have

$$\{\widetilde{u}_j, \widetilde{v}_k\} = \{exp(\widehat{f}\,)(x_j), exp(\widehat{f}\,)(y_k)\} = exp(\widehat{f}\,)(\{x_j, y_k\}) = exp(\widehat{f}\,)(\delta_k^j) = \delta_k^j.$$

In a similar way one gets $\{\widetilde{u}_j, \widetilde{u}_k\} \equiv 0,\;\; \{\widetilde{v}_j, \widetilde{v}_k\} \equiv 0$.

Hence, by Definition 4, (9) is a formal canonical transformation. $\Box$

According to [6, 7, 8, 9], let us introduce the following definition.

Definition 5 *A formal canonical transformation of the form* (9) *is called a Lie transform.*

Proposition 5 *Lie transforms form a group.*

Proof By definition,

$$exp(\widehat{f}\,)\,exp(\,\widehat{g}\,) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\widehat{f}^{\,k} \left(\sum_{l=0}^{\infty} \frac{1}{l!}\,\widehat{g}^{\,l} \right) = \sum_{k,l=0}^{\infty} \frac{\widehat{f}^{\,k}\,\widehat{g}^{\,l}}{k!\,l!}.$$

Substituting this operator series into the logarithm series (see [10, Lecture 4])

$$\ln Z = (Z - E) - \frac{1}{2}(Z - E)^2 + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(Z - E)^k,$$

where E is the identical operator, gives the operator series

$$\ln(exp(\widehat{f}\,)\,exp(\,\widehat{g}\,)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum \frac{\widehat{f}^{\,p_1}\,\widehat{g}^{\,q_1} \cdots \widehat{f}^{\,p_k}\,\widehat{g}^{\,q_k}}{p_1!q_1!\ldots p_k!q_k!},$$

where the inner sum is taken over all sets $(p_1, \ldots, p_k, q_1, \ldots, q_k)$ of nonnegative integers satisfying conditions

$$p_1 + q_1 \geq 1, \ldots, p_k + q_k \geq 1.$$

This series is called the Campbell-Hausdorff series. By [10, Lecture 4, Proposition B], each homogeneous term of the Campbell-Hausdorff series as a series in variables $\widehat{f}$, $\widehat{g}$ can be represented in the form of a Lie polynomial, i. e. as a linear combination of the Lie brackets of these variables.

Since by the Jacobi's identity the Lie bracket of $\widehat{f}$ and $\widehat{g}$ satisfies $[\widehat{f}, \widehat{g}\,] = \widehat{f}\,\widehat{g} - \widehat{g}\,\widehat{f} = \widehat{\{f, g\}}$, and $\operatorname{ord}\{f, g\} \geq 4$, there exists such a series h with $\operatorname{ord} h \geq 3$ that

$$exp(\widehat{f}\,)\,exp(\,\widehat{g}\,) = exp(\widehat{h}).$$

Thus, composition of Lie transforms is a Lie transform.

Associativity of the composition is obvious.

Identity element is represented by $exp(\,\widehat{0}\,)$.

By the 4^{th} property of the operator $exp(\widehat{f}\,)$, the inverse of the Lie transform exists and is also a Lie transform. □

As it was shown in Proposition 2, the formal canonical transformations transform the Hamiltonian by the law $\widetilde{H}(x, y) = H(\widetilde{u}, \widetilde{v})$. In case the transformation is of the form (9), the last is equivalent to

$$\widetilde{H}(x, y) = exp(\widehat{f}\,)(H(x, y)) \tag{10}$$

by the 1^{st} and the 2^{nd} property of the operator $exp(\widehat{f}\,)$.

Let us write down the first few terms of the series in the right part of (10):

$$\widetilde{H} = H + \{f,\ H\} + \frac{1}{2}\{f,\ \{f,\ H\}\} + \ldots. \tag{11}$$

Since we assume that $\operatorname{ord}(f) \geq 3$, the Lie transforms do not change the unperturbed Hamiltonian, i. e. in (10) $\widetilde{H}_r = H_r$.

2 Formal Equivalence of Hamiltonian Systems

2.1 Resonance Equation

Denote the space of polynomials in variables x, y over $\mathbb{K}$ by $\mathfrak{P}$.

Let $z = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n)$.

Formula

$$\langle\langle P, Q\rangle\rangle = P(D)\overline{Q}(z)|_{z=0} \quad P, Q \in \mathfrak{P},$$

where $\overline{Q}$ is a polynomial obtained from Q by the complex conjugation of coefficients, defines an inner product on $\mathfrak{P}$ with the following properties (see for instance [11, Chapter 0, §5]):

1) $\langle\langle x^p y^q, x^{p'} y^{q'}\rangle\rangle = p!q!\delta^p_{p'}\delta^q_{q'}$,

2) $\langle\langle PQ, R\rangle\rangle = \langle\langle P, Q^* R\rangle\rangle \quad (P, Q, R \in \mathfrak{P})$,

where $\delta^p_{p'} = \prod_{j=1}^n \delta^{p_j}_{p'_j}$, $\delta^q_{q'} = \prod_{j=1}^n \delta^{q_j}_{q'_j}$ are the products of the Kronecker deltas and $Q^* = \overline{Q}(D)$ is the differential operator adjoint of Q with respect to the inner product $\langle\langle\, .\, ,\, .\,\rangle\rangle$.

By the 1^{st} property, the space $\mathfrak{P}$ is the direct sum of the orthogonal subspaces $\mathfrak{P}^k$ of homogeneous polynomials of degree k.

Consider a Hamiltonian

$$H(x, y) = H_r(x, y) + \sum_{|p+q|\geq r+1} h^{(p,q)} x^p y^q \qquad (r \geq 2). \tag{12_r}$$

By the 2^{nd} property of the inner product $\langle\langle\, .\, ,\, .\,\rangle\rangle$, the operator $\widehat{H}^*_r$ adjoint of $\widehat{H}_r$ has the form

$$\widehat{H}^*_r = \sum_{j=1}^n y_j(\partial H_r/\partial x_j)^* - x_j(\partial H_r/\partial y_j)^* \ \text{ or } \ \widehat{H}^*_r = \sum_{j=1}^n (\partial H_r/\partial x_j)^* y_j - (\partial H_r/\partial y_j)^* x_j. \tag{13}$$

Here, the equality (13_2) is obtained from (13_1) by using Leibniz's law.

Definition 6 *Equation*

$$\widehat{H}^*_r P = 0, \quad P \in \mathfrak{P}. \tag{14}$$

is called the resonance equation for H_r. Its solutions are called resonant polynomials.

Denote the space of resonant polynomials by $\mathfrak{J}$.

Since the unperturbed Hamiltonian H_r is a homogeneous polynomial, the space $\mathfrak{J}$ is the direct sum of the orthogonal subspaces $\mathfrak{J}^k$ of homogeneous resonant polynomials of degree k.

2.2 Equivalence of Hamiltonian Systems Relatively to the Group of Lie Transforms

Definition 7 *We say that two Hamiltonians H and H' with* $\operatorname{ord} H, \operatorname{ord} H' = r$ *and* $H_r = H'_r$ *are equivalent in the order $r + m$ $(m \geq 1)$ if there exists such a Lie transform that transformed Hamiltonian $\widetilde{H}$ satisfies*

$$\widetilde{H}_k = H'_k \qquad (k = \overline{r, r+m}\,).$$

Denote the space of series in variables x, y over $\mathbb{K}$ by Φ.

Consider the spaces

$$J_s = \{\psi \in \Phi : \ \mathrm{ord}\,\psi \geq 3, \ \{\psi, H_{r+l}\} = 0 \ \ (l = \overline{0, s}\,)\} \qquad (s = \overline{0, m-2}\,).$$

Denote the subspace of homogeneous elements of J_s of degree k by J_s^k.

Lemma 1 *Let* $f = \sum_{k=3}^{m+1} f_k + \varphi$, *where* $m \geq 1$, $f_k \in J_{m-k+1}^k$ *and* $\varphi \in \Phi$, $\mathrm{ord}\,\varphi \geq m+2$. *Then* $H_r, \ldots, H_{r+m-1}$ *are invariant under the transformation* (9) *and*

$$\widetilde{H}_{r+m} = H_{r+m} + \sum_{k=3}^{m+1} \{f_k, H_{r+m-k+2}\} + \{\varphi_{m+2}, H_r\}. \tag{15}$$

Proof Let $\psi^1, \ldots, \psi^s$ ($s \geq 0$) be a set of series (empty set when $s = 0$) such that $\mathrm{ord}\,\psi^\nu \geq 3$ ($\nu = \overline{1, s}\,$), and let $\psi \in J_{m-k+1}$, $\mathrm{ord}\,\psi = k$ ($k \in \{3, \ldots, m+1\}$). Then

$$\mathrm{ord}\,\{\psi^1, \{\psi^2, \ldots \{\psi^s, \{\psi, H_{r+l}\}\}\}\} \geq r + m + s \quad (\forall\, l \geq 0).$$

Indeed, as far as $\mathrm{ord}\,0 = +\infty$, we may assume that

$$\{\psi^1, \{\psi^2, \ldots \{\psi^s, \{\psi, H_{r+l}\}\}\}\} \not\equiv 0.$$

Meanwhile, $l \geq m - k + 2$ because $\psi \in J_{m-k+1}$ and $\{\psi, H_{r+l}\} = 0$ for all $l = \overline{0, m-k+1}$. Therefore,

$$\mathrm{ord}\,\{\psi^1, \{\psi^2, \ldots \{\psi^s, \{\psi, H_{r+l}\}\}\}\} = \sum_{\nu=1}^{s} \mathrm{ord}\,\psi^\nu + \mathrm{ord}\,\psi - 2s - 2 + r + l \geq$$
$$\geq 3s + k - 2s - 2 + r + m - k + 2 = r + m + s.$$

By the inequality obtained, all the Poisson brackets in (11) have orders greater than or equal to $r + m$, so $H_r, \ldots, H_{r+m-1}$ are invariant under the considered transformation.

In addition the multiple Poisson brackets of indices greater than one have the orders greater than $r + m$. This proves (15). □

Theorem 1 *Let* $H, H' \in \Phi$, $\mathrm{ord}\,H, \mathrm{ord}\,H' = r$ *and* $H_k = H'_k$ ($k = \overline{r, r+m-1}\,$), *and let there exist* $f_k \in J_{m-k+1}^k$ ($k = \overline{3, m+1}\,$) *such that for any resonant polynomial* $P \in \mathfrak{J}^{r+m}$,

$$\langle\langle P, H'_{r+m}\rangle\rangle = \langle\langle P, H_{r+m}\rangle\rangle + \langle\langle P, \sum_{k=3}^{m+1} \{f_k, H_{r+m-k+2}\}\rangle\rangle. \tag{16}$$

Then H *and* H' *are equivalent in the order* $r + m$.

Proof Let $m = 1$ and $\langle\langle P, H_{r+1} - H'_{r+1}\rangle\rangle = 0$ for all $P \in \mathfrak{J}^{r+1}$.

Let us find a Lie transform that preserves H_r and transforms H_{r+1} to H'_{r+1}.

The subspace $\mathfrak{J}^{r+1} \subset \mathfrak{P}^{r+1}$ is a complementary subspace to the image of the linear map $\widehat{H}_r$ from $\mathfrak{P}^3$ to $\mathfrak{P}^{r+1}$. Since $\mathfrak{P}^{r+1}$ is finite-dimensional, $Im(\widehat{H}_r|_{\mathfrak{P}^3}) = (Im(\widehat{H}_r|_{\mathfrak{P}^3}))^{\perp\perp} = (\mathfrak{J}^{r+1})^{\perp}$. Therefore, there exists such a polynomial $\varphi \in \mathfrak{P}^3$ that $\{H_r, \varphi\} = H_{r+1} - H'_{r+1}$. By this equality and (15), the desired transformation has the form $exp(\widehat{\varphi})$.

In case $m \geq 2$, the result is proved in a similar manner. □

Part II

Hamiltonian Normal Forms

3 Hamiltonian Metanormal Form

Definition 8 *We say that an element $h^{(p,q)}x^py^q$ of the perturbation of Hamiltonian* (12_r) *is nonresonant if $\langle\langle P, x^py^q\rangle\rangle = 0$ for all $P \in \mathfrak{J}$. Otherwise, we say that the element $h^{(p,q)}x^py^q$ is resonant.*

By Definition 8, the perturbation $H - H_r$ can be uniquely decomposed into the sum of resonant and nonresonant parts which consist of resonant and nonresonant terms respectively.

Definition 9 *We say that a series H of the view* (12_r) *or the corresponding Hamiltonian system* (3) *is a Hamiltonian Metanormal Form* (*or* HMNF *for short*) *if the perturbation $H - H_r$ consists of resonant elements only.*

Theorem 2 *There exists a formal canonical transformation that transforms* (12_r) *to* HMNF.

Proof is done by induction on the order m.

Denote the resonant part of H_{r+1} by H'_{r+1}.

By Definition 8, $\langle\langle P, H_{r+1} - H'_{r+1}\rangle\rangle = 0$ for all $P \in \mathfrak{J}^{r+1}$.

The last equality is identical to (16) with $m = 1$. Therefore by Theorem 1, the Hamiltonian H is equivalent to its HMNF in the order $r + 1$.

Let us assume that H is equivalent to its HMNF in the order $r + m - 1$. Without loss of generality assume that H is identical to its HMNF in the orders $r + k$ $(k = \overline{1, m-1}\,)$.

Denote the resonant part of H_{r+m} by H'_{r+m}.

By Definition 7, $\langle\langle P, H_{r+m} - H'_{r+m}\rangle\rangle = 0$ for all $P \in \mathfrak{J}^{r+m}$.

The last equality is identical to (16) with $f_3, \ldots, f_{m+1} = 0$.

Thus by Theorem 1, H is equivalent to its HMNF in the order $r + m$. Furthermore, by Lemma 1, the corresponding transformation is of the view (9) with ord $f \geq m + 2$.

Step by step, by eliminating the nonresonant elements of degree $r+1, r+2, \ldots$, we construct a sequence of Lie transforms $exp(\widehat{f^m}\,)$ with $f^m \in \Phi$ and ord $f^m \geq m + 2$.

The composition of these transformations converges in Φ, i. e. for any $\varphi \in \Phi$ the summand φ_k of any fixed degree k does not change starting with a sufficiently far step. Therefore by Propositions 4 and 5, the limit transformation $\ldots exp(\widehat{f^m}\,)\ldots exp(\widehat{f^2}\,)exp(\widehat{f^1}\,)$ is a formal canonical transformation. □

Remark 1 HMNF is an intermediate normal form (meta- from Greek $\mu\varepsilon\tau\acute{\alpha}$ = "after", "beyond", "with", "adjacent"). Further we will show that not only resonant elements of the perturbation of (12_r), but also some resonant elements can be eliminated by formal canonical transformations.

Remark 2 The fact that the unperturbed part H_r of the Hamoltonian is of arbitrary degree $r \geq 2$ is the advantage of the introduced definition of HMNF. In the next two sections two other definitions of Hamiltonian normal forms (HNF) will be given: HNF by Bruno and HNF by Meyer both of which assume that the unperturbed part of the Hamiltonian is of degree two.

4 HNF by Bruno, Relationship to HMNF

In [5] it is proved that the quadratic part of the Hamiltonian (12_2) can be transformed to

$$H_2 = \frac{1}{2}\, z^T G z,$$

where $z = (x, y)$, and $G = \begin{pmatrix} 0 & C^T \\ C & D \end{pmatrix}$ is a block matrix in which $C = \{C^{(1)}, \ldots, C^{(s)}\}$ is a Jordan matrix of order n with Jordan blocks $C^{(k)}$ of order $l^{(k)}$ $(k = \overline{1, s})$, and $D = \{D^{(1)}, \ldots, D^{(s)}\}$ is a block-diagonal matrix with blocks $D^{(k)} = \sigma^{(k)} \Delta^{(k)}$ of order $l^{(k)}$, by complex linear canonical transformations. Here, $\Delta^{(k)} = \mathrm{diag}\{1, 0, \ldots, 0\}$, and $\sigma^{(k)} = 0$ if the eigenvalue $\lambda^{(k)}$ corresponding to the block $C^{(k)}$ is nonzero. In other notation

$$H_2 = \sum_{j=1}^{n} \lambda_j x_j y_j + \sum_{j=1}^{n-1} \varepsilon_j x_j y_{j+1} + \frac{1}{2} \sum_{j=1}^{n} \sigma_j y_j^2, \tag{17}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of C, and

$$\varepsilon_j = \begin{cases} 0, & \text{if } \lambda_j \neq \lambda_{j+1}; \\ 0 \text{ or } 1, & \text{if } \lambda_j = \lambda_{j+1}, \end{cases} \qquad \sigma_j = \begin{cases} 0, & \text{if } \lambda_j \neq 0 \text{ or } \varepsilon_{j-1} \neq 0; \\ 0 \text{ or } 1, & \text{if } \lambda_j = 0 \text{ and } \varepsilon_{j-1} = 0 \end{cases} \quad (\varepsilon_0 = 0). \tag{18}$$

Definition 10 *We say that Hamiltonian* (12_2) *or the corresponding Hamiltonian system* (3) *is a Hamiltonian Normal Form by Bruno* (HNFB) *if the unperturbed Hamiltonian H_2 has the form* (17), *and in the perturbation $H - H_2$ a coefficient $h^{(p,q)} = 0$ if $\langle p - q, \lambda \rangle \neq 0$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\langle \xi, \eta \rangle = \xi_1 \eta_1 + \ldots + \xi_n \eta_n$.*

In [5] it is also proved that every Hamiltonian of order greater than or equal to two can be transformed to HNFB by a complex formal canonical transformation.

Theorem 3 HMNF *with the unperturbed Hamiltonian* (17) *is* HNFB.

Proof By (13) and (17), the operator $\widehat{H}_2^*$ is of the view

$$\widehat{H}_2^* = \sum_{j=1}^{n} \overline{\lambda}_j \left(y_j \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial x_j} \right) + \sum_{j=1}^{n-1} \varepsilon_j \left(y_j \frac{\partial}{\partial y_{j+1}} - x_{j+1} \frac{\partial}{\partial x_j} \right) - \sum_{j=1}^{n} \sigma_j x_j \frac{\partial}{\partial y_j}.$$

Consider the action of $\widehat{H}_2^*$ on $x^p y^q$.

$$\widehat{H}_2^* x^p y^q = \langle q - p, \overline{\lambda} \rangle x^p y^q + \sum_{j=1}^{n-1} \varepsilon_j (q_{j+1} x^p y^{q+e_j-e_{j+1}} - p_j x^{p-e_j+e_{j+1}} y^q) + \sum_{j=1}^{n} \sigma_j q_j x^{p+e_j} y^{q-e_j}. \tag{19}$$

As follows by (18) and (19), the operator $\widehat{H}_2^*$ preserves degree $|p+q|$ and a value $\langle q-p,\overline{\lambda}\rangle$, i.e. the operator $\widehat{H}_2^*$ maps the subspace

$$\mathfrak{P}_a^k = Lin\{x^p y^q : |p+q| = k, \langle q-p,\overline{\lambda}\rangle = a\} \subset \mathfrak{P}^k$$

to itself.

Let us introduce the following order on the set of multiple indices $\mathfrak{A} = \{(p,q) : |p+q| = k\}$: $(p',q') \prec (p'',q'')$ if $|p'| < |p''|$, or $|p'| = |p''|$ and p'' precedes p' in the lexicographic order, or $p' = p''$ and q' precedes q'' in the lexicographic order.

Let us choose the basis in $\mathfrak{P}_a^k$ consisting of monomials $x^p y^q$ arranged by ascending indices in the sense of the order introduced.

In (19) q precedes $q+e_j-e_{j+1}$ in lexicographic order, so $(p,q) \prec (p,q+e_j-e_{j+1})$, $p-e_j+e_{j+1}$ precedes p in lexicographic order, so $(p,q) \prec (p-e_j+e_{j+1},q)$, and $|p| < |p+e_j|$, so $(p,q) \prec (p+e_j,q-e_j)$.

Thus, $\widehat{H}_2^*$ maps every element of the considered basis into the subspace of $\mathfrak{P}_a^k$ based on the basis elements with indices greater than or equal to the initial indices (in the sense of the order introduced). Hence, in this basis the operator $\widehat{H}_2^*$ is represented by the upper triangular matrix with the entry a on the main diagonal.

There is a resonant polynomial in $\mathfrak{P}_a^k$ if and only if the determinant of this matrix is zero, which is equivalent to $a = 0$, and in its turn this is equivalent to the equality $\langle p-q,\lambda\rangle = 0$.

By Definition 8, this implies that only elements $h^{(p,q)}x^p y^q$ with $\langle p-q,\lambda\rangle = 0$ can be resonant. □

Let us show now that HMNF may have a simpler form than HNFB.

Consider an arbitrary Hamiltonian (12_2) with the following quadratic unperturbed part

$$H_2 = x_1 y_2 + y_1^2/2 \quad (n=2). \tag{20}$$

By Definition 10, it is HNFB, since $\lambda_1, \lambda_2 = 0$.

Proposition 6 HMNF *of the Hamiltonian* (12_2) *with* H_2 *from* (20) *has the form*

$$H = H_2 + \sum_{\substack{p_2 \geq q_1 + 2q_2 \\ (p_1,q_1) \neq (1,0)}} h^{(p,q)} x^p y^q. \tag{21}$$

Proof In this case, the resonance equation is of the view

$$\widehat{H}_2^* P = y_1 \frac{\partial P}{\partial y_2} - x_1 \frac{\partial P}{\partial y_1} - x_2 \frac{\partial P}{\partial x_1} = 0, \quad P \in \mathfrak{P}.$$

The last equation has three independent solutions: $P_1 = x_2$, $P_2 = x_2 y_1 - x_1^2/2$, $P_3 = x_2^2 y_2 + x_1 x_2 y_1 - x_1^3/3$, i.e. every resonant polynomial $P \in \mathfrak{P}$ is a polynomial in P_1, P_2, P_3. Therefore, every resonant element of $H - H_2$ is a product of the terms of P_1, P_2, P_3 with nonnegative integer indices, and so every resonant element is of the form (we drop the constant multiplier)

$$x_2^{i_1}(x_2 y_1)^{i_2} x_1^{2i_3}(x_2^2 y_2)^{i_4}(x_1 x_2 y_1)^{i_5} x_1^{3i_6} = x_1^{p_1} x_2^{p_2} y_1^{q_1} y_2^{q_2},$$

where $p_1 = 2i_3 + i_5 + 3i_6$, $p_2 = i_1 + i_2 + 2i_4 + i_5$, $q_1 = i_2 + i_5$, $q_2 = i_4$.

These indices satisfy the condition $p_2 \geq q_1 + 2q_2$. In addition $q_1 \neq 0$ when $p_1 = 1$. Thus, we obtain HMNF (21). □

5 HNF by Meyer, Relationship to HMNF

Consider a real Hamiltonian (12_2) with the unperturbed part written in the form

$$H_2 = \frac{1}{2} z^T G z \quad (G \text{ is a symmetric matrix}), \tag{22}$$

Consider also a linear Hamiltonian system of differential equations with the Hamiltonian H_2 from (22):

$$\dot{z} = Az, \tag{23}$$

where $A = IG$ and I is from (5).

In [8] it is proved that if the matrix A is symmetric, then the Hamiltonian (12_2) with the unperturbed part (22) can be transformed to the Hamiltonian H' such that

$$H_2' = H_2; \quad \forall k \geq 3,\ \forall t \in \mathbb{R},\ \forall z \in \mathbb{R}^{2n}:\ H_k'(e^{At}z) \equiv H_k'(z)$$

by a real formal canonical near-identity transformation $z = w + W(w)$ with $\operatorname{ord} W \geq 2$.

Later, in [9], this result was extended to the case of an arbitrary matrix $A = IG$.

Definition 11 *We say that a real Hamiltonian* (12_2) *with the unperturbed part* (22) *or the corresponding Hamiltonian system* (3) *is a Hamiltonian Normal Form by Meyer* (HNFM) *if*

$$\forall k \geq 3,\ \forall t \in \mathbb{R},\ \forall z \in \mathbb{R}^{2n}:\ H_k(e^{A^T t}z) \equiv H_k(z) \quad (A = IG). \tag{24}$$

By [9, Theorem 10.4.2], an arbitrary real Hamiltonian (12_2) can be transformed to HNFM by a real formal canonical near-identity transformation $z = w + W(w)$ with $\operatorname{ord} W \geq 2$.

Theorem 4 HNFM *is an* HMNF *with the perturbation consisting of the resonant polynomials.*

Proof Let the Hamiltonian (12_2) with the unperturbed part (22) be a HNFM.

The linear system of differential equations adjoint of (23) is of the view

$$\dot{z} = A^T z \tag{25}$$

and is Hamiltonian as well with the Hamiltonian $H_2^T(z) = \dfrac{1}{2} z^T G' z$, where $G' = IGI$.

The equality (24) means that homogeneous summands of HNFM are the integrals of the linear system (25). Since the system (25) is Hamiltonian with the Hamiltonian H_2^T, the last is equivalent to the identities

$$\{H_2^T, H_k\} = 0 \quad (k \geq 3). \tag{26}$$

By [9, Lemma 10.4.2], the operator $\widehat{H}_2^T$ is the adjoint one of $\widehat{H}_2$ with respect to the inner product $\langle\langle\,.\,,\,.\,\rangle\rangle$, i.e. $\widehat{H}_2^T = \widehat{H}_2^*$. From this and (26) one can see that the homogeneous summands H_k ($k \geq 3$) of the perturbation of HNFM are solutions of the resonance equation for H_2. Hence, by Definition 9 and the 1^{st} property of the inner product $\langle\langle\,.\,,\,.\,\rangle\rangle$, the HNFM considered is an HMNF and its perturbation $H - H_2$ consists of the resonant polynomials. □

Coefficients of the resonant elements in the perturbation of HNFM are defined by the solutions of the resonance equation, while in HMNF they have arbitrary values.

For example, the perturbation of the HNFM of the Hamiltonian (12_2) with H_2 from (20) is a series in resonant polynomials P_1, P_2, P_3 from the proof of Proposition 6, which is a particular case of (21).

6 Hamiltonian Normal Form

6.1 NF of the Hamiltonian with a Homogeneous Unperturbed Part

Let $\{P_i\}_{i=1}^k$ be a basis of the space $\mathfrak{J}^{r+m}$ $(m \geq 1)$ of homogeneous solutions of the resonance equation (14).

Definition 12 *We say that a set of monomials* $\mathfrak{R}^m = \{R_i = x^{p^i} y^{q^i} : \ |p^i + q^i| = r + m\}_{i=1}^k$, *where* p^i, q^i *are multiple indices, forms a minimal resonant set in the order* $r+m$ *if determinant of the matrix of inner products* $A = \{a_{ij} = \langle\langle P_i, R_j\rangle\rangle\}_{i,j=1}^k$ *is nonzero. We call the set* $\mathfrak{R} = \cup_{m\geq 1}\mathfrak{R}^m$ *a minimal resonant set.*

Definition 13 *We say that Hamiltonian* (12_r) *or the corresponding Hamiltonian system* (3) *is a Hamiltonian Normal Form* (HNF) *if the perturbation* $H - H_r$ *contains only resonant elements belonging to some minimal resonant set* $\mathfrak{R}$, *i. e.* $h^{(p,q)} = 0$ *if* $x^p y^q \notin \mathfrak{R}$.

Remark 3 According to the theory of nonHamiltonian normal forms, in case $r = 2$ and the unperturbed Hamiltonian is of the view (17) with $\lambda \neq 0$, HNF may be called resonant (RHNF), and in case $r \geq 3$, or $r = 2$ and the unperturbed Hamiltonian is of the view (17) with $\lambda = 0$, we may use the term "generalized Hamiltonian normal form" (GHNF).

Theorem 5 *Let* $\mathfrak{R} = \cup_{m\geq 1}\mathfrak{R}^m$ *be a minimal resonant set. Then there is a formal canonical transformation that transforms the Hamiltonian* (12_r) *to* HNF *in which* $h^{(p,q)} = 0$ *if* $x^p y^q \notin \mathfrak{R}$.

Proof Denote $c = \{c_i = \langle\langle H_{r+m}, P_i\rangle\rangle\}_{i=1}^{k_m}$, $H'_{r+m} = \sum_{i=1}^{k_m} b_i R_i^m$, where $R_i^m \in \mathfrak{R}^m$, $b = A^{-1}c$, and A is the matrix from Definition 12.

Then $\langle\langle H'_{r+m} - H_{r+m}, P_i\rangle\rangle = \sum_{j=1}^{k_m} a_{ij} b_j - c_i = 0$ for all $i = \overline{1, k_m}$.

The remaining part of the proof is similar to the proof of Theorem 2. □

Let us introduce an example of HNF that has a simpler form than HMNF.

Proposition 7 *The Hamiltonian* (12_2) *with the unperturbed part* (20) *can be transformed to the following* HNF

$$H = H_2 + \sum_{p_1=0,\ p_2 \geq q_1 + 2q_2} h^{(p,q)} x^p y^q \tag{27}$$

by a formal canonical transformation.

Proof Indeed, in the proof of Proposition 6 it was shown that polynomials $P_1^{i_1} P_2^{i_2} P_3^{i_3}$, where $P_1 = x_2$, $P_2 = x_2 y_1 - x_1^2/2$, $P_3 = x_2^2 y_2 + x_1 x_2 y_1 - x_1^3/3$, form a basis of the space $\mathfrak{J}$.

The minimal resonant set can be chosen as a set of monomials of the view

$$x_2^{i_1}(x_2 y_1)^{i_2}(x_2^2 y_2)^{i_3} = x_1^{p_1} x_2^{p_2} y_1^{q_1} y_2^{q_2},$$

where $p_1 = 0$, $p_2 = i_1 + i_2 + 2i_3$, $q_1 = i_2$, $q_2 = i_3$, because every such monomial has a nonzero inner product with the only basis element $P_1^{i_1} P_2^{i_2} P_3^{i_3}$. Hence, by Theorem 5, the proof is complete. □

Remark 4 Unlike HMNF (21) and the corresponding HNFM which perturbation depends on the resonant polynomials P_1, P_2, P_3, the perturbation of HNF (27) is independent of x_1.

6.2 NF of the Hamiltonian with a Quasi-Homogeneous Unperturbed Part

Till now a homogeneous polynomial H_r of degree $r \geq 2$ was chosen as the unperturbed part of the Hamiltonian. However, the less variables it depends on, or identically, the more components of the unperturbed part of the corresponding Hamiltonian system are identically zero, the less means for reducing the terms in each order of the perturbation do we have.

This fact is well-known and is illustrated in the theory of generalized (nonHamiltonian) normal forms and will be verified in section 7.4.

The situation seems perfect when the polynomial H_r is ultimately simplified by a linear canonical transformation and depends on all variables. Such an unperturbed Hamiltonian will be naturally called a nondegenerate one.

To obtain a nondegenerate unperturbed Hamiltonian one may use the following recipe that is widely used in the theory of generalized normal forms. Namely, one can artificially supplement H_r with some summands of the perturbation of the Hamiltonian (12_r) and add some summands of H_r to the perturbation so that after introducing new, generalized, degrees, the new unperturbed Hamiltonian is homogeneous in a definite sense and is of less generalized degree than the generalized degrees of perturbation terms.

Let us introduce several definitions from [1, 3] modified for the case when the considered system of differential equations is Hamiltonian.

Definition 14 *Vector $\gamma = (\alpha, \beta)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, is called a weight of variable $z = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ if the components of γ are positive coprime integers such that $\alpha_i + \beta_i = \delta$ $(\delta \geq 2,\ i = \overline{1, n})$.*

Let $\gamma = (\alpha, \beta)$ be a weight of variable $z = (x, y)$.

Definition 15 *We say that a monomial $Z^{(p,q)} x^p y^q$ is of generalized degree $k = \langle p, \alpha \rangle + \langle q, \beta \rangle$ with weight γ.*

Definition 16 *Polynomial $Q(z)$ is called a quasi-homogeneous polynomial* (QHP) *of generalized degree k with weight γ and is denoted by $Q_\gamma^{[k]}(z)$ if it consists of monomials of generalized degree k with weight γ only.*

All definitions of the previous sections can be transfered to the case of a quasi-homogeneous unperturbed Hamiltonian by the following recipe. Instead of the ordinary degrees one should consider generalized degrees with weight γ, and everywhere the term "homogeneous" should be replaced with "quasi-homogeneous with weight γ." In particular, by the order of f one should understand the least r such that $f_\gamma^{[r]} \not\equiv 0$.

Further we omit the words "with weight γ" for short.

The Poisson bracket of two QHPs of generalized degrees k, l is again a QHP of generalized degree $k + l - \delta$. Hence, for every series f of the order greater than δ the Lie transform $exp(\widehat{f}\,)$ is defined on the space of series.

By the introduced definitions and by the properties of the inner product $\langle\langle\,.\,,\,.\,\rangle\rangle$, the space $\mathfrak{P}$ is the direct sum of the orthogonal subspaces $\mathfrak{P}_\gamma^{[k]}$ of QHPs of generalized degree k.

The resonance equation for a quasi-homogeneous unperturbed Hamiltonian $H_\gamma^{[x]}$ is of the form

$$\left(\widehat{H}_\gamma^{[x]}\right)^* P = 0, \quad P \in \mathfrak{P},$$

and the space $\mathfrak{J}$ of its solutions is the direct sum of the orthogonal subspaces $\mathfrak{J}_\gamma^{[k]}$ of quasi-homogeneous solutions of generalized degree k.

Everywhere instead of the spaces $\mathfrak{P}^k$ and $\mathfrak{J}^k$ one should consider the spaces $\mathfrak{P}_\gamma^{[k]}$ and $\mathfrak{J}_\gamma^{[k]}$.

In definitions of HNF and HMNF, H_r should be replaced with $H_\gamma^{[x]}$, and in the definition of the minimal resonant set, $\mathfrak{R}^m$ should be replaced with

$$\mathfrak{R}_\gamma^{[m]} = \{R_i = x^{p^i} y^{q^i} : \langle \alpha, p^i \rangle + \langle \beta, q^i \rangle = r + m\}_{i=1}^k .$$

With a view of the above, Theorems 2 and 5 also transfer to the case of a quasi-homogeneous unperturbed Hamiltonian.

7 HNFs of Some One Degree of Freedom Systems

Here we will consider the case of $n = 1$ and $\mathbb{K} = \mathbb{R}$. Still, H_r is the unperturbed part of the Hamiltonian (12_r) : $H(x, y) = H_r(x, y) + \sum_{|p+q| \geq r+1} h^{(p,q)} x^p y^q \quad (r \geq 2)$.

In the formulas for HNF and HMNF given below it is assumed that the perturbation summands are of greater degrees than the degree of the unperturbed Hamiltonian, so we omit the corresponding inequalities in the summation limits for short.

7.1 The Unperturbed Hamiltonian is a Monomial in One Variable

Let the unperturbed Hamiltonian H_r be independent of one of the variables, for example, of x.

Theorem 6 HNF *with the degenerate unperturbed Hamiltonian* $H_r = hy^r$ $(h \in \mathbb{R}\backslash\{0\})$ *coincides with the corresponding* HMNF *and has the form*

$$H = hy^r + \sum_{q \leq r-2} h^{(p,q)} x^p y^q. \tag{28}$$

Proof Indeed, according to (13), $\widehat{H}_r^* = -x(\partial H_r/\partial y)^* = -hrx\partial^{r-1}/\partial y^{r-1}$, and the resonance equation (14) is equivalent to $\partial^{r-1} P/\partial y^{r-1} = 0$. Its solutions are the polynomials with the degree of y less than or equal to $r - 2$. Hence, by Definition 9, we obtain HMNF (28).

Monomials $\{P_i = x^{p_i} y^{q_i} |\ p_i = r + m + 1 - i,\ q_i = i - 1\}_{i=1}^{r-1}$ form a basis in the space of solutions of (14) of degree $r + m$ and by Definition 11, they form a minimal resonant set in the order $r + m$. Therefore, the obtained HMNF is HNF by Definition 13. □

Rewrite the Hamiltonian (28) as a sum of homogeneous polynomials:

$$H = hy^r + \sum_{k=r+1}^{\infty} \sum_{q=0}^{r-2} h^{(k-q,q)} x^{k-q} y^q.$$

Let us consider two important particular cases of the HNF (28). Namely, $r = 2$ and $r = 3$.

1) In case $r = 2$, the perturbation in (28) is independent of y, and the HNF itself coincides with the corresponding HNFM (see [9, p. 263]) and has the form

$$H = hy^2 + \sum_{k=3}^{\infty} h^{(k,0)} x^k. \tag{28_2}$$

2) Let $r = 3$ and $h = 1/3$. Then HNF (28) has the form

$$H = y^3/3 + \sum_{k=4}^{\infty} (h^{(k,0)} x^k + h^{(k-1,1)} x^{k-1} y). \tag{28_3}$$

The corresponding Hamiltonian system is of the view

$$\dot{x} = y^2 + \sum_{k=3}^{\infty} h^{(k,1)} x^k, \quad \dot{y} = -\sum_{k=3}^{\infty} ((k+1) h^{(k+1,0)} x^k + k h^{(k,1)} x^{k-1} y). \tag{$28_3^{\mathtt{s}}$}$$

According to [2, Theorem 11], arbitrary (nonHamiltonian) two-dimensional system of ordinary differential equations with the unperturbed part $(y^2, 0)$ is formally equivalent to the following generalized normal forms (GNF):

$$\dot{x} = y^2 + \sum_{p=2}^{\infty} (X^{(p+1,0)} x^{p+1} + X^{(p,1)} x^p y), \quad \dot{y} = \sum_{p=2}^{\infty} (Y^{(p,1)} x^p y + Y^{(p+1,0)} x^{p+1}); \tag{$28_{3.1}^{\mathtt{gnf}}$}$$

$$\dot{x} = y^2 + \sum_{p=2}^{\infty} X^{(p+1,0)} x^{p+1}, \quad \dot{y} = \sum_{p=2}^{\infty} (Y^{(p-1,2)} x^{p-1} y^2 + Y^{(p,1)} x^p y + Y^{(p+1,0)} x^{p+1}). \tag{$28_{3.2}^{\mathtt{gnf}}$}$$

The first component of the perturbation in ($28_3^{\mathtt{s}}$) coincides with the first component of the perturbation in GNF ($28_{3.2}^{\mathtt{gnf}}$), and the second one coincides with the second component of the perturbation in GNF ($28_{3.1}^{\mathtt{gnf}}$).

Thus, we come to a conclusion that due to the hamiltonicity of the initial system with the unperturbed part $(y^2, 0)$, it becomes possible to reduce it to a GNF with fewer resonant terms in each order of the perturbation.

Note that in the definition of HNF we consider the equivalence relatively to the group of formal canonical transformations, while in the definition of GNF (see [1]) equivalence relatively to a wider group of invertible formal transformations is considered.

7.2 The Unperturbed Hamiltonian is a Monomial in Two Variables

Theorem 7 HNF *with the nondegenerate unperturbed Hamiltonian* $H_r = hx^m y^l$, *where* $m + l = r$, $d = \gcd(m, l)$ $(m, l \geq 1,\ h \in \mathbb{R}\backslash\{0\})$, *coincides with the corresponding* HMNF *and has the form*

$$H = hx^m y^l + \sum_{p \leq m-2} h^{(p,q)} x^p y^q + \sum_{\substack{p \geq m-1 \\ q \leq l-2}} h^{(p,q)} x^p y^q + \sum_{j=d}^{\infty} h^{(jm/d-1,\, jl/d-1)} x^{jm/d-1} y^{jl/d-1}. \tag{29}$$

Proof According to (13), $\widehat{H}_r^* = (\partial H_r/\partial x)^* y - (\partial H_r/\partial y)^* x = hm\partial_x^{m-1}\partial_y^l y - hl\partial_x^m\partial_y^{l-1}x$, and the resonance equation (14) is equivalent to

$$\partial_x^{m-1}\partial_y^{l-1}(m\partial_y yP - l\partial_x xP) = 0,$$

where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$. We are looking for solutions of this equation in the form $P = x^p y^q$. Substituting into the resonance equation gives

$$\partial_x^{m-1}\partial_y^{l-1}(m(q+1) - l(p+1))x^p y^q = 0.$$

This equality is satisfied in one of the following mutually exclusive cases:

1) $p \le m-2$;

2) $p \ge m-1$, $q \le l-2$;

3) $p = jm/d - 1$, $q = jl/d - 1$, where $j \ge d$.

The obtained monomial solutions form a basis of the space $\mathfrak{J}$. Hence, by Definition 9, we obtain HMNF (29).

As well as in the previous example, the minimal resonant set in each order is unique and consists of the monomial solutions found. Therefore, the obtained HMNF is HNF. □

Rewrite the Hamiltonian (29) as a sum of homogeneous polynomials:

$$H = hx^m y^l + \sum_{k=r+1}^{\infty}\left(\sum_{p=0}^{m-2} h^{(p,k-p)}x^p y^{k-p} + x^{m-1}\sum_{q=0}^{l-2} h^{(k+m-1-q,q)}x^{k-q}y^q + \eta_k(x,y)\right),$$

where

$$\eta_{k-2}(x,y) = \begin{cases} h^{(km/r-1,kl/r-1)}x^{km/r-1}y^{kl/r-1}, & \text{if } k \,\vdots\, (r/d); \\ 0, & \text{if } k \not\vdots\, (r/d). \end{cases}$$

Let us consider two important particular cases of the HNF (29). Namely, $r = 2$ and $r = 3$.

1) Let $r = 2$. Then in HNF (29) the indices l, $m = 1$; therefore, $H_2 = hxy$, the first two sums in the right part of (29) disappear, and the third one consists of powers of the product xy. Hence, HNF (29) has the form

$$H = hxy + \sum_{j=2}^{\infty} h^{(j,j)}x^j y^j. \qquad (29_2)$$

The obtained result agrees with [9, Corollary 10.4.1]. In this case, HNF (29_2) coincides with HNFB and HNFM.

2) Let $r = 3,\ l = 1,\ m = 2,\ h = -1/2$. Then $H_3 = -x^2y/2$, and HNF (29) has the form

$$H = -\frac{1}{2}x^2 y + \sum_{k=4}^{\infty} h^{(0,k)}y^k + \sum_{j=1}^{\infty} h^{(2j+1,j)}x^{2j+1}y^j, \qquad (29_3)$$

and the Hamiltonian system in normal form generated by HNF (29_3) is of the view

$$\dot{x} = -\frac{1}{2}x^2 + \sum_{k=3}^{\infty}(k+1)h^{(0,k+1)}y^k + \sum_{j=1}^{\infty} jh^{(2j+1,j)}x^{2j+1}y^{j-1},\ \dot{y} = xy - \sum_{j=1}^{\infty}(2j+1)h^{(2j+1,j)}x^{2j}y^j. \qquad (29_3^{\mathrm{s}})$$

It seems interesting to compare the system ($29^{\mathfrak{s}}_3$) with a GNF that has the same Hamiltonian unperturbed part as in ($29^{\mathfrak{s}}_3$) and arbitrary (nonHamiltonian) perturbation.

The unperturbed part of ($29^{\mathfrak{s}}_3$) — a vector-polynomial $(-x^2/2, xy)$ — is one of the nineteen linearly inequivalent canonical forms to which the unperturbed part of a two-dimensional system represented by an arbitrary nondegenerate vector quadratic polynomial can be reduced by a linear nonsingular transformation (see [4, Section 5.2], [2, § 2]). Namely, according to the classification given in [2], $(-x^2/2, xy) = \mathrm{CF}^1_1$ with $u = -1/2$. In [4, § 5] all GNFs formally equivalent to the systems with the unperturbed part $(\alpha x^2, xy)$ are given explicitly.

7.3 The Unperturbed Hamiltonian is an Irreducible Binomial with Coprime Indices

Consider an irreducible binomial $(h_1/\beta)x^\beta - (h_2/\alpha)y^\alpha$ in which $h_1h_2 \neq 0$ and $\gcd(\alpha, \beta) = 1$. By Definition 16, such a binomial is a QHP of generalized degree $\alpha\beta$ with weight (α, β).

Consider a Hamiltonian H_γ with a quasi-homogeneous nondegenerate unperturbed part

$$H^{[\chi]}_\gamma = (h_1/\beta)x^\beta - (h_2/\alpha)y^\alpha, \tag{30}$$

where $\gcd(\alpha, \beta) = 1$, $h_1h_2 \neq 0$, $\chi = \alpha\beta$, $\gamma = (\alpha, \beta)$.

One can assume that it was obtained from the degenerate unperturbed Hamiltonian $H_r = hy^r$ (if $\alpha < \beta$) by adding the term of the Hamiltonian (28) depending on x.

The resonance equation for the quasi-homogeneous unperturbed Hamiltonian $H^{[\chi]}_{(\alpha,\beta)}$ is of the form

$$h_1 y \partial_x^{\beta-1} P + h_2 x \partial_y^{\alpha-1} P = 0, \quad P \in \mathfrak{P}. \tag{31}$$

By the notation given in 6.2, $\mathfrak{I}^{[k]}_{(\alpha,\beta)}$ is the space of quasi-homogeneous solutions of (31) of generalized degree k with weight (α, β).

Lemma 2 *$x^py^q \in \mathfrak{I}^{[k]}_{(\alpha,\beta)}$ if and only if $k = \alpha p + \beta q$, $p \leq \beta - 2$ and $q \leq \alpha - 2$.*

Proof The assertion is verified by direct substitution x^py^q into (31). □

Lemma 3 *Let $x^py^q \in \mathfrak{I}^{[k]}_{(\alpha,\beta)}$, $x^{p'}y^{q'} \in \mathfrak{I}^{[k']}_{(\alpha,\beta)}$. Then either $(p, q) = (p', q')$ or $k \not\equiv k' \mod \chi$.*

Proof Let $\alpha p + \beta q \equiv \alpha p' + \beta q' \mod \chi$. Then $\alpha p \equiv \alpha p' \mod \beta$ and $\beta q \equiv \beta q' \mod \alpha$. Since α and β are relatively prime, the last equalities are equivalent to the following ones

$$p \equiv p' \mod \beta, \quad q \equiv q' \mod \alpha. \tag{32}$$

Now let $x^py^q \in \mathfrak{I}^{[k]}_{(\alpha,\beta)}$, $x^{p'}y^{q'} \in \mathfrak{I}^{[k']}_{(\alpha,\beta)}$, and $k \equiv k' \mod \chi$. By Lemma 2, $p, p' \leq \beta - 2$, $q, q' \leq \alpha - 2$. From this and from (32) it follows that $(p, q) = (p', q')$. □

Lemma 4 *Let $P \in \mathfrak{I}^{[k]}_{(\alpha,\beta)}$ and $P \not\equiv 0$. Then there exists a unique $Q \in \mathfrak{I}^{[k+\chi]}_{(\alpha,\beta)}$ such that*

$$xP = h_1 \partial_x^{\beta-1} Q, \quad yP = -h_2 \partial_y^{\alpha-1} Q. \tag{33}$$

Moreover, if $\mathfrak{I}^{[k]}_{(\alpha,\beta)} \neq \{0\}$, then there exists such $k' \geq 0$ that $k \equiv k' \mod \chi$ and $\mathfrak{I}^{[k']}_{(\alpha,\beta)}$ contains a monomial solution of the equation (31), and there is an isomorphism of the spaces $\mathfrak{I}^{[k]}_{(\alpha,\beta)}$ and $\mathfrak{I}^{[k']}_{(\alpha,\beta)}$.

Proof Existence of Q. Let

$$Q = \frac{1}{h_1} \underbrace{\int \cdots \int}_{\beta-1 \text{ times}} xP dx^{\beta-1} + \sum_{j=0}^{\beta-2} R_j(y) x^j,$$

where R_j are arbitrary polynomials. Then the first equality in (33) is satisfied identically, and after substituting Q into the second equality by the equation (31), we get

$$h_2 \partial_y^{\alpha-1} Q = -y \underbrace{\int \cdots \int}_{\beta-1 \text{ times}} \partial_x^{\beta-1} P dx^{\beta-1} + \sum_{j=0}^{\beta-2} h_2 x^j \partial_y^{\alpha-1} R_j(y).$$

For any primitive in the first term, one can choose such polynomials R_j that the second equality in (33) is satisfied, which proves the existence of the polynomial Q satisfying (33). In its turn, (33) with the quasi-homogeneity of P implies that Q can be chosen quasi-homogeneous.

Multiplying the first equality from (33) by y and subtracting the second one multiplied by x yields that Q satisfies the equation (31).

Uniqueness of Q. Let $Q, Q' \in \mathfrak{I}^{[k+\chi]}_{(\alpha,\beta)}$ satisfy the relations (33) with the same $P \in \mathfrak{I}^{[k]}_{(\alpha,\beta)}$, $P \not\equiv 0$. Then $\partial_x^{\beta-1}(Q - Q') = 0$ and $\partial_y^{\alpha-1}(Q - Q') = 0$. From this by Lemma 2, solutions Q and Q' can differ only by monomial solutions of the resonance equation (31), and from the quasi-homogeneity of $Q - Q'$ and from Lemma 3, we get

$$Q - Q' = Cx^p y^q, \tag{34}$$

where $k + \chi = \alpha p + \beta q$, $p \leq \beta - 2$ and $q \leq \alpha - 2$.

Denote $P_1 = P$. Let us construct a sequence $P_j \in \mathfrak{I}^{[k-(j-1)\chi]}_{(\alpha,\beta)}$ by sequentially applying the relations (33):

$$xP_{j+1} = h_1 \partial_x^{\beta-1} P_j, \quad yP_{j+1} = -h_2 \partial_y^{\alpha-1} P_j.$$

By the resonance equation (31), all P_j are uniquely determined by P_1.

There is such a number $m \in \{1, \ldots, [k/\chi] + 1\}$ that $P_m \not\equiv 0$ and $P_{m+1} \equiv 0$.

It follows from the relations (33) that P_m is a monomial solution of generalized degree $k - (m-1)\chi < k + \chi$. By Lemma 3, from this in (34) $C = 0$, which is equivalent to $Q = Q'$.

Incidentally, for every nonmonomial QHP $Q \in \mathfrak{I}_{(\alpha,\beta)}$ (Q is nonmonomial due to $P \not\equiv 0$ and Lemma 2) we have found a unique monomial solution P_m such that Q is obtained from P_m by applying relations (33) m-times, which gives the one-one correspondence between the solutions $Q \in \mathfrak{I}^{[k+\chi]}_{(\alpha,\beta)}$ and the monomial solutions of the equation (31) of generalized degree $k - (m-1)\chi$.

Since the relations (33) are linear by P and Q, this correspondence is an isomorphism of spaces. □

Theorem 8 HMNF *with the quasi-homogeneous unperturbed Hamiltonian* (30) *is of the view*

$$H = (h_1/\beta)x^\beta - (h_2/\alpha)y^\alpha + \sum_{\substack{p \not\equiv -1 \bmod \beta \\ q \not\equiv -1 \bmod \alpha}} h^{(p,q)} x^p y^q \quad (\alpha p + \beta q \geq \chi + 1), \tag{35}$$

and all the HNF*s are given by*

$$H = H^{[\chi]}_{(\alpha,\beta)} + \sum_{k=\chi+1}^{\infty} H^{[k]}_{(\alpha,\beta)}, \tag{36}$$

where $H^{[k]}_{(\alpha,\beta)} = h^{(p,q)} x^p y^q$ *with* $k = \alpha p + \beta q \geq \chi + 1$ *and* $p \not\equiv -1 \bmod \beta$, $q \not\equiv -1 \bmod \alpha$.

Proof Let the monomial $x^p y^q$ appear in $P \in \mathfrak{J}^{[k]}_{(\alpha,\beta)}$ $(k = \alpha p + \beta q \geq 0)$ with a nonzero coefficient. Namely, $\langle\langle P, x^p y^q \rangle\rangle \neq 0$.

By Lemma 4, there exists a unique $Q \in \mathfrak{J}^{[k+\chi]}_{(\alpha,\beta)}$ such that the equalities (33) are satisfied. Hence, $\langle\langle xP, x^{p+1}y^q \rangle\rangle \overset{(33)}{=} \langle\langle h_1 \partial_x^{\beta-1} Q, x^{p+1} y^q \rangle\rangle \neq 0$ and $\langle\langle yP, x^p y^{q+1} \rangle\rangle \overset{(33)}{=} \langle\langle -h_2 \partial_y^{\alpha-1} Q, x^p y^{q+1} \rangle\rangle \neq 0$. By the 2^{nd} property of the inner product $\langle\langle\, . \,,\, . \,\rangle\rangle$, we get

$$\langle\langle Q, x^{p+\beta} y^q \rangle\rangle \neq 0, \quad \langle\langle Q, x^p y^{q+\alpha} \rangle\rangle \neq 0. \tag{37}$$

Let monomial $x^{p+\beta} y^q$ appear in a nonmonomial $Q \in \mathfrak{J}^{[k+\chi]}_{(\alpha,\beta)}$ with a nonzero coefficient, i. e. $\langle\langle Q, x^{p+\beta} y^q \rangle\rangle \neq 0$. Then by Lemma 4, there exists $P \in \mathfrak{J}^{[k]}_{(\alpha,\beta)}$, $P \not\equiv 0$ satisfying (33). By applying relations (33) and the 2^{nd} property of the inner product $\langle\langle\, . \,,\, . \,\rangle\rangle$, we get $\langle\langle h_1 \partial_x^{\beta-1} Q, x^{p+1} y^q \rangle\rangle \overset{(33)}{=} \langle\langle xP, x^{p+1} y^q \rangle\rangle \neq 0$, which implies $\langle\langle P, x^p y^q \rangle\rangle \neq 0$. Hence, $\langle\langle Q, x^p y^{q+\alpha} \rangle\rangle \neq 0$ by (37).

It can be proved in the same manner that $\langle\langle Q, x^p y^{q+\alpha} \rangle\rangle \neq 0$ implies $\langle\langle P, x^p y^q \rangle\rangle \neq 0$ and $\langle\langle Q, x^{p+\beta} y^q \rangle\rangle \neq 0$.

Therefore, for every p and q such that $\alpha p + \beta q \geq \chi + 1$, monomials $x^p y^q$, $x^{p+\beta} y^q$, $x^p y^{q+\alpha}$ are either all in one resonant or all in one nonresonant.

From this, in view of Lemma 4, we obtain that the resonant elements must be of the form $x^{p+j\beta} y^{q+k\alpha}$, where $x^p y^q \in \mathfrak{J}_{(\alpha,\beta)}$.

Hence, by Lemma 2, HMNF with the unperturbed part (30) has the form (35).

By Lemmas 3 and 4, it follows that for every $k \geq 0$, the inequality $\dim \mathfrak{J}^{[k]}_{(\alpha,\beta)} \leq 1$ is satisfied.

Hence, every minimal resonant set can be obtained by choosing the only resonant element in each generalized degree of the perturbation of HMNF (35). Eventually, we get HNF (36). □

Corollary 1 *The set* $\mathfrak{R} = \{x^{p+k\beta} y^q |\ p \leq \beta - 2,\ q \leq \alpha - 2,\ k \geq 0\}$ *represents a minimal resonant set for the unperturbed Hamiltonian* (30), *and the series*

$$H = (h_1/\beta)x^\beta - (h_2/\alpha)y^\alpha + \sum_{\substack{p \not\equiv -1 \bmod \beta \\ q \leq \alpha - 2}} h^{(p,q)} x^p y^q \tag{38}$$

is a HNF corresponding to the minimal resonant set $\mathfrak{R}$.

7.4 Comparison of HNFs with Degenerate and Nondegenerate Unperturbed Hamiltonians

Consider the particular case of Hamiltonians from the sections 7.1, 7.3, namely, the Hamiltonian

$$H = y^2/2 - x^3/3 + h^{(2,1)}x^2y + h^{(1,2)}xy^2 + h^{(0,3)}y^3 + \sum_{p+q\geq 4} h^{(p,q)}x^py^q.$$

The homogeneous unperturbed Hamiltonian has the form $H_2 = y^2/2$. It does not depend on x, and hence, by Theorem 6, HNF coincides with HMNF and is of the view

$$H = y^2/2 + \sum_{k=3}^{\infty} h^{(k,0)}x^k. \tag{39}$$

On the other hand, one can choose a nondegenerate QHP $H^{[6]}_{(2,3)} = y^2/2 - x^3/3$ as the unperturbed part of H. Then by Theorem 8 with $\chi = 6$, $(\alpha, \beta) = (2, 3)$, we obtain the following HMNF:

$$H = y^2/2 - x^3/3 + \sum_{\substack{p \not\equiv 2 \bmod 3 \\ q \equiv 0 \bmod 2}} h^{(p,q)}x^py^q \qquad (2p + 3q \geq 7).$$

HNF (36) corresponding to the minimal resonant set $\mathfrak{R}$ chosen in Corollary 1 with $\chi = 6$, $(\alpha, \beta) = (2, 3)$ is of the view

$$H = y^2/2 - x^3/3 + \sum_{j=1}^{\infty}(h^{(3j+1,0)}x^{3j+1} + h^{(3j+3,0)}x^{3j+3}) \tag{40}$$

according to (38). The corresponding system of differential equations is of the form

$$\dot{x} = y, \quad \dot{y} = x^2 - \sum_{j=1}^{\infty} \left((3j+1)h^{(3j+1,0)}x^{3j} + (3j+3)h^{(3j+3,0)}x^{3j+2}\right). \tag{41}$$

Comparing expressions (39) and (40), one can see that accounting the term $-x^3/3$ together with a proper choice of a minimal resonant set allows us to strengthen the Hamiltonian normal form.

In [3], two-dimensional system of ordinary differential equations with the linear-quadratic unperturbed part (y, x^2) is considered relatively to the group of invertible formal transformations. According to [3, §5, Theorem 4], such a system can be transformed to a GNF with the same unperturbed part in which for every $k \geq 1$, all coefficients of the forms $Y^{[6k-2]}$, $Y^{[6k-1]}$, $Y^{[6k+1]}$ are zero, and all coefficients, except maybe one in each of the forms $Y^{[6k-4]}$, $Y^{[6k-3]}$, $Y^{[6k]}$, are zero. Such a GNF differs from the HNF (41) in which after the reexpansion by the generalized degrees, by the notation given in [3], vectors $Y^{[6k-3]}$ and $Y^{[6k+1]}$ can be nonzero, and the other vectors are zero.

The difference between the normal forms is explained by the fact that in one case, the conditions on the normal form are: the Hamiltonian character of the initial system on the one hand, and the equivalence relatively to the group of formal canonical transformations on the other hand. In other case, the condition of hamiltonicity is withdrawn, at the same time equivalence relatively to a wider group of invertible formal transformations is considered.

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