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THE EXACT CONTROLLABILITY PROBLEM FOR THE SECOND ORDER LINEAR HYPERBOLIC EQUATION

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Abstract

We consider a problem of exact controllability in the processes described by the second order linear hyperbolic equation with boundary control. Using Hilbert uniqueness method [1], we introduce an auxiliary boundary value problem. By means of this problem it is shown that after certain threshold time moment the considered system is controllable. Unlike [2] we consider nonhomogeneous hyperbolic equation. Note that different approaches have been applied to the solution of such kind of problems in, for instance, [3, 4].

Key words: controllability problem, linear hyperbolic equation, Hilbert uniqueness method.

1. Problem Statement

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary Γ, $x = (x_1, ...x_n)$ be an arbitrary point of domain Ω. Let $T > 0$ be a given number, $0 \leq t \leq T$, $Q = \Omega \times (0, T)$ be a cylinder, $S = \Gamma \times (0, T)$ be a lateral surface of the cylinder Q.

Let some process be described by the initial boundary value problem in Q for the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}\left(x,t\right) \frac{\partial u}{\partial x_j} \right) = f\left(x,t\right), \quad (x,t) \in Q, \tag{1}$$

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$$u|_S = v(x,t), \quad (x,t) \in S, \tag{2}$$

$$u|_{t=0} = u_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x), \quad x \in \Omega. \tag{3}$$

The exact controllability problem for (1)-(3) is formulated as follows:

Given T find a Hilbert space H, such that for each initial pair $\{u_0, u_1\} \in H$ there exists a control $v \in L^2(S)$ such that the solution of (1)-(3) satisfies the stabilization conditions

$$u|_{t=T} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=T} = 0, \quad x \in \Omega. \tag{4}$$

Note that the similar problem has been considered in [5], where the equation (1) contains additional terms, which are the solution and its first derivatives. However the coefficients of (1) in [5] do not depend on t. The technique of proofs in [5] is based on the results of the theory of pseudodifferential operators. As it is known this technique is enough complicated. We use the Hilbert uniqueness method introduced by Lions [1] and applied in [2] which is more practical and simple. We find the concrete value for the threshold time moment T_0, whereas in [5] the existence of T_0 is shown theoretically.

2. Denotations and some assumptions

Let R^n be an n - dimensional Euclidean space and let be

$$x^0 \in R^n, \ m(x) = x - x^0 = \left(x_1 - x_1^0, ..., x_n - x_n^0\right), \ m_k(x) = x_k - x_k^0.$$

Let $R\left(x^0\right)$ be a radius of the minimal ball with center at x^0, containing Ω. By $\nu(x)$ we denote the unit exterior normal to Γ. Denote $\Gamma\left(x^0\right) = \{x \in \Gamma | \, (m(x), \nu(x)) > 0\}$, $\Gamma_*\left(x^0\right) = \{x \in \Gamma | \, (m(x), \nu(x)) \le 0\}$, where $(m(x), \nu(x))$ is an inner product in R^n,

$$S\left(x^0\right) = \Gamma\left(x^0\right) \times (0,T), S_*\left(x^0\right) = \Gamma_*\left(x^0\right) \times (0,T), S = S\left(x^0\right) \cup S_*\left(x^0\right).$$

Denote

$$A(t)\,u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right).$$

Assume that $a_{ij}(x,t) = a_{ji}(x,t)$, for all $(x,t) \in Q$ and for all $\xi \in R^n, (x,t) \in Q$, $\sum\limits_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha \sum\limits_{i=1}^{n} \xi_i^2$, $\alpha = const > 0$ and $a_{ij} \in C^1(\overline{Q})$, $i,j = 1,...,n$.

Let there exist a number $\delta, 0 < \delta < 1$ such that

$$(1-\delta)\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j - \frac{1}{2}\sum_{k=1}^{n}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_k}a_{ij}(x,t)\,m_k\xi_i\xi_j \geq 0 \,(\text{see [6]})$$

for all $\xi \in R^n$, $(x,t) \in Q$.

Assume that $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$. Here we use the denotations from [7].

By $a(t;\Phi,\Psi)$ we denote the following bilinear form:

$$a(t;\Phi,\Psi) = \sum_{i,j=1}^{n}\int\limits_{\Omega} a_{ij}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}dx.$$

Let

$$\beta(t) \equiv \max_{1\leq i,\ j\leq n}\left\|\frac{\partial a_{ij}}{\partial t}\right\|_{C(\overline{\Omega})},\ T_0 = \frac{R(x^0)}{\delta}C_\alpha C_1^2,$$

$$C_\alpha = \max\left\{1,\frac{1}{\alpha}\right\}, C_1 = exp(\frac{n}{\alpha}\int\limits_0^T \beta(t)\,dt).$$

Below we show that for $T > T_0$ the system is controllable, therefore T_0 is called a threshold time moment.

By a solution of problem (1)-(3), for the given control $v \in L^2(S)$ we mean a function $u = u(x,t)$ from $L^2(Q)$ satisfying the integral identity

$$\int\limits_Q u\left[\frac{\partial^2 g}{\partial t^2} + A(t)\,g\right]dxdt =$$

$$= \int\limits_Q fgdxdt - \int\limits_S v\frac{\partial g}{\partial \nu_A}ds + \langle u_1(x), g(x,0)\rangle - \int\limits_\Omega u_0(x)\frac{\partial g(x,0)}{\partial t}dx,$$

$$\forall g \in C^2(\overline{Q}), g(x,T) = \frac{\partial g(x,T)}{\partial t} = 0,\ \ g|_S = 0.$$

Here $\langle .,.\rangle$ means the value of the functional from $H^{-1}(\Omega)$ on the element from $H_0^1(\Omega)$,

$$\frac{\partial}{\partial \nu_A} \equiv \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial}{\partial x_j} \cos(\nu, x_i)$$

is co-normal with respect to A derivative, $\cos(\nu, x_i)$ is the i-th direction cosine of the exterior normal to the boundary Γ of the domain Ω.

Problem (1)-(3) has a unique weak solution $u(x,t)$, determined by means of transposition (see [8]). Note that such a solution possesses the following properties

$$u \in C\left([0,T]; L^2(\Omega)\right), \ \frac{\partial u}{\partial t} \in C\left([0,T]; H^{-1}(\Omega)\right) \ \text{(see [9])}.$$

3. Main result

Theorem 3.1. *Let $T > T_0$. Then for each pair $\{u_0, u_1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $v \in L^2(S)$ such that the corresponding solution of problem (1)-(3) satisfies the conditions (4).*

Proof. To prove the theorem we use Hilbert uniqueness method [1]. Let as take $\varphi_0 \in H_0^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$ and consider the problem

$$\frac{\partial^2 \varphi}{\partial t^2} + A(t)\varphi = 0 \text{ in } Q, \tag{5}$$

$$\varphi|_S = 0, \tag{6}$$

$$\varphi|_{t=0} = \varphi_0(x), \ \left.\frac{\partial \varphi}{\partial t}\right|_{t=0} = \varphi_1(x) \ \text{ in } \Omega. \tag{7}$$

Then for the unique solution of problem (5)-(7) the condition $\frac{\partial \varphi}{\partial \nu} \in L^2(S)$ (see [9], [10]) is satisfied.

Consider the following problem

$$\frac{\partial^2 \psi}{\partial t^2} + A(t)\psi = f \text{ in } Q, \tag{8}$$

$$\psi = \begin{cases} \frac{\partial \varphi}{\partial \nu} & \text{on } S\left(x^0\right), \\ 0 & \text{on } S_*\left(x^0\right), \end{cases} \tag{9}$$

$$\psi|_{t=T} = 0, \ \left.\frac{\partial \psi}{\partial t}\right|_{t=T} = 0 \ \text{ in } \ \Omega. \tag{10}$$

Problem (8)-(10) also possesses a unique weak solution $\psi(x,t)$ determined by means of transposition (see [8]), and moreover

$$\psi \in C\left([0,T];L^2(\Omega)\right), \ \frac{\partial\psi}{\partial t} \in C\left([0,T];H^{-1}(\Omega)\right) \ \text{(see [9])}. \tag{11}$$

For $\varphi_0 \in H_0^1(\Omega), \varphi_1 \in L^2(\Omega)$ we solve problem (5)-(7) and obtain $\dfrac{\partial\varphi}{\partial\nu} \in L^2(S)$. Then we solve problem (8)-(10) and show that (11) is valid. Therefore we determine the mapping

$$\wedge : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega),$$

given by the equality

$$\wedge\{\varphi_0,\varphi_1\} = \left\{\frac{\partial\psi(x,0)}{\partial t}, -\psi(x,0)\right\}. \tag{12}$$

Smoothing all the data of the problems (5)-(7) and (8)-(10), we obtain that the solutions of the smoothed problems belong at least to space $H^2(Q)$. Then multiplying the both hand sides of the smoothed equation (5) by the $\psi(x,t)$, solution of the smoothed problem (8)-(10), integrating on the domain Q, taking into account the boundary conditions (6),(7),(9),(10) and then passing to the limit with respect to the smoothing parameter, we obtain

$$\left\langle \frac{\partial\psi(x,0)}{\partial t}, \varphi_0(x) \right\rangle - \int_\Omega \psi(x,0)\,\varphi_1(x)\,dx =$$

$$= \int_{S(x^0)} \sum_{i,j=1}^n a_{ij}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds - \int_Q f(x,t)\,\varphi(x,t)\,dxdt. \tag{13}$$

It follows from (12) and (13) that

$$\langle \wedge\{\varphi_0,\varphi_1\}, \{\varphi_0,\varphi_1\}\rangle = \int_{S(x^0)} \sum_{i,j=1}^n a_{ij}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds - \int_Q f\varphi\,dxdt, \tag{14}$$

where $\langle \wedge\{\varphi_0,\varphi_1\}, \{\varphi_0,\varphi_1\}\rangle$ means duality relation between $H^{-1}(\Omega) \times L^2(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$.

In $H_0^1(\Omega) \times L^2(\Omega)$, consider the quadratic form

$$\|\{\varphi_0,\varphi_1\}\|_F^2 = \int_{S(x^0)} \sum_{i,j=1}^n a_{ij}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds.$$

Let as show that there exist such constants $M_1, M_2 > 0$ that

$$(T - T_0)\, M_1 \left\| \{\varphi_0,\varphi_1\} \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \le \int\limits_{S(x^0)} \sum_{i,j=1}^{n} a_{ij}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds \le$$

$$\le M_2 \left\| \{\varphi_0,\varphi_1\} \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} . \tag{15}$$

In lemma 3.2 (section 3, [2]) it is proved that

$$\int\limits_{S(x^0)} \sum_{i,j=1}^{n} a_{i\,j}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds \le C \left\| \{\varphi_0,\varphi_1\} \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} . \tag{16}$$

And in lemma 3.3 (section 3, [2]) it is shown that

$$(T - T_0)E_0 \le \frac{R(x^0)C_1}{2\delta} \int\limits_{S(x^0)} \sum_{i,j=1}^{n} a_{i\,j}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds$$

or

$$(T - T_0)\frac{2\delta E_0}{R(x^0)C_1} \le \int\limits_{S(x^0)} \sum_{i,j=1}^{n} a_{i\,j}\nu_i\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds. \tag{17}$$

Let as denote an energy integral corresponding to the equation (5) by

$$E(t) = \frac{1}{2}\int\limits_{\Omega} \left[\left|\frac{\partial\varphi\,(x,t)}{\partial t}\right|^2 + \sum_{i,j=1}^{n} a_{i\,j}(x,t)\frac{\partial\varphi(x,t)}{\partial x_i}\ \frac{\partial\varphi(x,t)}{\partial x_j} \right] dx,$$

similarly to [8]. Then

$$E_0 = E\,(0) = \frac{1}{2}\int\limits_{\Omega} (\varphi_1^2(x) + \sum_{i,j=1}^{n} a_{i\,j}(x,0)\frac{\partial\varphi_0(x)}{\partial x_i}\ \frac{\partial\varphi_0(x)}{\partial x_j})dx.$$

¿From the coerciveness condition on the coefficients $a_{ij}(x,t)$ it follows that

$$E_0 \ge \frac{1}{2}\int\limits_{\Omega} \left[\varphi_1^2(x) + \alpha\sum_{i=1}^{n} \left(\frac{\partial\varphi_0(x)}{\partial x}\right)^2 \right] dx \ge M_\alpha \int\limits_{\Omega} \left[\varphi_1^2(x) + \sum_{i=1}^{n} \left(\frac{\partial\varphi_0(x)}{\partial x}\right)^2 \right] dx,$$

where

$$M_\alpha = \frac{1}{2}\min\{1,\alpha\}.$$

Then

$$E_0 \geq M_\alpha \left\|\{\varphi_0,\varphi_1\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}.$$

From (17) one can get

$$(T-T_0)\frac{2\delta\, M_\alpha}{R(x^0)C_1}\left\|\{\varphi_0,\varphi_1\}\right\|_{\mathrm{H}_0^1(\Omega)\times L^2(\Omega)} \leq \int\sum_{i,j}^{n} a_{ij}\nu_i\nu_j\left(\frac{\partial\varphi}{\partial\nu}\right)^2 ds. \qquad (18)$$

Thus from (16) and (18) the validity of the inequalities (15) follows.

Inequalities (15) show that for $T > T_0$ the norm $\left\|\{\varphi_0,\varphi_1\}\right\|^2_F$ is equivalent (see [11]) to the norm in $H_0^1(\Omega)\times L^2(\Omega)$ defined by the equality

$$\left\|\{\varphi_0,\varphi_1\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} = \int_\Omega \sum_{i=1}^{n}\left(\frac{\partial\varphi_0(x)}{\partial x_i}\right)^2 dx + \int_\Omega (\varphi_1(x))^2\, dx.$$

Also the inequalities (15) show that $F = H_0^1(\Omega)\times L^2(\Omega)$ for $T > T_0$. Note that $F' = H^{-1}(\Omega)\times L^2(\Omega)$ is a space conjugated to F, the operator $\wedge$ is continuous by the norm $\|\cdot\|_F$.

Considering (5)-(7) we obtain the existence of such $M_3 > 0$ that

$$\|\varphi\|_X \leq M_3\left(\|\varphi_0\|_{H_0^1(\Omega)} + \|\varphi_1\|_{L^2(\Omega)}\right), \qquad (19)$$

where

$$X = \left\{\varphi|\varphi\in C\left([0,T];H^{-1}(\Omega)\right), \frac{\partial\varphi}{\partial t}\in C\left([0,T];L^2(\Omega)\right)\right\} \text{ (see [8], [9])}.$$

Since

$$\left|\int_Q f\varphi dxdt\right| \leq \|f\|_{L^2(Q)}\cdot\|\varphi\|_{L^2(Q)},$$

then by (19) it follows that there exists such $M_4 > 0$ that

$$-\int_Q f\varphi dxdt \geq -\|f\|_{L^2(Q)}\cdot\|\varphi\|_{L^2(Q)} \geq -M_4\left(\|\varphi_0\|_{H_0^1(\Omega)} + \|\varphi_1\|_{L^2(\Omega)}\right). \qquad (20)$$

Then as one may obtain from (14),(15) and (20) the operator $\wedge : F \to F'$ is coercive, therefore it is an isomorphism between F and conjugated F'. This shows that for the given pair $\{u_1(x), -u_0(x)\} \in F' = H^{-1}(\Omega) \times L^2(\Omega)$ there exists a unique pair $\{\varphi_0, \varphi_1\} \in F = H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$\wedge \{\varphi_0, \varphi_1\} = \{u_1(x), -u_0(x)\}. \tag{21}$$

Then from (12) and (21)we obtain that the solution $\psi(x,t)$ of problem (8)-(10) satisfies to the conditions

$$\psi(x,0) = u_0(x), \ \frac{\partial \psi(x,0)}{\partial t} = u_1(x).$$

Thus, the unique solution $\psi(x,t)$ of problem (8)-(10) corresponding to the control

$$v = \begin{cases} \dfrac{\partial \varphi}{\partial \nu} & \text{on } S\left(x^0\right), \\ 0 & \text{on } S_*\left(x^0\right) \end{cases}$$

coincides with the solution $u(x,t)$ of problem (1)-(3). It shows that $u(x,t)$ satisfies the stabilization conditions (4). The theorem 3.1 is proved.

In the theorem 3.1 it is assumed that $T > T_0$. It may be shown that for a certain class of functions $a_{ij}(x,t)$ this inequality has a solution. For example, if $a_{ij}(x,t)$, $i,j = \overline{1,n}$ do not depend on t, then $\beta(t) \equiv 0$, therefore $C_1 = 1$. Then the inequality $T > T_0$ turns to $T > \frac{R(x^0)}{\delta} C_\alpha$.

Remark 3.1. In the paper, some inaccuracies of the paper [2] are corrected, namely, on page 478 of that paper, in formula (5) for the value of the constant T_0 the multiplier 2 is unnecessary, the value of the constant C_1 is not shown. In formula (10) on page 479, instead of

$$\psi = \begin{cases} a_{ij}\nu_i\nu_j \dfrac{\partial \varphi}{\partial \nu} & \text{on } S\left(x^0\right), \\ 0 & \text{on } S_*\left(x^0\right) \end{cases}$$

should be

$$\psi = \begin{cases} \dfrac{\partial \varphi}{\partial \nu} & \text{on } S\left(x^0\right), \\ 0 & \text{on } S_*\left(x^0\right). \end{cases}$$

4. Proof of the formula for C_1

Let

$$E(t) = \frac{1}{2}\int\limits_{\Omega}\left[\left|\frac{\partial\varphi(x,t)}{\partial t}\right|^2 + \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial\varphi(x,t)}{\partial x_j}\right]dx$$

be an energy integral corresponding to the equation (5). Using the equality ([8], page 297)

$$2E(t) = 2E_0 + \int\limits_0^t \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial\varphi(x,t)}{\partial x_j}dx$$

and coerciveness condition on the coefficients $a_{ij}(x,t), i,j = \overline{1,n}$ we obtain

$$E(t) \leq E_0 + \frac{n}{\alpha}\int\limits_0^t \beta(s)E(s)\,ds.$$

¿From this considering Gronwall's lemma one can get

$$E(t) \leq C_1 E_0, \forall t \in [0,T],$$

where

$$C_1 = \exp\left(\frac{n}{\alpha}\int\limits_0^T \beta(t)\,dt\right).$$

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