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FRACTIONAL EVOLUTION INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

$M.CHANDRASEKARAN^1$, $V.DHANAPALAN^2$

[1]Higher College of Technology, Muscat, Sultanate of Oman.
[2]Government College of Technology, Coimbatore-641013, Tamilnadu, India.

***ABSTRACT*:** In this work, we use the theory of resolvent operators, the fractional powers of operators, fixed point technique and the Gelfand- Shilov principle to establish the existence and uniqueness of local mild and then classical solutions of a class of nonlinear fractional evolution integro-differential equations with nonlocal conditions in Banach space.

Keywords : Fractional parabolic equation, fractional powers, mild and classical solutions, localexistence, resolvent operators.
2000 Mathematics *Classification: 47H10, 34G20, 47D03, 45N05.*

1. Introduction

The aim of this paper is to study the nonlocal fractional Cauchy problem of the form

$$\frac{d^\alpha u(t)}{dt^\alpha} + A(t)u(t) = f\left(t, u(t), \int_{t_0}^{t} k\big(t,s,u(s)\big)ds, \int_{t_0}^{t} k_1\big(t,s,u(s)\big)ds\right) + \int_{t_0}^{t} B(t-s)g(s,u(s)ds, \qquad t \in [0,b], \tag{1.1}$$

$$u(t_0) + h(u) = u_0, \tag{1.2}$$

in a Banach space X, where $0 < \alpha \le 1$, $0 \le t_0 < t$. Let J denote the closure of the interval $[t_0, T)$, $t_0 < T \le \infty$. We assume that $-A(t)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is assumed also that $-A(t)$ generates an evolution operator in the Banach space X, the function B is real valued and locally integrable on $[t_0, \infty) \times X$ into X and $h: C(J, X) \to \overline{D(A)}$ is a given function.

Recently, fractional differential equations have attracted many authors [8, 12, 14-18, 20, 22, 25]. This is mostly because it efficiently describes many phenomena arising in engineering, physics, economy and science. In fact it can find several applications in viscoelasticity, electrochemistry, electromagnetic, and so on. The existence of solutions to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [9]. Subsequently many authors extended the work to various kinds of nonlinear evolution equations [17, 18, 22, 25]. Deng [13] indicated that using the nonlocal condition $u(t_0) + h(u) = u_0$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube, can give better results than using the usual local Cauchy problem $u(t_0) = u_0$.Also for several works (first order differential equations with initial conditions) concerned with this kind of research, we refer to [3-7, 11, 21, 29].

The results obtained in this paper are generalizations of the results given by Debbouche [11] Bahuguna [2], El-Borai[17,18], Pazy [26] and Yan [28].

Our work is organized as follows. Section 2 is devoted to the review of some essential results in fractional calculus and also to the resolvent operators and the fractional powers of operators which will be used in this work to obtain our main results. In section 3, we establish the existence of a unique local mild solution of (1.1), (1.2). In Section 4, We study the regularity of the mild solution of the considered problem and show under the additional condition of Hölder continuity on B that this mild solution is in fact the classical solution.

2. Preliminaries

Following Gelfand and Shilov [20], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds.$$

and the fractional derivative of the function f of order $0 < \alpha < 1$ as

$$_aD_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha} u(s)ds.$$

where f is an abstract continuous on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [11].

Definition 2.1 By a classical solution of (1.1), (1.2) on J, we mean a function u with values in X such that:

(1) u is continuous function on $[t_0, T)$ and $u(t) \in D(A)$,

(2) $\frac{d^\alpha u}{dt^\alpha}$ exists and is continuous on $[t_0, T), 0 < \alpha < 1$, and u satisfies (1.1) on $(t_0, T)$and the nonlocal condition (1.2).

By a local classical solution of (1.1), (1.2) on J, we mean that there exist a T_0, $t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a classical solution of (1.1), (1.2).

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $-A(t)$ is aclosed operator, it follows that $-A(t)$ is in the set of bounded operators from E to X.

Definition 2.2(See[8,28]). A resolvent operator for problem (1.1), (1.2) is abounded operator-valued function $R(t,s) \in B(X), 0 \le s \le t < T$,the space of bounded linear operators on X, having the following properties:

(i) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, $0 \le s < T$, $\|R(t,s)\| \le Me^{\beta(t-s)}$ for some constant M and β.

(ii) $R(t,s)E \subset E, R(t,s)$ is strongly continuous in s and t on E.

(iii) For $x \in X, R(t,s)x$ is continuously differentiable in $s \in [0,T)$ and

$$\frac{\partial R}{\partial s}(t,s)x = R(t,s)A(s)x.$$

(iv) For $x \in X, s \in [0,T), R(t,s)x$ is continuously differentiable in $t \in [s,T)$ and

$$\frac{\partial R}{\partial s}(t,s)x = -A(t)R(t,s)x.$$

with $\frac{\partial R}{\partial s}(t,s)x$ and $\frac{\partial R}{\partial t}(t,s)x$ are strongly continuous on $0 \le s \le t < T$.
Here $R(t,s)$ can be extracted from the evolution operator of the generator $-A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space.

Definition 2.3 (See[8, 22, 25]). A continuous function $u: J \to X$ is said to be a mild solution of problem (1.1), (1.2) if for all $u_0 \in X$, it satisfies the integral equation

$$u(t) = R(t,t_0)[u_0 - h(u)] + \frac{1}{\Gamma\alpha}\int_{t_0}^{t}(t-s)^{\alpha-1}R(t,s)[f\left(s,u(s),\int_{t_0}^{s}k\big(s,\tau,u(\tau)\big)d\tau,\int_{t_0}^{s}k_1\big(s,\tau,u(\tau)\big)d\tau\right) + \int_{u_0}^{s}B(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds \quad (2.1)$$

By a local mild solution of (1.1), (1.2) on J, we mean that there exist a $T_0, t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a mild solution of (1.1),(1.2).

We define the fractional power $A^{-q}(t)$ by $A^{-q}(t) = \frac{1}{\Gamma\alpha}\int_0^\infty x^{q-1}R(x,t)dx, q > 0$. For $0 < q \le 1$, $A^q(t)$ (we denote by A^q for short) is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X. This implies that $D(A^q)$ endowed with the graph norm $\|u\|_{D(A)} = \|u\| + \|A^q u\|, u \in D(A^q)$ is a Banach space as clearly $A^q = A^{-(-q)}$ because A^{-q} is one to one. Since $0 \in \rho(A)$, A^q is invertible, and is graph norm is equivalent to the norm $\|u\|_q = \|A^q u\|$.

Thus $D(A^q)$ equipped with norm $\|\cdot\|_q$ is a Banach space denoted by X_q. For more details we refer to [1,11]. To state and prove the main results of this paper, we shall require the following assumption on the maps f, k, k_1 and g:

(F) Let U be an open subset of $[0,\infty) \times X_q \times X_q$, and for every $(t,x,y) \in U$ there exist a neighborhood $V \subset U$ of (t,x,y) and constants $L > 0, 0 < \mu < 1$ such that

$$\|f(s_1,u_1,v_1) - f(s_2,u_2,v_2)\| \le L\big[|s_1 - s_2|^\mu + \|u_1 - u_2\|_q + \|v_1 - v_2\|_q\big],$$

for all (s_1,u_1,v_1) and (s_2,u_2,v_2) in V.

(G) Let D be an open subset of $[0,\infty) \times [0,\infty) \times X_q$, and for every $(t,s,x) \in D$ there exist a neighborhood $E \subset D$ of (t,s,x) and constants $L_1 > 0, L_0 > 0$, $0 < \mu < 1$ such that

$$\|k(t_1,s_1,u_1) - k(t_2,s_2,u_2)\| \le L_1\big[|t_1 - t_2|^\mu + |s_1 - s_2|^\mu + \|u_1 - u_2\|_q\big],$$

$$\|k_1(t_1,s_1,u_1) - k_1(t_2,s_2,u_2)\| \le L_0\big[|t_1 - t_2|^\mu + |s_1 - s_2|^\mu + \|u_1 - u_2\|_q\big],$$

for all (t_1,s_1,u_1) and (t_2,s_2,u_2) in E.

(H) Let P be an open subset of $[0,\infty) \times X_q$, and for every $(t,x) \in P$ there exist a neighborhood $W \subset P$ of (t,x) and constants $L_2 > 0, 0 < \mu < 1$ such that

$$\|g(t_1,x) - g(t_2,y)\| \le L_2\big[|t_1 - t_2|^\mu + \|x - y\|_q\big],$$

for all (t_1,x) and (t_2,y) in W.

3. Local Mild Solutions

To establish local existence of the considered problem, we assume that $-A$ is invertible and $t_0 < T < \infty$, see [19, 23, 24, 27]. According to [26, Section 2.6], we can deduce the following.

Lemma 3.1. Let $A(t)$ be the infinitesimal generator of a resolvent operator $R(t,s)$. We denote by $\rho[A(t)]$ the resolvent set of $A(t)$, then

(a) $R(t,s): X \to D(A^q)$ for every $0 \le s \le t < T$ and $q > 0$,

(b) For every $u \in D(A^q)$, we have $R(t,s)A^q(t)u = A^q(t)R(t,s)u$,

(c) The operator $A^q R(t,s)$ is bounded and $\|A^q R(t,s)\| \le M_{q,\beta}(t-s)^{-q}$.

Let $Y = C([t_0, t_1]; X_q)$ be endowed with the supremum norm

$$\|y\|_\infty = \sup_{t_0 \le t \le t_1} \|y(t)\|_q \,, y \in Y.$$

Then Y is a Banach space. The function $h: Y \to X_q$ is continuous and there exists a number b such that $\|R(t,t_0)\| < 1/_{2b}$ and $\|h(x) - h(y)\|_q \le b\|x - y\|_\infty$ for all $x, y \in Y$. Note that, if $z \in Y$, then $A^{-q}z \in Y$.

Theorem 3.2. Suppose that the operator $-A(t)$ generates the resolvent operator $R(t,s)$ with $\|R(t,s)\| \le Me^{\beta(t-s)}$and that $0 \in \rho[-A(t)]$. If the maps f, k and g satisfy (F),(G),(H) and the real-valued map B is integrable on J, then (1.1), (1.2) has a unique local mild solution for every $u_0 \in X_q$.

Proof: We fix a point (t_0, u_0) in the open subset U of $[0, \infty) \times X_q$ and choose $t_0' > t_0$ and $\epsilon > 0$ such that (F),(G) and (H) holds for the functions f , k and g on the set

$$V = \{(t,x) \in U: t_0 \le t_0', \|x - u_0\|_q \le \epsilon\}. \tag{3.1}$$

Let

$$N_1 = \sup_{t_0 \le t \le t_1'} \left\| f\left(t, u_0, \int_{t_0}^{t} k(t,s,u_0)ds\,, \int_{t_0}^{t} k_1(t,s,u_0)ds\right) \right\|, \quad N_2 = \sup_{t_0 \le t \le t_1'} \|g(t,u_0)\|$$

Set

$$\lambda = \sup_{x \in Y} \|h(x)\|_q$$

and choose $t_1 > t_0$ such that

$$\|R(t,t_0) - I\| \left[\|u_0\|_q + \lambda\right] \le \frac{\epsilon}{2}, t_0 \le t \le t_1.$$

and

$$t_1 - t_0 < \min\left\{t_1' - t_0, [\tfrac{\epsilon}{2} M_{q,\beta}^{-1} \Gamma(\alpha)(\alpha - q)\{(L\epsilon + L_1\epsilon\, b + L_0\epsilon\, b + N_1) + a_T(L_2\epsilon + N_2)\}^{-1}]^{\frac{1}{\alpha - q}}\right\}, \tag{3.4}$$

where

$$a_T = \textstyle\int_0^T |B(s)|\, ds. \tag{3.5}$$

We define a map on Y by $\phi y = \tilde{y}$, where $\tilde{y}$ is given by

$$\tilde{y}(t) = R(t,t_0)A^q\left[u_0 - h\left(A^{-q}y\right)\right]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} A^q R(t,s) \times [\, f\left(s, A^{-q}y(s), \int_{t_0}^{s} k\left(s,\tau, A^{-q}y(\tau)\right)d\tau\,, \int_{t_0}^{s} k_1\left(t,\tau, A^{-q}y(\tau)\right)d\tau\right)$$
$$+ \int_{u_0}^{s} B(s-\tau) g\left(\tau, A^{-q}y(\tau)\right)d\tau] ds.$$

For every $y \in Y$, $\phi y(t_0) = A^q\left[u_0 - h\left(A^{-q}y\right)\right]$, and for $t_0 \le s \le t \le t_1$ we have

$$\phi y(t) - \phi y(s)$$
$$= \left[R(t,t_0) - R(s,t_0)\right]A^q\left[u_0 - h\left(A^{-q}y\right)\right]$$
$$+ \frac{1}{\Gamma(\alpha)}\int_s^t (t-s)^{\alpha-1}A^q R(t,\tau)$$
$$\times f\left(\tau, A^{-q}y(\tau), \int_{t_0}^{\tau} k(\tau,\mu,A^{-q}y(\mu))d\mu, \int_{t_0}^{\tau} k_1(\tau,\mu,A^{-q}y(\mu))d\mu\right)$$
$$+ \int_{t_0}^{\tau} B(\tau-\eta)g\left(\eta, A^{-q}y(\eta)\right)d\eta]d\tau$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{s} (t-\tau)^{\alpha-1}A^q[R(t,\tau) - R(s,\tau)]$$
$$\times f\left(\tau, A^{-q}y(\tau), \int_{t_0}^{\tau} k(\tau,\mu,A^{-q}y(\mu))d\mu, \int_{t_0}^{\tau} k_1(\tau,\mu,A^{-q}y(\mu))d\mu\right)$$
$$+ \int_{t_0}^{\tau} B(\tau-\eta)g\left(\eta, A^{-q}y(\eta)\right)d\eta]d\tau.$$

It follows from(F) ,(G) and (H) on the functions f , k, k_1 and g, Lemma 3.1.c and (3.5) that $\phi: Y \to Y$.
Let S be the nonempty closed and bounded set given by

$$S = \left\{y \in Y: y(t_0) = A^q\left[u_0 - h\left(A^{-q}y\right)\right], \left\|y(t) - A^q\left[u_0 - h\left(A^{-q}y\right)\right]\right\| \le \epsilon\right\}. \qquad (3.6)$$

Then for $y \in S$, we have

$$\left\|\phi y(t) - A^q\left[u_0 - h\left(A^{-q}y\right)\right]\right\| \le \|R(t,t_0) - I\|\left\|A^q\left[u_0 - h\left(A^{-q}y\right)\right]\right\|$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-s)^{\alpha-1}$$
$$\times \|A^q R(t,s)\| \left\| f\left(s, A^{-q}y(s), \int_{t_0}^{s} k(s,\tau,A^{-q}y(\tau))d\tau, \int_{t_0}^{s} k_1(s,\tau,A^{-q}y(\tau))d\tau\right)\right.$$
$$\left. - f\left(s, u_0, \int_{t_0}^{s} k(s,\tau,u_0)d\tau, \int_{t_0}^{s} k_1(s,\tau,u_0)d\tau\right)\right\| ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-s)^{\alpha-1}\|A^q R(t,s)\| \int_{u_0}^{\tau} |B(s-t)| \left\|g\left(\tau, A^{-q}y(\tau)\right) - g(\tau,u_0)\right\| d\tau\, ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-s)^{\alpha-1}\|A^q R(t,s)\| \left\| f\left(s, u_{0,} \int_{t_0}^{s} k(s,\tau,u_0)d\tau, \int_{t_0}^{s} k_1(s,\tau,u_0)d\tau\right)\right\| ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-s)^{\alpha-1}\|A^q R(t,s)\| \int_{t_0}^{\tau} |B(s-t)| \|g(\tau,u_0)\| d\tau\, ds.$$

Using Lemma 3.1.c, (3.3) and (3.4) we get

$$\left\|\phi y(t) - A^q\left[u_0 - h\left(A^{-q}y\right)\right]\right\|$$
$$\le \frac{\epsilon}{2} + \frac{M_{q,\beta}}{\Gamma(\alpha)}(\alpha-q)^{-1}\{(L\epsilon + L_1\epsilon\, b + L_0\epsilon\, b + N_1) + a_T(L_2\epsilon + N_2)\}(t_1-t_0)^{\alpha-q} \le \epsilon \qquad (3.7)$$

Then $\phi: S \to S$. Now we shall show that ϕ is a strict contraction on S which will ensure the existence of a unique continuous function satisfying (1.1) and (1.2). If $y \,\epsilon\, S$ and $z \,\epsilon\, S$, then

$$\|\phi y(t)-\phi z(t)\| = \|\tilde{y}(t)-\tilde{z}(t)\|$$
$$\leq \|R(t,t_0)\|\left\|h\left(A^{-q}z\right)-h\left(A^{-q}y\right)\right\|_q$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}$$
$$\times\left\|A^qR(t,s)\right\|\left\|f\left(s,A^{-q}y(s),\int_{t_0}^{s}k(s,\tau,A^{-q}y(\tau))d\tau,\int_{t_0}^{s}k_1(s,\tau,A^{-q}y(\tau))d\tau\right)\right.$$
$$\left.-f\left(s,A^{-q}z(s),\int_{t_0}^{s}k(s,\tau,A^{-q}z(\tau))d\tau,\int_{t_0}^{s}k_1(s,\tau,A^{-q}z(\tau))d\tau\right)\right\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left\|A^qR(t,s)\right\|$$
$$\times\left[\int_{u_0}^{\tau}|B(s-t)|\left\|g\left(\tau,A^{-q}y(\tau)\right)-g\left(\tau,A^{-q}z(\tau)\right)\right\|d\tau\right]ds.$$

Using assumption (F),(G),(H) on f,k,k_1 and g, (3.5), Lemma 3.1.c and (3.4) respectively, we get

$$\|\phi y(t)-\phi z(t)\|$$
$$\leq b\|R(t,t_0)\|\|y-z\|_\infty$$
$$+\frac{1}{\Gamma(\alpha)}[(L+L_1\,b+L_0\,b+L_2a_T)\int_{t_0}^{t}(t-s)^{\alpha-1}\left\|A^qR(t,s)\right\|ds]\|y-z\|_Y$$
$$\leq b\|R(t,t_0)\|\|y-z\|_Y$$
$$+\frac{1}{\Gamma(\alpha)}(L+L_1\,b+L_0\,b+L_2a_T)M_{q,\beta}(\alpha-q)^{-1}(t_1-t_0)^{\alpha-q}\|y-z\|_Y$$

$$<\frac{1}{2}\|y-z\|_Y+\frac{1}{\epsilon\Gamma(\alpha)}[(L\epsilon+L_1\epsilon\,b+L_0\epsilon\,b\ \ +N_1)$$
$$+a_T(L_2\epsilon+N_2)]M_{q,\beta}(\alpha-q)^{-1}(t_1-t_0)^{\alpha-q}\|y-z\|_Y$$
$$<\frac{1}{2}\|y-z\|_Y+\frac{1}{2}\|y-z\|_Y.$$

Thus ϕ is a strict contraction map from S into S and therefore by the Banach contraction principle there exists a unique fixed point y in S such that

$$\phi y = y = \tilde{y}. \tag{3.8}$$

Let $u=A^{-q}y$. Using lemma 3.1.b, we have

$$u(t)=R(t,t_0)[u_0-h(u)]$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}R(t,s)[f\left(s,u(s),\int_{t_0}^{s}k\big(s,\tau,u(\tau)\big)d\tau,\int_{t_0}^{s}k_1\big(s,\tau,u(\tau)\big)d\tau\right)$$
$$+\int_{t_0}^{s}B(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds \tag{3.9}$$

for every $t\in[t_0,t_1]$. Hence u is a unique local mild solution of (1.1),(1.2). □

4. Local Classical Solutions

In this Section, We establish the regularity of the mild solutions of (1.1), (1.2). Let J denote the closure of the interval $[t_0,T), t_0<T\leq\infty$. In addition to the hypotheses mentioned in the earlier sections, we assume on the kernel B, that

(I) There exist constants $L_3\geq 0$ and $0<p\leq 1$ such that

$$|B(t_1)-B(t_2)|\leq L_3|t_1-t_2|^p,$$

for all $t_1,t_2\in J$.

Theorem 4.1. Suppose that $-A(t)$ generates the resolvent operator $R(t,s)$ such that $\|R(t,s)\| \leq Me^{\beta(t-s)}$ and $0 \in \rho[-A(t)]$. Further, suppose that the maps f, k, k_1 and g satisfy (F), (G) and (H) and the kernel B satisfies (I). Then (1.1), (1.2) has aunique local classical solution for $u_0 \in X_q$.

Proof: From Theorem 3.2, it follows that there exist T_0, $t_0 < T_0 < T$ and a function u such that u is a unique mild solution of (1.1), (1.2) on solution $J_o = [t_0, T)$ given by (3.9). Let $v(t) = A^q u(t)$. Then

$$v(t) = R(t,t_0)A^q[u_0 - h(u)] + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}A^q R(t,s)[\tilde{f}(s) + \int_{t_0}^{s}B(s-\tau)\tilde{g}(\tau)d\tau]ds, \tag{4.1}$$

where $\tilde{f}(t) = f\left(t, A^{-q}v(t), \int_{to}^{t}k(t,s,A^{-q}v(s))ds, \int_{to}^{t}k_1(t,s,A^{-q}v(s))ds\right)$, $\tilde{g}(t) = g\left(t, A^{-q}v(t)\right)$. Since $u(t)$ is continuous on J_o and the maps , k, k_1 and g satisfy (F) , (G) and (H), it follows that $\tilde{f}$ and $\tilde{g}$ are continuous, and therefore bounded on J_o. Let

$$M_1 = \sup_{t\in J_o}\|\tilde{f}(t)\|$$

and

$$M_2 = \sup_{t\in J_o}\|\tilde{g}(t)\| \tag{4.2}$$

By using the same method as in [13, Theorem 3.2], we can prove that $v(t)$ is locally Hölder continuous on J_o. Then there exist a constant C such that for every $t_0' > t_0$, we have

$$\|v(t_1) - v(t_2)\| \leq C|t_1 - t_2|^p, \tag{4.3}$$

for all $t_0 < t_0' < t_1, t_2 < T_0$.

Now, assumption (F), (G) and (H) with (4.3) implies that there exist constants $h_1, h_2 \geq 0$ and $0 < \gamma, \eta < 1$ such that for all $t_0 < t_0' < t_1, t_2 < T_0$, we have

$$\|\tilde{f}(t_1) - \tilde{f}(t_2)\| \leq h_1|t_1 - t_2|^{\gamma},$$

$$\|\tilde{g}(t_1) - \tilde{g}(t_2)\| \leq h_2|t_1 - t_2|^{\eta},$$

Which shows that $\tilde{f}$ and $\tilde{g}$ are locally Hölder continuous on J_o. Let

$$\omega(t) = \tilde{f}(t) + \int_{t_0}^{t}B(s-\tau)\tilde{g}(\tau)d\tau.$$

It is clear that $\omega(t)$ is locally Hölder continuous on J_o. For $t_2 < t_1$, we have

$$\|\omega(t_1) - \omega(t_2)\| \leq C^*|t_1 - t_2|^{\beta},$$

For some constants $C^* \geq 0$ and $0 < \beta < 1$. Consider the Cauchy problem

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} + A(t)v(t) = \omega(t), t > t_0, \tag{4.4}$$

$$v(t_0) = u_0 - h(u). \tag{4.5}$$

From Pazy [25], the problem (4.4), (4.5) has a unique solution v on J_o into X given by

$$v(t) = R(t,t_0)[u_0 - h(u)] + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}R(t,s)\omega(s)ds, \tag{4.6}$$

for $t > t_0$, Each term on the right-hand side belongs to $D(A)$, hence belongs to $D(A^q)$.

Applying A^q on both sides of (4.6) and using the uniqueness of $v(t)$, we have that $A^q v(t) = u(t)$. It follows that u is the classical solution of (1.1),(1.2) on J_o. Thus u is the unique local classical solution of (1.1),(1.2) on J. □

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