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A NEW APPROACH OF REFRACTION FOR 3D FIELD IN ANISOTROPIC PERMANENT MAGNETS WITH RANDOM MAGNETIZATION MAIN DIRECTIONS

IOAN BERE

Prof., Dr. Sci., "Politehnica" University of Timisoara
Department of Electrotechnics,
2 Vasile Parvan blvd., 300223
Timisoara, Romania
E-mail: ioan.bere@et.upt.ro

Abstract

Using a new permeability - defined for anisotropic permanent magnets - we will demonstrate the refraction theorems of the three–dimensional (3D) magnetic field lines at the separation surface of two anisotropic materials with permanent magnetization (two permanent magnets), which have random magnetization main directions. Also, the general forms of demonstrated theorems are particularized for diverse concrete cases and an example is given to illustrate the new defined quantities.

Keywords: nonlinear and anisotropic permanent magnets, another permeability, random magnetization, refraction theorems

1. INTRODUCTION

For materials with permanent magnetization, the relation law between flux density $\boldsymbol{B}$, magnetic field intensity $\boldsymbol{H}$ and magnetization $\boldsymbol{M}$ [1,2] it's

$$\boldsymbol{B} = \mu_0\, \boldsymbol{H} + \mu_0\, \boldsymbol{M}_\tau + \mu_0\, \boldsymbol{M}_p\,, \tag{1}$$

where μ_0 is the vacuum permeability. The separation in temporary ($\boldsymbol{M}_\tau$) and permanent ($\boldsymbol{M}_p$) components of magnetization $\boldsymbol{M}$ is unique only if $\boldsymbol{M}_p$ is independent of $\boldsymbol{H}$ and $\boldsymbol{M}_\tau$ – depending on $\boldsymbol{H}$ – is null at the same time with $\boldsymbol{H}$. The value of $\boldsymbol{B}$ for $\boldsymbol{H} = 0$ represents the remanent flux density ($\boldsymbol{B}_r$), that is

$$\boldsymbol{B}_r = \boldsymbol{B}|_{H=0} = \mu_0\, \boldsymbol{M}_p\,. \tag{2}$$

From relation (1) it follows that for materials with $\boldsymbol{M}_p \neq 0$ (permanent magnets), the "classical" tensor of magnetic permeability $[\mu]$ ($\boldsymbol{B} = [\mu]\boldsymbol{H}$) is not univocally determined by material, because $\boldsymbol{M}_p$ could have more values at the same material (for diverse minor cycles of hysteresis which are possible, $\boldsymbol{B}_r = \mu_0 \boldsymbol{M}_p$ can have more values). In this context it is useful to define another permeability for permanent magnets, which helps overcome the above-mentioned difficulty.

2. ANOTHER PERMEABILITY FOR ANISOTROPIC PERMANENT MAGNETS

The temporary magnetization value of anisotropic materials is depending on the direction of magnetic field, and the temporary magnetization low is

$$\boldsymbol{M}_\tau = [\chi]_m \boldsymbol{H} , \tag{3}$$

were, for the nonlinear materials, the components of magnetic susceptivity tensor $[\chi]_m$ are depending on the magnetic field intensity components. Consequently, in case on the nonlinear anisotropic permanent magnets, rel. (1) becomes

$$\boldsymbol{B} = \mu_0 ([1] + [\chi]_m) \boldsymbol{H} + \boldsymbol{B}_r , \tag{4}$$

were, the tensor's components are nonlinear functions depending on the components of $\boldsymbol{H}$.

If we introduce the calculation quantity

$$\boldsymbol{B}_p = \boldsymbol{B} - \boldsymbol{B}_r = \boldsymbol{B} - \mu_0 \boldsymbol{M}_p , \tag{5}$$

rel.(4) becomes

$$\boldsymbol{B}_p = \mu_0 ([1] + [\chi]_m) \boldsymbol{H}. \tag{6}$$

From rel. (4, 5, 6), the relative $[\mu_{rp}]$ and absolute $[\mu_p]$ calculation tensors permeability of permanent magnets are defined with these relations :

$$[\mu_{rp}] = ([1]+[\chi]_m); \; [\mu_p] = \mu_0 [\mu_{rp}]. \tag{7}$$

Introducing $\boldsymbol{B}_p$ vector (rel.5) and the new permeability $[\mu_p]$ (rel.7), for permanent magnets we obtain relation

$$\boldsymbol{B}_p = [\mu_p] \boldsymbol{H} , \tag{8}$$

which, *formally, is similarly with the "classical" relation* $\boldsymbol{B} = [\mu] \boldsymbol{H}$, written for the materials without permanent magnetization. For isotropic materials, even they are with permanent magnetization (isotropic permanent magnets), rel. (8) becomes $\boldsymbol{B}_p = \mu_p \boldsymbol{H}$, which is showing that the lines spectra of $\boldsymbol{B}_p$ and $\boldsymbol{H}$ are the same in this case. We known that for permanent magnets (even isotropic one) the spectra lines of $\boldsymbol{B}$ and $\boldsymbol{H}$ are different [1, 4].

Since the definition relation of $[\mu_p]$ contain also permanent magnetization $\boldsymbol{M}_p$, using $\boldsymbol{B}_p$ and $[\mu_p]$, we have advantageously taken into account the non-linearity of the demagnetization curves of permanent magnets, for any minor hysteresis cycle could be.

It's known that following the magnetization main directions [2], tensor $[\chi]_m$ has only three components. If we note these three directions (generally, non-rectangular) with x , y , z index, from rel. (4) results

$$B_v = \mu_0 (1+\chi_{mv}) H_v + B_{rv} ; \; v = x , y , or \; z , \tag{9}$$

and all three components of tensor $[\mu_{rp}]$ are

$$\mu_{rpv}=(B_v - B_{rv})/\mu_0 H_v = B_{pv} /\mu_0 H_v; \; v = x, y, z. \tag{10}$$

If we take into account that for the point of function of a permanent magnet $B < B_r$ (respectively $B_v < B_{rv}$) and $H < 0$ (the demagnetization curve is in the second quadrant of the hysterezis cycle), results that the components of tensor $[\mu_{rp}]$ are positive and scalar quantities. It's interesting to specify if we known all the three hysteresis cycles following the magnetization main axes, we should determine the nonlinear functions $\mu_{rpv}(H_v)$. For these three main directions x, y, z, the nonlinear function plots will have similar forms, but they will be quantitative different, as like as the demagnetization curves following the main three directions of the anisotropic magnet are different between them.

The defining relative permeability of permanent magnets it's an useful operation. For example, since the system is generally nonlinear, the numerical solution for magnetic field problem in permanent magnets it's obtained with an iterative process (see [5] for isotropic materials). The parameter after which the convergence of the problem is followed could be the relative permeability, defined with rel. (7), respectively rel. (10). It's evident that for anisotropic materials the convergence of the calculation is made with components of tensors $[\mu_p]$, respectively $[\mu_{rp}]$. Through this defined calculation quantity we take univocally and advantageously into account the nonlinearity of the demagnetization curves, indifferently if we talk about the major or minor demagnetization curves (for any permanent magnetization $\boldsymbol{M}_p$).

3. THE REFRACTION THEOREMS

We consider two different permanent magnets 1 and 2, at rest, separate by smooth surface S_{12} (Fig. 1). The general demonstration is referring to the refraction of the magnetic field lines of $\boldsymbol{H}$ and of calculation flux density $\boldsymbol{B}_p$ (defined by rel. 5), for 3D field in anisotropic permanent magnets, having random magnetization main directions. For magnet 1 these directions are noted with (x_1, y_1, z_1) and unit vectors $\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1$, and for magnet 2 are noted with $(x_2, y_2, z_2,)$ and unit vectors $\boldsymbol{i}_2$, $\boldsymbol{j}_2$, $\boldsymbol{k}_2$. These axes system (after magnetization main directions of the two permanent magnets) are, generally, non-rectangular.

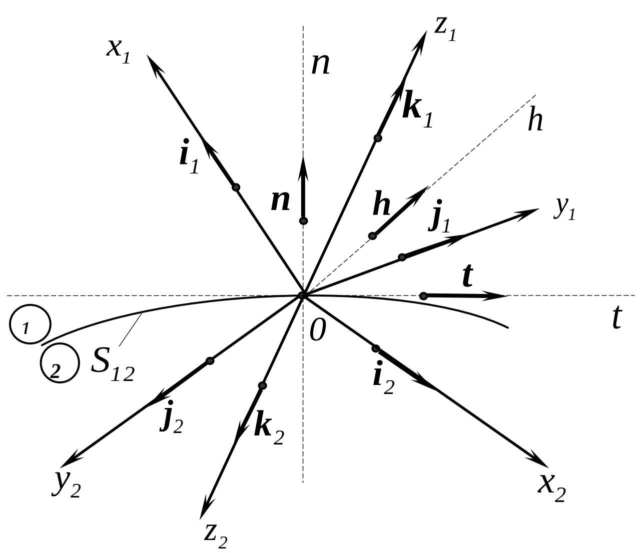

Fig.1. Reference axes systems 3D

In order to utterance of normal and tangent components of the magnetic field state quantities at the separation surface S_{12}, for both media we attach a rectangular axes system (n, t, h) with unit vectors $\boldsymbol{n}, \boldsymbol{t},$ and $\boldsymbol{h}$. Unit vector $\boldsymbol{n}$ is perpendicular on S_{12} in point O, unit vector $\boldsymbol{t}$ is tangent on S_{12} in point O and situated in the plane of $\boldsymbol{H}$ vectors, and unit vector $\boldsymbol{k}$ is orthogonal on the plane determined by $\boldsymbol{n}$ and $\boldsymbol{t}$.

It's known that in permanent magnets the spectra lines of flux density $\boldsymbol{B}$, of magnetic field intensity $\boldsymbol{H}$ and of magnetization $\boldsymbol{M}_p$ are different, both for anisotropic media and for isotropic media.

In order to write the projections on axes of quantities $\boldsymbol{B}_p$, $\boldsymbol{H}$ and $\boldsymbol{M}_p$, we introduce the angles:

- the angles between $\boldsymbol{B}_{p1}$, respectively $\boldsymbol{B}_{p2}$ and the axes of system (n, t, h) :

$$\alpha_{\lambda n} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{n})\ ;\ \alpha_{\lambda t} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{t})\ ;\ \alpha_{\lambda h} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{h})\ ;\ \lambda = 1 \text{ or } 2\ ; \tag{11}$$

- the angles between $\boldsymbol{B}_{p1}$, respectively $\boldsymbol{B}_{p2}$ and the main directions of magnetization:

$$\alpha_{\lambda x} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{i}_\lambda)\ ;\ \alpha_{\lambda y} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{j}_\lambda)\ ;\ \alpha_{\lambda z} = \sphericalangle(\boldsymbol{B}_{p\lambda}, \boldsymbol{k}_\lambda)\ ;\ \lambda = 1, 2\ ; \tag{12}$$

- the angles between $\boldsymbol{H}_1$ respectively $\boldsymbol{H}_2$ and the axes of system (n, t, h) :

$$\beta_{\lambda n} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{n})\ ;\ \beta_{\lambda t} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{t})\ ;\ \beta_{\lambda h} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{h})\ ;\ \lambda = 1, 2\ ; \tag{13}$$

- the angles between $\boldsymbol{H}_1$, respectively $\boldsymbol{H}_2$ and the main directions of magnetizations:

$$\beta_{\lambda x} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{i}_\lambda)\ ;\ \beta_{\lambda y} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{j}_\lambda)\ ;\ \beta_{\lambda z} = \sphericalangle(\boldsymbol{H}_\lambda, \boldsymbol{k}_\lambda)\ ;\ \lambda = 1, 2\ ; \tag{14}$$

- the angles between $\boldsymbol{M}_{p1}$, respectively $\boldsymbol{M}_{p2}$ and the axes of system (n, t, h) :

$$\gamma_{\lambda n} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{n})\ ;\ \gamma_{\lambda t} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{t})\ ;\ \gamma_{\lambda h} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{h})\ ;\ \lambda = 1,\ 2\ ; \quad (15)$$

- the angles between $\boldsymbol{M}_{p1}$, respectively $\boldsymbol{M}_{p2}$ and the main directions of magnetization :

$$\gamma_{\lambda x} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{i}_\lambda)\ ;\ \gamma_{\lambda y} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{j}_\lambda)\ ;\ \gamma_{\lambda z} = \sphericalangle\ (\boldsymbol{M}_{p\lambda}\ ,\ \boldsymbol{k}_\lambda)\ ;\ \lambda = 1,\ 2\ ; \quad (16)$$

- the angles between the axes of rectangular system (n, t, h) and the main directions of magnetization (x_1, y_1, z_1) – in medium 1, respectively (x_2, y_2, z_2) – in medium 2 :

$$\varphi_{\lambda nx} = \sphericalangle\ (\boldsymbol{n}\ ,\ \boldsymbol{i}_\lambda)\ ;\ \varphi_{\lambda ny} = \sphericalangle\ (\boldsymbol{n}\ ,\ \boldsymbol{j}_\lambda)\ ;\ \varphi_{\lambda nz} = \sphericalangle\ (\boldsymbol{n}\ ,\ \boldsymbol{k}_\lambda)\ ;\ \lambda = 1,\ 2\ ; \quad (17)$$

$$\varphi_{\lambda tx} = \sphericalangle\ (\boldsymbol{t}\ ,\ \boldsymbol{i}_\lambda)\ ;\ \varphi_{\lambda ty} = \sphericalangle\ (\mathbf{t}\ ,\ \boldsymbol{j}_\lambda)\ ;\ \varphi_{\lambda tz} = \sphericalangle\ (\boldsymbol{t}\ ,\ \boldsymbol{k}_\lambda)\ ;\ \lambda = 1,\ 2\ . \quad (18)$$

Because the 3D systems (n, t, h) and (x_λ, y_λ, z_λ), with $\lambda = 1$ or 2, generally have a random position, the angles "φ" must be defined. For example: $\alpha_{1n} = \sphericalangle(\boldsymbol{B}_{p1}, \boldsymbol{n})$, $\alpha_{1x} = \sphericalangle\ (\boldsymbol{B}_{p1}\ ,\ \boldsymbol{i}_1)$; because $\boldsymbol{B}_{p1}$, $\boldsymbol{n}$ and $\boldsymbol{i}_1$ are not in the same plane, the angle $\varphi_{1nx} = \sphericalangle\ (\boldsymbol{n}\ ,\ \boldsymbol{i}_1)$ can't obtain from a combination between the angles α_{1n} and α_{1x}. Also, it is remark that between the angles of $\boldsymbol{B}_p$, $\boldsymbol{H}$ and $\boldsymbol{M}_p$ vectors with the normal, respectively with the tangent directions, the sum is not 90^0, because it's 3D system. Namely, $\alpha_{\lambda n} + \alpha_{\lambda t} \neq 90^0$, because $\boldsymbol{B}_{p1}$, $\boldsymbol{n}$ and $\boldsymbol{t}$ generally are not in the same plane. Similarly, $\beta_{\lambda n} + \beta_{\lambda t} \neq 90^0$ and $\gamma_{\lambda n} + \gamma_{\lambda t} \neq 90^0$ $(\lambda = 1,\ 2)$.

The normal components of magnetic flux density $\boldsymbol{B}$ of the separation surface S_{12} it's preserved (the local form of magnetic flux law), i.e.

$$B_{1n} = B_{2n} = B_n\ . \quad (19)$$

Considering that the separation surface (at rest) is not containing a current skin-deep repartition, result the conservation of the tangent components of $\boldsymbol{H}$ (the local form of magnetic circuit law):

$$H_{1t} = H_{2t} = H_t\ . \quad (20)$$

For 3D field in anisotropic media with permanent magnetization, from rel. (6, 7) considering for both media, results

$$\boldsymbol{B}_{p\lambda} = [\mu_{p\lambda}]\ \boldsymbol{H}_\lambda\ ; \quad \lambda = 1,\ 2\ , \quad (21)$$

where $[\ \mu_{p\lambda}\ ] = [\ \mu_{p\lambda x} \quad \mu_{p\lambda y} \quad \mu_{p\lambda z}\ ]$ are the tensors for calculation absolute permeability.

If we emphasize the components following the magnetization main directions (see also rel. (10), where $\mu_0\,\mu_{rpv} = \mu_{pv}$), rel. (21) becomes

$$B_{p\lambda v} = \mu_{p\lambda v} H_{\lambda v}\ \ ;\ \lambda = 1,\ 2\ \ ;\ v = x,\ y,\ z\ . \quad (22)$$

We can see that between $\boldsymbol{B}_p$ and $\boldsymbol{H}$ components could be written relations like (22) only following the magnetization main directions (x_λ, y_λ, z_λ), but not following rectangular directions (n, t, h) [1, 2, 5].

With the projections following the magnetization main directions, in both media, we can write these relations:

$$\boldsymbol{B}_{p\lambda} = B_{p\lambda x}\,\boldsymbol{i}_\lambda + B_{p\lambda y}\,\boldsymbol{j}_\lambda + B_{p\lambda z}\,\boldsymbol{k}_\lambda\ ,\ \ \boldsymbol{H}_\lambda = H_{\lambda x}\,\boldsymbol{i}_\lambda + H_{\lambda y}\,\boldsymbol{j}_\lambda + H_{\lambda z}\,\boldsymbol{k}_\lambda\ ,\ \ \boldsymbol{M}_{p\lambda} = M_{p\lambda x}\,\boldsymbol{i}_\lambda + M_{p\lambda y}\,\boldsymbol{j}_\lambda + M_{p\lambda z}\,\boldsymbol{k}_\lambda\ , \quad (23)$$

where $\lambda = 1,\ 2$. Because it is the general case (anisotropic permanent magnets having random magnetization main directions), we remark that [3] : $B_{p\lambda} \neq (B^2_{p\lambda x}+B^2_{p\lambda y}+B^2_{p\lambda z})^{1/2}$; $H_\lambda \neq (H^2_{\lambda x}+H^2_{\lambda y}+H^2_{\lambda z})^{1/2}$; $M_{p\lambda} \neq (M^2_{p\lambda x}+M^2_{p\lambda y}+M^2_{p\lambda z})^{1/2}$.

If we write the vectors $\boldsymbol{B}_{p\lambda}$, $\boldsymbol{H}_\lambda$ and $\boldsymbol{M}_{p\lambda}$ depending on the components following the rectangular system $(n,\ t,\ h)$, we could write the relations :

$$\begin{aligned} \boldsymbol{B}_{p\lambda} &= B_{p\lambda n}\,\boldsymbol{n} + B_{p\lambda t}\,\boldsymbol{t} + B_{p\lambda h}\,\boldsymbol{h} = B_{p\lambda}\,(\cos\alpha_{\lambda n}\,\boldsymbol{n} + \cos\alpha_{\lambda t}\,\boldsymbol{t} + \cos\alpha_{\lambda h}\,\boldsymbol{h});\\ \boldsymbol{H}_\lambda &= H_{\lambda n}\,\boldsymbol{n} + H_{\lambda t}\,\boldsymbol{t} + H_{\lambda h}\,\boldsymbol{h} = H_\lambda\,(\cos\beta_{\lambda n}\,\boldsymbol{n} + \cos\beta_{\lambda t}\,\boldsymbol{t} + \cos\beta_{\lambda h}\,\boldsymbol{h});\\ \boldsymbol{M}_{p\lambda} &= M_{p\lambda n}\,\boldsymbol{n} + M_{p\lambda t}\,\boldsymbol{t} + M_{p\lambda h}\,\boldsymbol{h} = M_{p\lambda}\,(\cos\gamma_{\lambda n}\,\boldsymbol{n} + \cos\gamma_{\lambda t}\,\boldsymbol{t} + \cos\gamma_{\lambda h}\,\boldsymbol{h})\ , \end{aligned} \quad (24)$$

where $\lambda = 1,\ 2,$ for the two media.

3.1. The refraction theorem of magnetic field intensity lines *H*

The normal component of flux density $\boldsymbol{B}$ in medium 1 we can write as sum of the projections on normal direction of three components (B_{1x} , B_{1y} , B_{1z}) following the magnetization main directions :

$$B_{1n} = B_{1xn} + B_{1yn} + B_{1zn}\ . \quad (25)$$

Writing rel. (5) for medium 1 ($\boldsymbol{B}_1 = \boldsymbol{B}_{p1} + \mu_0 \boldsymbol{M}_{p1}$), the three components are :

$$B_{1\nu n} = B_{p1\nu n} + \mu_0 M_{p1\nu n}\ ;\ \nu = x, y, z,. \tag{26}$$

where $B_{p1\nu n}$ are the projections on normal axe (n) of the components $B_{p1\nu}$ of the calculation flux density $\boldsymbol{B}_{p1}$ following the magnetization main directions (x_1, y_1, z_1) of the medium 1; $M_{p1\nu n}$ - similarly, but regarding permanent magnetization $\boldsymbol{M}_{p1}$ of medium 1. These components are illustrated in rel. (27) - for $\boldsymbol{B}_{p1}$ - and in rel. (28) for $\boldsymbol{M}_{p1}$.

$$\boldsymbol{B}_{p1} = \boldsymbol{B}_{p1x} + \boldsymbol{B}_{p1y} + \boldsymbol{B}_{p1z}\ ,\ \ \boldsymbol{B}_{p1\nu} = B_{p1\nu n}\,\boldsymbol{n} + B_{p1\nu t}\,\boldsymbol{t} + B_{p1\nu h}\,\boldsymbol{h}\ ,\ \nu = x, y, z; \tag{27}$$

$$\boldsymbol{M}_{p1} = \boldsymbol{M}_{p1x} + \boldsymbol{M}_{p1y} + \boldsymbol{M}_{p1z}\ ,\ \ \boldsymbol{M}_{p1\nu} = M_{p1\nu n}\,\boldsymbol{n} + M_{p1\nu t}\,\boldsymbol{t} + M_{p1\nu h}\,\boldsymbol{h}\ ;\ \nu = x, y, z. \tag{28}$$

From rel.(25) and (26) we obtain

$$B_{1n}=B_{p1xn}+B_{p1yn}+B_{p1zn}+\mu_0(M_{p1xn}+M_{p1yn}+M_{p1zn}), \tag{29}$$

where the components are :

$$B_{p1\nu n}=B_{p1\nu}\cos\varphi_{1n\nu}\ ;\ \ M_{p1\nu n}=M_{p1\nu}\cos\varphi_{1n\nu};\ \nu = x, y, z. \tag{30}$$

Taking into account these and rel. (22), expression (29) becomes

$$\begin{aligned} B_{\mathbf{1n}}&=B_{p1x}\cos\varphi_{1nx}+B_{p1y}\cos\varphi_{1ny}+B_{p1z}\cos\varphi_{1nz}+\mu_0(M_{p1xn}+M_{p1yn}+M_{p1zn})=\\ &=\mu_{p1x}H_{1x}\cos\varphi_{1nx}+\mu_{p1y}H_{1y}\cos\varphi_{1ny}+\mu_{p1z}H_{1z}\cos\varphi_{1nz}+\mu_0(M_{p1xn}+M_{p1yn}+M_{p1zn}). \end{aligned} \tag{31}$$

Similarly, for normal components of flux density in medium 2 we can write

$$B_{2n} = B_{2xn} + B_{2yn} + B_{2zn}\ . \tag{32}$$

Writing rel. (5) for medium 2 ($\boldsymbol{B}_2 = \boldsymbol{B}_{p2} + \mu_0 \boldsymbol{M}_{p2}$), the three components are :

$$B_{2\nu n} = B_{p2\nu n} + \mu_0 M_{p2\nu n}\ ;\ \nu = x, y, z, \tag{33}$$

where $B_{p2\nu n}$ are the projections on normal axe (n) of the components $B_{p2\nu}$ of the calculation flux density $\boldsymbol{B}_{p2}$, following the magnetization main directions (x_2, y_2, z_2) of the medium 2; $M_{p2\nu n}$ - similarly, but regarding permanent magnetization $\boldsymbol{M}_{p2}$ of medium 2. These components are illustrated in rel. (34) - for $\boldsymbol{B}_{p2}$ - and in rel. (35) for $\boldsymbol{M}_{p2}$.

$$\boldsymbol{B}_{p2} = \boldsymbol{B}_{p2x} + \boldsymbol{B}_{p2y} + \boldsymbol{B}_{p2z}\ ,\ \ \boldsymbol{B}_{p2\nu} = B_{p2\nu n}\,\boldsymbol{n} + B_{p2\nu t}\,\boldsymbol{t} + B_{p2\nu h}\,\boldsymbol{h}\ ,\ \nu = x, y, z; \tag{34}$$

$$\boldsymbol{M}_{p2} = \boldsymbol{M}_{p2x} + \boldsymbol{M}_{p2y} + \boldsymbol{M}_{p2z}\ ,\ \ \boldsymbol{M}_{p2\nu} = M_{p2\nu n}\,\boldsymbol{n} + M_{p2\nu t}\,\boldsymbol{t} + M_{p2\nu h}\,\boldsymbol{h}\ ;\ \nu = x, y, z. \tag{35}$$

From rel.(32) and (33) we obtain

$$B_{2n} = B_{p2xn}+B_{p2yn}+B_{p2zn}+\mu_0(M_{p2xn}+M_{p2yn}+M_{p2zn})\ , \tag{36}$$

where the components are :

$$B_{p2\nu n}=B_{p2\nu}\cos\varphi_{2n\nu}\ ;\ \ M_{p2\nu n}=M_{p2\nu}\cos\varphi_{2n\nu};\ \nu = x, y, z. \tag{37}$$

Taking into account these and rel. (22), expression (36) becomes

$$\begin{aligned} B_{\mathbf{2n}}&=B_{p2x}\cos\varphi_{2nx}+B_{p2y}\cos\varphi_{2ny}+B_{p2z}\cos\varphi_{2nz}+\mu_0(M_{p2xn}+M_{p2yn}+M_{p2zn})=\\ &=\mu_{p2x}H_{2x}\cos\varphi_{2nx}+\mu_{p2y}H_{2y}\cos\varphi_{2ny}+\mu_{p2z}H_{2z}\cos\varphi_{2nz}+\mu_0(M_{p2xn}+M_{p2yn}+M_{p2zn}). \end{aligned} \tag{38}$$

By replacing (31) and (38) in (19) we obtain

$$\begin{aligned} &(\mu_{p1x}H_{1x}\cos\varphi_{1nx} - \mu_{p2x}H_{2x}\cos\varphi_{2nx}) + (\mu_{p1y}H_{1y}\cos\varphi_{1ny} - \mu_{p2y}H_{2y}\cos\varphi_{2ny}) + (\mu_{p1z}H_{1z}\cos\varphi_{1nz} - \mu_{p2z}H_{2z}\cos\varphi_{2nz}) +\\ &+ \mu_0\,[(M_{p1xn} - M_{p2xn}) + (M_{p1yn} - M_{p2yn}) + (M_{p1zn} - M_{p2zn})] = 0\ . \end{aligned} \tag{39}$$

If we emphasize the projections on normal direction of the components following the main magnetization axes for $\boldsymbol{H}_\lambda$ $(\lambda = 1, 2)$, from (39) results

$$(\mu_{p1x} H_{1xn} - \mu_{p2x} H_{2xn}) + (\mu_{p1y} H_{1yn} - \mu_{p2y} H_{2yn}) + (\mu_{p1z} H_{1zn} - \mu_{p2z} H_{2zn}) + \\ + \mu_0 [(M_{p1xn} - M_{p2xn}) + (M_{p1yn} - M_{p2yn}) + (M_{p1zn} - M_{p2zn})] = 0 , \qquad (40)$$

where $H_{\lambda vn} = H_{\lambda v} \cos \varphi_{\lambda nv}$, with $\lambda = 1, 2$ and $v = x, y, z$. Components $B_{p\lambda vn}$, $H_{\lambda vn}$, and $M_{p\lambda vn}$ are positive or negative depending on the concrete laying of vectors $\boldsymbol{B}_{p\lambda}$, $\boldsymbol{H}_\lambda$ and $\boldsymbol{M}_{p\lambda}$ in comparison with the axes system.

Consequently, the normal components of magnetic field intensity $\boldsymbol{H}$ (components which are not conserved) are respecting rel.(40); this relation will be named *the theorem of the 3D magnetic field intensity lines refraction, in anisotropic permanent magnets with random magnetization main directions.*

3.2. The refraction theorem of calculation flux density lines $\boldsymbol{B}_p$

The tangent component of $\boldsymbol{H}$ in medium 1 could be written as a sum of projections on tangent direction of the three components (H_{1x}, H_{1y}, H_{1z}) following the magnetization main directions:

$$H_{1t} = H_{1xt} + H_{1yt} + H_{1zt} . \qquad (41)$$

Regarding the meaning of H_{1xt} , H_{1yt} and H_{1zt} , as well as rel.(22), expression (41) becomes

$$H_{1t} = H_{1x} \cos \varphi_{1tx} + H_{1y} \cos \varphi_{1ty} + H_{1z} \cos \varphi_{1tz} = \\ = (B_{p1x} / \mu_{p1x}) \cos \varphi_{1tx} + (B_{p1y} / \mu_{p1y}) \cos \varphi_{1ty} + (B_{p1z} / \mu_{p1z}) \cos \varphi_{1tz} . \qquad (42)$$

Alike, for the tangent component of $\boldsymbol{H}$ from medium 2 we can write

$$H_{2t} = H_{2xt} + H_{2yt} + H_{2zt} = H_{2x} \cos \varphi_{2tx} + H_{2y} \cos \varphi_{2ty} + H_{2z} \cos \varphi_{2tz} = \\ = (B_{p2x} / \mu_{p2x}) \cos \varphi_{2tx} + (B_{p2y} / \mu_{p2y}) \cos \varphi_{2ty} + (B_{p2z} / \mu_{p2z}) \cos \varphi_{2tz} . \qquad (43)$$

By replacing (42) and (43) in (20) we can write

$$[(B_{p1x} / \mu_{p1x}) \cos \varphi_{1tx} - (B_{p2x} / \mu_{p2x}) \cos \varphi_{2tx}] + [(B_{p1y} / \mu_{p1y}) \cos \varphi_{1ty} - (B_{p2y} / \mu_{p2y}) \cos \varphi_{2ty}] + \\ + [(B_{p1z} / \mu_{p1z}) \cos \varphi_{1tz} - (B_{p2z} / \mu_{p2z}) \cos \varphi_{2tz}] = 0. \qquad (44)$$

If we put into evidence the projections on the tangent of the components following magnetization main axes for $\boldsymbol{B}_{p\lambda}$ $(\lambda = 1, 2)$, from rel.(44) results

$$(B_{p1xt} / \mu_{p1x} - B_{p2xt} / \mu_{p2x}) + (B_{p1yt} / \mu_{p1y} - B_{p2yt} / \mu_{p2y}) + (B_{p1zt} / \mu_{p1z} - B_{p2zt} / \mu_{p2z}) = 0, \qquad (45)$$

where $B_{p\lambda vt} = B_{p\lambda v} \cos \varphi_{\lambda tv}$, with $\lambda = 1, 2$ and $v = x, y, z$. Components $B_{p\lambda vt}$ and $H_{\lambda vt}$ are positive or negative depending on the concrete laying of vectors $\boldsymbol{B}_{p\lambda}$ and $\boldsymbol{H}_\lambda$ in comparison with the axes systems.

Consequently, the tangent components of calculation flux density $\boldsymbol{B}_p$ are respecting relation (45); this relation will be named *the theorem of 3D refraction of calculation magnetic flux density $\boldsymbol{B}_p$ lines, in anisotropic permanent magnets with random magnetization main directions.* We should remark that the theorem (45) has a simple form than the refraction theorem of magnetic flux density lines, which we had been considered the "classical" quantities $\boldsymbol{B}$ and $[\mu]$ (see [4], rel. 27).

4. PARTICULAR CASES OF THE REFRACTION THEOREMS

4.1. 3D fields in isotropic permanent magnets

For isotropic media, the calculation permeability in two materials is:

$$\mu_{p1x} = \mu_{p1y} = \mu_{p1z} = \mu_{p1} ; \quad \mu_{p2x} = \mu_{p2y} = \mu_{p2z} = \mu_{p2} . \qquad (46)$$

If we take into account rel.(46), theorem (40) for refraction of magnetic field intensity lines becomes

$$\mu_{p1}(H_{1xn} + H_{1yn} + H_{1zn}) - \mu_{p2}(H_{2xn} + H_{2yn} + H_{2zn}) + \\ + \mu_0 [(M_{p1xn} + M_{p1yn} + M_{p1zn}) - (M_{p2xn} + M_{p2yn} + M_{p2zn})] = 0 . \qquad (47)$$

Considering the significations from theorem (40), expression (47) may be written shortly in this way :

$$\mu_{p1} H_{1n} = \mu_{p2} H_{2n} - \mu_0 (M_{p1n} - M_{p2n}) . \qquad (48)$$

In case of isotropic permanent magnets, rel. (5) and (7), become $\boldsymbol{B}_{p\lambda} = \boldsymbol{B}_\lambda - \mu_0 \boldsymbol{M}_{p\lambda} = \mu_{p\lambda} \boldsymbol{H}_\lambda$ $(\lambda = 1, 2)$. In this case, we can write the relations $B_{p\lambda n} = \mu_{p\lambda} H_{\lambda n} = B_{\lambda n} - \mu_0 M_{p\lambda n}$ or $B_{\lambda n} = \mu_{p\lambda} H_{\lambda n} + \mu_0 M_{p\lambda n}$ $(\lambda = 1, 2)$. Consequently, after a regrouping of the terms in rel. (48), we track down rel. (19), as we expect.

Alike, taking into account rel. (46), theorem (45) for refraction of calculation flux density lines becomes

$$(B_{p1xt} + B_{p1yt} + B_{p1zt})/\mu_{p1} - (B_{p2xt} + B_{p2yt} + B_{p2zt})/\mu_{p2} = 0 \,. \tag{49}$$

Considering the significations from theorem (45), expression (49) may be written shortly:

$$B_{p1t} / \mu_{p1} = B_{p2t} / \mu_{p2} \,. \tag{50}$$

Rel. (48) and (50) are the theorem of refraction for $\boldsymbol{H}$, respectively $\boldsymbol{B}_p$, in 3D field for isotropic permanent magnets. We can remark that, for the tangent components of $\boldsymbol{B}_p$, *theorem* (50) *for permanent magnets has a similarly form (but another content) with "classical" theorem of refraction in materials without permanent magnetization.*

For isotropic permanent magnets, we can write rel. $\boldsymbol{B}_{p\lambda} = \mu_{p\lambda} \boldsymbol{H}_\lambda$ $(\lambda = 1, 2)$. That means, *for this case, vectorial quantity* $\boldsymbol{B}_p$, *defined in rel.*(5), *is refracting in the same way as magnetic field intensity* $\boldsymbol{H}$, *or* $\boldsymbol{B}_p$ *and* $\boldsymbol{H}$ *have the same direction.* In permanent magnets, field lines of "classical" $\boldsymbol{B}$ and field line of $\boldsymbol{H}$, generally are different [1, 2, 3]. Also, theorem (50) has more simple form than "classical" treatment, with $\boldsymbol{B}$ and $\boldsymbol{H}$ (s. [4], rel. 32). So, *the introduction of new quantities* $\boldsymbol{B}_p$ *and* $[\mu_p]$ *are helping us to express the refraction theorem in a more simple form.*

4.2. 3D fields in isotropic permanent media without permanent magnetization

In this case, from rel.(5) we obtain $\boldsymbol{B}_p = \boldsymbol{B}$ (for $\boldsymbol{M}_p = 0$). Also, from rel.(7), for isotropic media we can write $\mu_p = B_p / H = B / H$. So $\mu_p = \mu$, which means that the calculation permeability is identical with the "classical" permeability, if the media is without permanent magnetization.

Particularizing rel.(48) and (59) for this case and taking into account of the previous observations, results

$$\mu_{p1} / \mu_{p2} = B_{p1t} / B_{p2t} = H_{2n} / H_{1n} = \mu_1/\mu_2 = B_{1t} / B_{2t} \,, \tag{51}$$

that is the "classical" form of the refraction theorem for the magnetic field lines. $\boldsymbol{B}_p$ and $\boldsymbol{H}$ have the same field lines because is an isotropic material. But $\boldsymbol{B}_p$ and $\boldsymbol{B}$ are identical (because $\boldsymbol{M}_p = 0$), that means $\boldsymbol{B}$ and $\boldsymbol{H}$ have the same field lines.

4.3. 2D fields in isotropic media with permanent magnetization

For 2D field, vectors $\boldsymbol{B}_p$, $\boldsymbol{H}$ and $\boldsymbol{M}_p$ have not the components after z axe. Rel.(48) and (50) are valid in this case, but z components absent from rel.(47) and (49). In this case $\alpha_{\lambda n} + \alpha_{\lambda t} = 90^0$ and $\beta_{\lambda n} + \beta_{\lambda t} = 90^0$. If we represent $\boldsymbol{B}_{p\lambda}$ and $\boldsymbol{H}_\lambda$ vectors, we obtain "classical" representation, but $\boldsymbol{B}$ replace with $\boldsymbol{B}_p$ (Fig. 2). Because are isotropic media, $\boldsymbol{B}_p$ and $\boldsymbol{H}$ have the same lines spectra, therefore $\alpha_{\lambda n} = \beta_{\lambda n}$ and $\alpha_{\lambda t} = \beta_{\lambda t}$ $(\lambda = 1, 2)$. We remark that vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ have not the same lines spectra, because $\boldsymbol{M}_p \neq 0$.

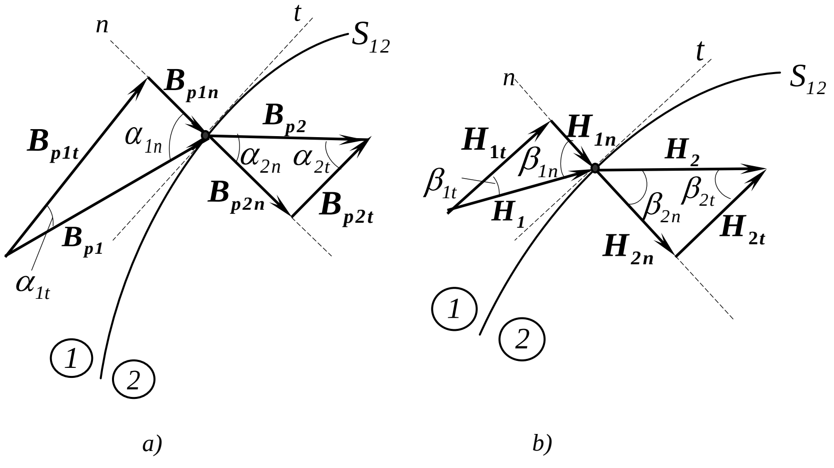

Fig.2. Continuity conditions for $\boldsymbol{B}_p$ and $\boldsymbol{H}$ (2D)

Because are isotropic media, $\boldsymbol{B}_p$ and $\boldsymbol{H}$ have the same lines spectra, therefore $\alpha_{\lambda n} = \beta_{\lambda n}$ and $\alpha_{\lambda t} = \beta_{\lambda t}$ $(\lambda = 1, 2)$. We remark that vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ have not the same lines spectra, because $\boldsymbol{M}_p \neq 0$.

4.4. 2D fields in isotropic media without permanent magnetization

In this case $\boldsymbol{B}_{p\lambda} = \boldsymbol{B}_\lambda$, $\mu_{p\lambda} = \mu_\lambda$, $\alpha_{\lambda n} = \beta_{\lambda n}$, $\alpha_{\lambda t} = \beta_{\lambda t}$ and $\alpha_{\lambda n} + \alpha_{\lambda t} = 90^0$ $(\lambda = 1, 2)$. With this, taking into account the "classical" representation for 2D fields refraction in isotropic media without permanent magnetization [1, 2], we can complete rel.(51), finding again the "classical" relations:

$$\mu_{p1}/\mu_{p2}=B_{p1t}/B_{p2t}=H_{2n}/H_{1n}=\mu_1/\mu_2=B_{1t}/B_{2t} = tg\ \alpha_{1n} / tg\ \alpha_{2n}=tg\ \beta_{1n} / tg\ \beta_{2n}. \quad (52)$$

That is the "classical" form of the refraction theorem for the magnetic field lines in isotropic media, without permanent magnetization, when $\boldsymbol{B}$ and $\boldsymbol{H}$ have the same lines spectra.

It's easy to remark that, from general expression of refraction theorems of $\boldsymbol{B}_p$ and $\boldsymbol{H}$ or from the particular forms already mentioned, we can obtain also other particular forms. Such cases are possible when one of the media has permanent magnetization and the other one does not (for example: permanent magnet – air gap, permanent magnet – common ferromagnetic material), when the permanent magnetization vectors have particular orientation, when the main directions of magnetization have particular orientation and so on.

5. EXAMPLES FOR THE FUNCTIONS $B_p(H)$ AND $\mu_{rp}(H)$

If the hysteresis cycle for the material of permanent magnets is known, we can determine the diagram of nonlinear function $B_p(H)$. After that, we have deduced nonlinear function $\mu_{rp}(H)$ (or $\mu_p(H)$) . For an anisotropic permanent magnet, it's necessary to known the hysteresis cycles after main directions of magnetization. In this case we can determine the diagrams of nonlinear function $B_{pv}(H)$ and $\mu_{rpv}(H)$, with $v = x, y, z$.

For example, in Fig.3 nonlinear functions $B_p(H)$ and $\mu_{rp}(H)$ are presented, for ALNICO 13/5, considering the major curve of demagnetization and isotropic material.

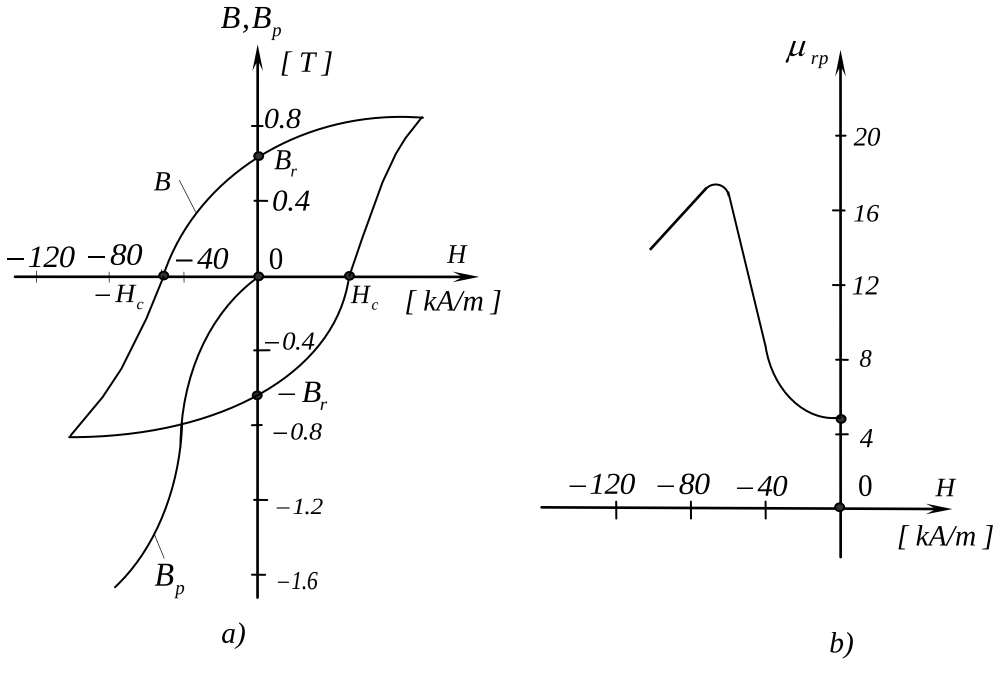

Fig.3. Examples for $B_p(H)$ and $\mu_{rp}(H)$

6. CONCLUSIONS

The introduction of permeability $[\mu_p]$ for permanent magnets and calculation flux density $\boldsymbol{B}_p$ - as we specify at par.2 – it's useful operation because the theorems of refraction have more simple form. Also, the solution of field problem in nonlinear and anisotropic systems with permanent magnets it could be done in an advantageous way.

For anisotropic media with random magnetization main directions and also with permanent magnetization, the refraction theorems for 3D field are given by rel.(40) (for magnetic field intensity $\boldsymbol{H}$), respectively rel. (45) (for calculation flux density $\boldsymbol{B}_p$). Starting from these general forms of the theorems, some particular forms have been deduced, which could be necessary for solve magnetic field problem in system with permanent magnets.

We can also remark that the similar theorems could be demonstrated for the electrical field lines refraction in media having permanent polarization.

REFERENCES

[1] ŞORA, C.: *Bazele electrotehnicii*, E.D.P., Bucureşti, 1982, pp (86 – 87; 263 – 269).

[2] MOCANU, C.I.: *Teoria câmpului electromagnetic*, E.D.P., Bucureşti, 1981, pp (220 – 225; 528 – 529).

[3] BACHMANN, K.H. a.a.: *Kleine Enzyklopadie der Mathematik*, Editura Tehnică, Bucureşti, 1980, pp (665 – 672)

[4] BERE, I. - BARBULESCU, E. : *Teoremele refracţiei liniilor câmpului magnetic 3D in magneţi permanenţi anizotropi având direcţiile principale de magnetizare oarecare*, E.E.A., vol. 50, nr. 4, Bucureşti, 2002, pp (5 – 10).

[5] BERE, I.: *Contribuţii la studiul câmpului magnetic prin metode numerice, cu aplicaţii la calculul unor sisteme cu magneţi permanenţi*, thesis, Timişoara, 1995.

[6] BERE, I.: *2D Magnetic field lines refraction in anisotropic materials with permanent magnetization, having orthogonal magnetization main directions*, Proceedings of 6th International Conference on Applied Electromagnetics, Nis, Serbia and Montenegro, 2003, pp (17-20).

[7] BERE, I.: *The theorems of refraction for 3D electric field in anisotropic dielectrics with permanent polarization, having random polarization main directions*, Proceedings of 7th International Power Systems Conference, Timisoara, Romania, 2007, pp (61-68).