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Exponential stabilization of linear systems with mixed delays in state and control

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Abstract

This paper considers the exponential stabilization problem for a class of linear systems with mixed time delays in both state and control. By using an improved Lyapunov-Krasovskii functional and memoryless controller, new conditions for the exponential stabilization of the system are derived in terms of linear matrix inequalities (LMIs). This approach allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. A numerical example is given to illustrate the effectiveness of our result.

Keywords: Exponential stabilization, mixed time delays, linear matrix inequalities.
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1 Introduction

Time delays in control inputs are often encountered in many practical systems because of transmission of the measurement information. The existence of these delays may be the source of instability and serious deterioration in the performance of the closed-loop systems. Therefore, the problem of stability and stabilization of control systems with input delays has been received considerable attention from many researchers (see, e.g. [1, 3, 5, 6, 8, 12, 17, 19, 20, 22] and references therein). The stability problem has been considered mostly for linear time-invariant control systems with state and control delays. The lead to memoryless controllers which means control law of the form $u(t) = Kx(t)$ ([8, 13, 22]), or to more general controllers with memory that include, nevertheless, an instantaneous feedback term $u(t) = Kx(t) + \sum_{i=1}^{m} K_i x(t - h_i)$. Another method to improve the control of linear time-invariant systems with input delay is the reduction method proposed in [1], which reduces the system under consideration to a delay-free ordinary system by certain state transformation. Based on the reduction method, Moon et. al. [12] proposed a delay feedback controller for the robust stability of linear uncertain systems with input delay. By using an improved state transformation, Chen and Zheng [3], Yue [19], Yue and Han [20], derived sufficient conditions for the robust stabilization of linear uncertain systems with unknown input delay in terms of LMI's but the system is required to be global controllable.

Recently, special interest has been devoted to the exponential stability and stabilization problem for linear time-delay systems [5, 6, 10, 11, 14, 15, 16, 18]. Based on linear matrix inequalities approach [2], a systematic procedure for finding exponential stability conditions has been proposed in [11] for LTI systems with constant delay. In [9, 15, 18], by using state transformation $\xi(t) = e^{\lambda t}x(t)$, delay-dependent conditions for robust exponential stability of linear uncertain systems with constant delay were given in terms of LMIs. By also using the state transformation method, [14] gives conditions for the exponential stability of non-autonomous systems with constant delay in terms of solution of Riccati-type differential equation, which is further improved in [16] by using Razumikhin method.

In this paper, the results of [9]-[13], [18]-[20] will be extended to linear systems with mixed delays in both state and control:

$$\dot{x}(t) = Ax(t) + Dx(t-\tau) + E\int_{t-\tau}^{t} x(s)ds + Bu(t) + Cu(t-r) + F\int_{t-r}^{t} u(s)ds.$$

By using an improved Lyapunov-Krasovskii functional, new delay-dependent conditions for the exponential stabilization are derived in terms of linear matrix inequalities. The conditions do not require any assumption about the controllability of the nominal system, e.g., neither (A, B) nor $(A + D, B)$ needs to be controllable. The approach also allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

The paper is organized as follows. Section 2 presents notations, definitions and some technical propositions needed for the proof of the main result. Delay-dependent conditions for exponential stabilization and an illustrated example are presented in Section 3. The paper ends with conclusions and cited references.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all non-negative real numbers; R^n denotes the n dimensional Euclidean space with the Euclidean norm $\|.\|$; $R^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions, I_m denotes the identity matrix in $R^{m\times m}$, $\lambda_{\max}(A), (\lambda_{\min}(A)$, resp.) denotes the maximal (the minimum, resp.) number of the real part of eigenvalues of A, A^T denotes the transpose of the matrix A, $A \geq 0(A > 0$, resp.) means A is semi-positive definite (positive definite, resp.), $A \geq B$ means $A - B \geq 0$.

Consider a class of linear systems with mixed delays in state and control of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dx(t-\tau) + E\int_{t-\tau}^{t} x(s)ds + Bu(t) \\ &\quad + Cu(t-r) + F\int_{t-r}^{t} u(s)ds, t \in R^+, \qquad (1) \\ x(t) &= \phi(t), \quad t \in [-h, 0], \quad h = \max\{\tau, r\}, \end{aligned}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, τ, r are time delays; $\phi(t) \in C([-h,0], R^n)$ is the initial function with the norm $\|\phi\|_h = \sup_{-h\leq s\leq 0}\|\phi(s)\|$; A, B, C, D, E, F are given real constant matrices with appropriate dimensions.

The unforced system (i.e. without control) is of the form

$$\dot{x}(t) = Ax(t) + Dx(t-\tau) + E\int_{t-\tau}^{t} x(s)ds. \qquad (2)$$

In this paper, a memoryless state feedback controller

$$u(t) = Kx(t), \quad t \in R^+,$$

is employed to stabilize system (1). Then, the closed-loop system of (1) is given by

$$\dot{x}(t) = [A + BK]x(t) + Dx(t-\tau) + CKx(t-r) + E\int_{t-\tau}^{t} x(s)ds + F\int_{t-r}^{t} Kx(s)ds. \tag{3}$$

Definition 2.1 *For given $\alpha > 0$. System (2) is said to be α-exponentially stable if there exists $N \geq 1$ such that every solution $x(t,\phi)$ of (2) satisfies*

$$\|x(t,\phi)\| \leq Ne^{-\alpha t}\|\phi\|_h, \quad t \geq 0.$$

System (1) is α-exponentially stabilizable if there exists a state feedback controller $u(t) = Kx(t)$ such that the closed-loop system (3) is α-exponentially stable.

The objective of this paper is to design the memoryless feedback control law that makes system (1) is exponential stabilizable. For this purpose, the following technical propositions are first introduced.

Proposition 2.1 *(Schur complement Theorem [2]) For any constant matrices X, Y, Z, where $X = X^T$, $Y = Y^T > 0$. Then $X - Z^TY^{-1}Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & Y \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} < 0.$$

Proposition 2.2 *(Matrix Cauchy inequality) For any symmetric positive definite matrix $M \in R^{n\times n}$ and $x, y \in R^n$, we have*

$$2\langle x, y\rangle \leq \langle Mx, x\rangle + \langle M^{-1}y, y\rangle.$$

Proposition 2.3 *([4]) For any symmetric positive definite matrix $W \in R^{n\times n}$, scalar $\nu \geq 0$, and vector function $\omega : [0,\nu] \to R^n$ such that the integrals concerned are well defined, then*

$$\left(\int_0^{\nu} \omega(s)ds\right)^T W \left(\int_0^{\nu} \omega(s)ds\right) \leq \nu \int_0^{\nu} \omega^T(s)W\omega(s)ds.$$

3 Main result

For $\alpha > 0, \tau > 0, r > 0$, symmetric positive definite matrices $P, Q, R \in R^{n\times n}$ and matrix $Y \in R^{m\times n}$ we denote

$$G = BY + Y^T B^T + e^{2\alpha r}\left(CC^T + rFF^T\right),$$
$$\Omega = AP + PA^T + G + Q + \tau R,$$
$$U = \begin{bmatrix} DP & EP & Y^T \end{bmatrix}, \quad \mu = (1+r)^{-1},$$
$$H = diag\left[e^{-2\alpha\tau}Q, \ \frac{1}{\tau}e^{-2\alpha\tau}R, \ \mu I_m\right],$$
$$\mathcal{M}(P,Q,R) = \begin{bmatrix} \Omega & U \\ U^T & -H \end{bmatrix}, \quad \mathcal{N}(P) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}.$$

The main result is stated in the following theorem.

Theorem 3.1 *Given $\alpha > 0$. System (1) is α-exponentially stabilizable if there exist symmetric positive definite matrices P, Q, R and matrix Y satisfy the following LMI*

$$\mathcal{M}(P,Q,R) + 2\alpha\mathcal{N}(P) < 0. \tag{4}$$

The state feedback controller is given by

$$u(t) = YP^{-1}x(t), \quad t \geq 0.$$

Moreover, every solution $x(t,\phi)$ of the closed-loop system satisfies

$$\|x(t,\phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|_h, \quad t \geq 0,$$

where

$$\alpha_2 = \lambda_{\min}^{-1}(P) + \left[\tau\lambda_{\max}(Q) + \frac{1}{2}\tau^2\lambda_{\max}(R) + \left(1 + \frac{1}{2}r^2\right)\lambda_{\max}(Y^TY)\right]\lambda_{\min}^{-2}(P),$$
$$\alpha_1 = \lambda_{\max}^{-1}(P). \tag{5}$$

Proof. Denote $X = P^{-1}, \bar{Q} = XQX, \bar{R} = XRX$ and $K = YX$. Consider the following Lyapunov-Krasovskii functional

$$V(x_t) = V_1 + V_2 + V_3 + V_4 + V_5,$$

where

$$
\begin{aligned}
V_1 &= x^T(t)Xx(t),\\
V_2 &= \int_{-\tau}^{0} e^{2\alpha s}x^T(t+s)\bar{Q}x(t+s)ds,\\
V_3 &= \int_{-\tau}^{0}\int_{s}^{0} e^{2\alpha\xi}x^T(t+\xi)\bar{R}x(t+\xi)d\xi ds,\\
V_4 &= \int_{-r}^{0} e^{2\alpha s}x^T(t+s)K^TKx(t+s)ds,\\
V_5 &= \int_{-r}^{0}\int_{s}^{0} e^{2\alpha\xi}x^T(t+\xi)K^TKx(t+\xi)d\xi ds.
\end{aligned}
$$

It's easy to verify that

$$
\alpha_1\|x(t)\|^2 \le V(x_t) \le \alpha_2\|x_t\|_h^2, \quad t \ge 0, \tag{6}
$$

where α_1, α_2 are defined in (5).

Taking derivative of V_1 along solutions of the closed-loop system (3), we get

$$
\begin{aligned}
\dot{V}_1 = x^T(t)\Big[A^TX + XA + X(BY + Y^TB^T)X\Big]x(t)&\\
+ 2x^T(t)XDx(t-\tau) + 2x^T(t)XCu(t-r)&\\
+ 2x^T(t)XE\int_{t-\tau}^{t} x(s)ds + 2x^T(t)XF\int_{t-r}^{t} u(s)ds.&
\end{aligned} \tag{7}
$$

Applying Proposition 2.2 and 2.3 gives

$$
2x^T(t)XDx(t-\tau) \le e^{2\alpha\tau}x^T(t)XD\bar{Q}^{-1}D^TXx(t) + e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau),
$$

$$
2x^T(t)XCu(t-r) \le e^{2\alpha r}x^T(t)XCC^TXx(t) + e^{-2\alpha r}\|u(t-r)\|^2.
$$

$$
\begin{aligned}
2x^T(t)XE\int_{t-\tau}^{t} x(s)ds &\le \tau e^{2\alpha\tau}x^T(t)XE\bar{R}^{-1}E^TXx(t)\\
&\quad + \frac{1}{\tau}e^{-2\alpha\tau}\left(\int_{t-\tau}^{t} x(s)ds\right)^T\bar{R}\left(\int_{t-\tau}^{t} x(s)ds\right)\\
&\le \tau e^{2\alpha\tau}x^T(t)XE\bar{R}^{-1}E^TXx(t)\\
&\quad + e^{-2\alpha\tau}\int_{t-\tau}^{t} x^T(s)\bar{R}x(s)ds,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
2x^T(t)XF\int_{t-r}^t u(s)ds &\le re^{2\alpha r}x^T(t)XFF^TXx(t) \\
&\quad + \frac{1}{r}e^{-2\alpha r}\left(\int_{t-r}^t u(s)ds\right)^T\left(\int_{t-r}^t u(s)ds\right) \\
&\le re^{2\alpha r}x^T(t)XFF^TXx(t) \\
&\quad + e^{-2\alpha r}\int_{t-r}^t \|u(s)\|^2 ds.
\end{aligned} \tag{9}
$$

Therefore, from (7) to (9) we have

$$
\begin{aligned}
\dot{V}_1 \le\ & x^T(t)\Big[A^TX + XA + X(BY + Y^TB^T)X\Big]x(t) \\
& + e^{2\alpha\tau}x^T(t)XD\bar{Q}^{-1}D^TXx(t) + e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau) \\
& + e^{2\alpha r}x^T(t)XCC^TXx(t) + e^{-2\alpha r}\|u(t-r)\|^2 \\
& + \tau e^{2\alpha\tau}x^T(t)XE\bar{R}^{-1}E^TXx(t) + e^{-2\alpha\tau}\int_{t-\tau}^t x^T(s)\bar{R}x(s)ds \\
& + re^{2\alpha r}x^T(t)XFF^TXx(t) + e^{-2\alpha r}\int_{t-r}^t \|u(s)\|^2 ds.
\end{aligned} \tag{10}
$$

Next, taking derivative of $V, i = 2, 3, 4, 5$, along solutions of the closed-loop system respectively, we obtain

$$
\begin{aligned}
\dot{V}_2 &= x^T(t)\bar{Q}x(t) - e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau) - 2\alpha V_2, \\
\dot{V}_3 &= \tau x^T(t)\bar{R}x(t) - \int_{-\tau}^0 e^{2\alpha s}x^T(t+s)\bar{R}x(t+s)ds - 2\alpha V_3 \\
&\le \tau x^T(t)\bar{R}x(t) - e^{-2\alpha\tau}\int_{t-\tau}^t x^T(s)\bar{R}x(s)ds - 2\alpha V_3, \\
\dot{V}_4 &= x^T(t)K^TKx(t) - e^{-2\alpha r}x^T(t-r)K^TKx(t-r) - 2\alpha V_4 \\
&= x^T(t)XY^TYXx(t) - e^{-2\alpha r}\|u(t-r)\|^2 - 2\alpha V_4, \\
\dot{V}_5 &= rx^T(t)K^TKx(t) - \int_{-r}^0 e^{2\alpha s}x^T(t+s)K^TKx(t+s) - 2\alpha V_5 \\
&\le rx^T(t)XY^TYXx(t) - e^{-2\alpha r}\int_{t-r}^t \|u(s)\|^2 ds - 2\alpha V_5.
\end{aligned} \tag{11}
$$

Combining (10) and (11) we get

$$\begin{aligned}\dot V(x_t) + 2\alpha V(x_t) &\le x^T(t)\left(A^TX + XA + 2\alpha X\right)x(t)\\ &+ x^T(t)X\left[BY + Y^TB^T + (1+r)Y^TY + Q + \tau R\right]Xx(t)\\ &+ e^{2\alpha\tau}x^T(t)X\left(D\bar{Q}^{-1}D^T + \tau E\bar{R}^{-1}E^T\right)Xx(t)\\ &+ e^{2\alpha r}x^T(t)X\left(CC^T + rFF^T\right)Xx(t)\\ &= \eta^T(t)\Big(AP + PA^T + 2\alpha P + G + Q + \tau R\Big)\eta(t) + \eta^T(t)\Phi\eta(t),\\ &= \eta^T(t)\Big(\Omega + 2\alpha P + \Phi\Big)\eta(t),\end{aligned} \tag{12}$$

where $\eta(t) = Xx(t), \Phi = UH^{-1}U^T$.

By schur complement theorem (Proposition 2.1), (4) implies that

$$\Omega + 2\alpha P + UH^{-1}U^T < 0.$$

Therefore, it follows from (12)

$$\dot V(x_t) + 2\alpha V(x_t) \le 0, \quad \forall t \ge 0,$$

and hence

$$V(x_t) \le V(\phi)e^{-2\alpha t} \le \alpha_2\|\phi\|_h^2 e^{-2\alpha t}, \quad t \ge 0.$$

Taking (6) into account we finally obtain

$$\|x(t,\phi)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|_h, \quad t \ge 0,$$

where α_1, α_2 are defined by (5). The proof is complete.

Remark 3.1 *For given $\alpha > 0$, the exponential stabilization condition is given in terms of LMIs, which can be solve by various efficient convex algorithms [2]. By iteratively solving the LMI given in Theorem 3.1 with respect to h, one can find the maximum upper bound of the delays that guarantees exponential stabilization of system* (1) *with decay rate α.*

Remark 3.2 *The linear control systems with input delay considered in previous works (e.g. [1, 3, 8, 12, 19, 20, 22]) are special cases of the system* (1)*, where* $D = E = 0, F = 0$.

Numerical example. Consider the following control system:

$$\begin{aligned}\dot x(t) = Ax(t) + Dx(t-\tau) + E\int_{t-\tau}^{t} x(s)ds\\ + Bu(t) + Cu(t-r) + F\int_{t-r}^{t} u(s)ds,\end{aligned} \tag{13}$$

where $\tau = 1, r = 1$ and

$$(A, D, E, B, C, F) = \left(\begin{bmatrix} -4 & 1 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

By the Kalman rank condition [21], neither system (A, B) nor $(A + D, B)$ is controllable system. However, for $\alpha = 0.5$, LMI (4) is feasible using LMI toolbox of Matlab. Therefore the control system (13) is exponentially stabilizable with decay rate $\alpha = 0.5$. The LMI (4) in Theorem 3.1 is satisfied with

$$P = \begin{bmatrix} 39.0742 & -5.7013 \\ -5.7013 & 35.9471 \end{bmatrix}, \quad Q = \begin{bmatrix} 59.6758 & 29.5378 \\ 29.5378 & 90.8470 \end{bmatrix},$$
$$R = \begin{bmatrix} 80.9893 & -56.5626 \\ -56.5626 & 85.4023 \end{bmatrix}, \quad Y = \begin{bmatrix} -1.2705 & 1.2689 \end{bmatrix}.$$

The state feedback controller which makes system (13) is exponentially stabilizable with decay rate $\alpha = 0.5$ is given by $u(t) = Kx(t), t \geq 0$, where

$$K = \begin{bmatrix} -0.0280 & 0.0309 \end{bmatrix}.$$

By computation we find that, every solution $x(t, \phi)$ of the closed-loop system satisfies

$$\|x(t, \phi)\| \leq 3.0577e^{-0.5t}\|\phi\|_1, \quad t \geq 0.$$

The time-simulation response of the solution of system (13) with initial condition $\phi(t) = (5, 5)^T, t \in [-1, 0]$ is shown by Figure 1.

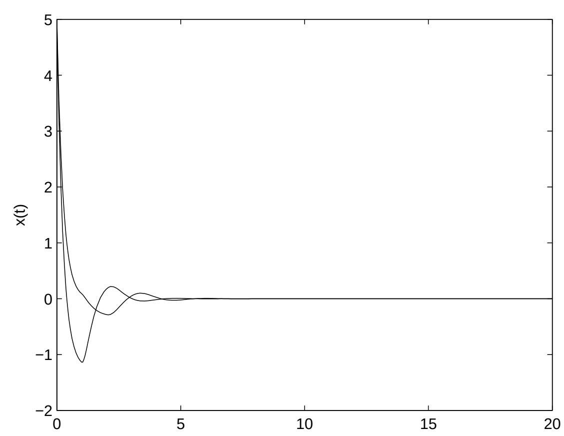

Fig. 1 Time-simulation response of solution

4 Conclusion

This paper has proposed new conditions for the exponential stabilization of linear systems with mixed delays in both state and control. Based on an improved Lyapunov-Krasovskii functional, the delay-dependent exponential stability conditions are derived in terms of linear matrix inequalities which allows to compute simultaneously the two bounds that characterize the exponential stability of the solution. A numerical example is given to show the effectiveness of the obtained result.

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