Exponential stability of linear uncertain polytopic systems with distributed time-varying delays

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Abstract

In this paper, a class of uncertain linear polytopic systems with distributed time varying delays is studied. By using an improved parameter dependent Lyapunov-Krasovskii functional approach and linear matrix inequality technique, delay-dependent sufficient conditions for exponential stability of the system are first established in terms of Mondie-Kharitonov type's linear matrix inequalities (LMIs). Numerical example is presented to demonstrate the effectiveness of the proposed conditions.

Keywords: polytopic uncertainty; exponential stability; distributed delays; linear matrix inequality.
2000MSC: 34D05, 34D20, 34K20, 34K35

1   Introduction

The stability analysis of linear time-delay systems is a topic of great practical importance, which has attracted a lot of interest over the decades (see e.g. [1, 2, 3, 5]). Also, systems uncertainties arise from many sources such as unavoidable approximation, data errors and aging of systems and so the stability issue of uncertain time delay systems has been investigated by many researchers (see

[6, 8, 9]), where the Lyapunov-Krasovskii functional method is certainly used as the main tool.

Recently, the stability analysis of linear system with polytopic type uncertainties has also received much attention (see e.g. [4, 8, 10]). However, the distributed delays are not considered in the mentioned papers. In practice, systems with distributed delays have many important applications in various areas (see [2, 3]). Theoretically, systems with distributed delays are much more complicated, especially for the case where the system matrices belong to some convex polytope. To the best of our knowledge, so far, no result on the stability for uncertain linear polytopic systems with distributed delays is available in the literature. This motivates our present investigation.

In this paper, we develop the robust stability problem for linear uncertain polytopic systems with discrete and distributed time varying delays. The novel feature of the results obtained in this paper is twofold. First, the system considered in this paper is convex polytopic uncertain subjected to discrete and distributed time varying delays. Second, by employing an improved parameter dependent Lyapunov-Krasovskii functional and linear matrix inequality technique, delay dependent sufficient conditions for the exponential stability of the system are obtained in terms of the Mondié-Kharitonov type's LMI conditions [7]. The approach also allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

The paper is organized as follows: Section 2 presents notations, definitions and a well-known technical proposition needed for the proof of the main result. Delay dependent exponential stability conditions of the system and numerical example is presented in Section 3. The paper ends with conclusions and cited references.

2 Preliminaries

The following notations will be employed throughout this paper: A^{T} denotes the transpose of A, $\lambda(A)$ denotes the set of all eigenvalues of A, $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$, $\lambda_{\min}(A) = \min\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$; matrix $Q \geq 0$ ($Q > 0$, resp.) means Q is semi positive definite matrix i.e. $\langle Qx, x \rangle \geq 0, \forall x \in R^n$ (positive definite, resp. i.e. $\langle Qx, x \rangle > 0, \forall x \in R^n, x \neq 0$), $A \geq B$ means $A - B \geq 0$; $C([a, b], R^n)$ denotes the set of all R^n−valued continuous functions on $[a, b]$; the segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t + s) : s \in t \in [-h, 0]\}$ with its norm $\|x_t\| = \sup_{s \in [-h, 0]} \|x(t + s)\|$.

Consider a linear uncertain polytopic system with discrete and distributed time varying delays of the form

$$\dot{x}(t) = A(\sigma)x(t) + B(\sigma)x(t - h(t)) + D(\sigma)\int_{t-\tau(t)}^{t} x(s)ds, \ t \geq 0, \quad (2.1)$$

$$x(t) = \phi(t), \quad t \in [-\bar{h}, 0],$$

where $x(t) \in R^n$ is the state, $h(t), \tau(t)$ are time varying delay functions which are continuous and satisfying

$$0 \leq h(t) \leq h, \quad 0 \leq \tau(t) \leq \tau \quad \dot{h}(t) \leq \mu < 1, \quad \dot{\tau}(t) \leq \delta < 1 \quad (2.2)$$

and $\bar{h} = \max\{h, \tau\}$. The system matrices $[A(\sigma), B(\sigma), D(\sigma)]$ are subject to uncertainties and belong to the polytope Ω given by

$$\Omega = \left\{ [A, B, D](\sigma) = \sum_{i=1}^{p} \sigma_i[A_i, B_i, D_i], \sigma_i \geq 0, \sum_{i=1}^{p} \sigma_i = 1 \right\},$$

where $A_i, B_i, D_i \in R^{n \times n}$, $i = 1, 2, \ldots, p$, are given real matrices; $\phi(t) \in C([-\bar{h}, 0], R^n)$ is the initial function with the norm $\|\phi\| = \sup_{-\bar{h} \leq s \leq 0} \|\phi(s)\|$.

Definition 2.1. *For a given $\alpha > 0$, system (2.1)is said to be α-exponentially stable if there exist a number $\gamma \geq 1$ such that every solution $x(t, \phi)$ of system (2.1) satisfies the following condition*

$$\|x(t, \phi)\| \leq \gamma \|\phi\| e^{-\alpha t}, \quad \forall t \geq 0.$$

3   Main result

For positive numbers α, h, τ, symmetric positive definite matrices $P_i, Q_i, R_i \in R^{n \times n}$ and semi-positive definite matrix $S \in R^{n \times n}$ we denote

$$P = \sum_{i=1}^{p} \sigma_i P_i, \quad Q = \sum_{i=1}^{p} \sigma_i Q_i, \quad R = \sum_{i=1}^{p} \sigma_i R_i,$$

$$\Gamma_{ij} = P_j A_i + A_i^{\mathsf{T}} P_j + Q_j + \tau^2 R_j,$$

$$\mathcal{M}_i(P_j, Q_j, R_j) = \begin{bmatrix} \Gamma_{ij} & P_j B_i & P_j D_i \\ B_i^{\mathsf{T}} P_j & -(1-\mu)e^{-2\alpha h} Q_j & 0 \\ D_i^{\mathsf{T}} P_j & 0 & -\dfrac{(1-\delta)}{\tau} e^{-2\alpha \tau} R_j \end{bmatrix}, \quad (3.1)$$

$$\mathbb{S} = \begin{bmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{N}(P_j) = \begin{bmatrix} P_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i,j = 1,2,\ldots,p,$$

$$\lambda_{\min}(P) = \min_{i=1,2,\ldots,p}\{\lambda_{\min}(P_i)\}, \quad \lambda_{\max}(P) = \max_{i=1,2,\ldots,p}\{\lambda_{\max}(P_i)\},$$

$$\lambda_{\max}(Q) = \max_{i=1,2,\ldots,p}\{\lambda_{\max}(Q_i)\}, \quad \lambda_{\max}(R) = \max_{i=1,2,\ldots,p}\{\lambda_{\max}(R_i)\},$$

$$\gamma_1 = \lambda_{\min}(P), \quad \gamma_2 = \lambda_{\max}(P) + h\lambda_{\max}(Q) + \frac{1}{2}\tau^2\lambda_{\max}(R). \tag{3.2}$$

The main result is stated in the following theorem.

Theorem 3.1. *For given $\alpha > 0$. System (2.1) is α-exponentially stable if there exist positive definite matrices $P_i, Q_i, R_i, i = 1, 2, \ldots, p$ and a semi-positive definite matrix S such that the following LMIs hold:*

i) $\mathcal{M}_i(P_i, Q_i, R_i) + 2\alpha\mathcal{N}(P_i) \le -\mathbb{S}, \quad i = 1, 2, \ldots, p,$

ii) $\mathcal{M}_i(P_j, Q_j, R_j) + \mathcal{M}_j(P_i, Q_i, R_i) + 2\alpha\mathcal{N}(P_i + P_j) \le \dfrac{2}{p-1}\mathbb{S},$
$i = 1, \ldots, p-1, j = i+1, \ldots, p.$

Moreover, every solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \le \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\|e^{-\alpha t}, \quad t \ge 0,$$

where γ_1, γ_2 are defined in (3.2).

Proof. Because $P_i > 0, Q_i > 0, R_i > 0, \sigma_i \ge 0, i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p}\sigma_i = 1$ we have $P = \sum_{i=1}^{p}\sigma_i P_i, Q = \sum_{i=1}^{p}\sigma_i Q_i, R = \sum_{i=1}^{p}\sigma_i R_i$ are symmetric positive definite matrices. Consider the following parameter dependent Lyapunov-Krasovskii functional for system (2.1)

$$
\begin{aligned}
V(x_t) = x^{\mathsf{T}}(t)Px(t) &+ \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^{\mathsf{T}}(s)Qx(s)ds \\
&+ \int_{t-\tau(t)}^{t}\int_{s}^{t} e^{2\alpha(\zeta-t)}x^{\mathsf{T}}(\zeta)Rx(\zeta)d\zeta ds.
\end{aligned}
\tag{3.3}
$$

It can be verified from (3.3) that

$$\gamma_1\|x(t)\|^2 \le V(x_t) \le \gamma_2\|x_t\|^2, \quad t \ge 0, \tag{3.4}$$

where γ_1, γ_2 are defined in (3.2).

Taking derivative of $V(x_t)$ along solutions of system (2.1), we get

$$
\begin{aligned}
\dot{V}(x_t) =\ & x^{\mathsf{T}}(t)[A^{\mathsf{T}}P + PA]x(t) + 2x^{\mathsf{T}}(t)PBx(t - h(t)) \\
& + 2x^{\mathsf{T}}(t)PD\int_{t-\tau(t)}^{t} x(s)ds + x^{\mathsf{T}}(t)Qx(t) \\
& - (1 - \dot{h}(t))e^{-2\alpha h(t)}x^{\mathsf{T}}(t - h(t))Qx(t - h(t)) \\
& - 2\alpha\int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^{\mathsf{T}}(s)Qx(s)ds \\
& + \tau(t)x^{\mathsf{T}}(t)Rx(t) - (1 - \dot{\tau}(t))\int_{t-\tau(t)}^{t} e^{2\alpha(s-t)}x^{\mathsf{T}}(s)Rx(s)ds \\
& - 2\alpha\int_{t-\tau(t)}^{t}\int_{s}^{t} e^{2\alpha(\zeta-t)}x^{\mathsf{T}}(\zeta)Rx(\zeta)d\zeta ds \\
\leq\ & x^{\mathsf{T}}(t)[A^{\mathsf{T}}P + PA + Q + \tau R]x(t) + 2x^{\mathsf{T}}(t)PBx(t - h(t)) \\
& + 2x^{\mathsf{T}}(t)PD\int_{t-\tau(t)}^{t} x(s)ds - (1 - \mu)e^{-2\alpha h}x^{\mathsf{T}}(t - h(t))Qx(t - h(t)) \\
& + 2x^{\mathsf{T}}(t)PD\int_{t-\tau(t)}^{t} x(s)ds - (1 - \delta)e^{-2\alpha\tau}\int_{t-\tau(t)}^{t} x^{\mathsf{T}}(s)Rx(s)ds \\
& - 2\alpha(V(x_t) - x^{\mathsf{T}}(t)Px(t)).
\end{aligned}
$$

$$(3.5)$$

By using the fact that

$$
-\int_{t-\tau(t)}^{t} x^{\mathsf{T}}(s)Rx(s)ds \leq -\frac{1}{\tau}\left(\int_{t-\tau(t)}^{t} x(s)ds\right)^{\mathsf{T}} R\left(\int_{t-\tau(t)}^{t} x(s)ds\right) \qquad (3.6)
$$

then from (3.5) and (3.6) we have

$$
\dot{V}(x_t) + 2\alpha V(x_t) \leq \xi^{\mathsf{T}}(t)\Xi\xi(t), \qquad (3.7)
$$

where

$$
\xi^{\mathsf{T}}(t) = \left[x^{\mathsf{T}}(t) \quad x^{\mathsf{T}}(t - h(t)) \quad \left(\int_{t-\tau(t)}^{t} x(s)ds\right)^{\mathsf{T}}\right],
$$

and

$$
\Xi = \begin{bmatrix} A^{\mathsf{T}}P + PA + 2\alpha P + Q + \tau R & PB & PD \\ B^{\mathsf{T}}P & -(1 - \mu)e^{-2\alpha h}Q & 0 \\ D^{\mathsf{T}}P & 0 & -\dfrac{1 - \delta}{\tau}e^{-2\alpha\tau}R \end{bmatrix}.
$$

By using properties

$$P = \sum_{i=1}^{p} \sigma_i P_i, \quad Q = \sum_{i=1}^{p} \sigma_i Q_i, \quad R = \sum_{i=1}^{p} \sigma_i R_i, \quad \sum_{i=1}^{p} \sigma_i = 1,$$

and from conditions (i) and (ii) of Theorem 3.1 we have

$$\Xi = \sum_{i=1}^{p} \sigma_i^2 \Big[\mathcal{M}_i(P_i, Q_i, R_i) + 2\sigma\mathcal{N}(P_i) \Big]$$

$$+ \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \sigma_i \sigma_j \Big[\mathcal{M}_i(P_j, Q_j, R_j) + \mathcal{M}_j(P_i, Q_i, R_i) + 2\alpha\mathcal{N}(P_i + P_j) \Big]$$

$$\leq -\sum_{i=1}^{p} \sigma_i^2 \mathbb{S} + \frac{2}{p-1} \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \sigma_i \sigma_j \mathbb{S}$$

$$= -\frac{1}{p-1} \Big[(p-1) \sum_{i=1}^{p} \sigma_i^2 - 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \sigma_i \sigma_j \Big] \mathbb{S}$$

It's easy to verify that

$$(p-1) \sum_{i=1}^{p} \sigma_i^2 - 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \sigma_i \sigma_j = \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} (\sigma_i - \sigma_j)^2 \geq 0.$$

Therefore it follows from (3.7) that

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq 0, \quad \forall t \geq 0,$$

and hence

$$V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \gamma_2 \|\phi\|^2 e^{-2\alpha t}, \quad t \geq 0.$$

Taking (3.4) into account, we finally obtain

$$\|x(t,\phi)\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}} e^{-\alpha t} \|\phi\|, \quad t \geq 0,$$

where γ_1, γ_2 are defined in (3.2). The proof of the theorem is completed.

Remark 3.1. *It is worth noting that the condition* (i) *means the asymptotic stability of each $i^{th}-$subsystem, the condition* (ii) *implies the asymptotic stability of the $ij^{th}-$subsystem and if $p = 1$ this condition is automatically removed.*

Remark 3.2. *As a consequent of theorem 3.1, if $\mu = 0$ and $D_i = 0, i = 1, 2, \ldots, p$ then the result of theorem 3.1 implies that of theorem 1 in [8].*

Example 3.1. Consider uncertain linear polytopic system with distributed time varying delays (2.1), where

$$A_1 = \begin{bmatrix} -20 & 1 \\ 0 & -15 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -30 & 0 \\ 1 & -10 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -40 & -1 \\ 0 & -40 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

and delay functions $h(t) = \sin^2 0.5t, \tau(t) = \cos^2 0.5t$. Then we have the upper bounds $h = \tau = 1$ and $\mu = 0.5, \delta = 0.5$. By using LMI toolbox of Matlab it can be verified that all LMIs in theorem 3.1 are satisfied with $\alpha = 0.5$ and

$$P_1 = \begin{bmatrix} 0.3993 & 0.0024 \\ 0.0024 & 0.4027 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.3555 & 0.0024 \\ 0.0024 & 0.4250 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.3246 & 0.0025 \\ 0.0025 & 0.3797 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 6.7947 & -0.1060 \\ -0.1060 & 4.5724 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 8.4561 & -0.1254 \\ -0.1254 & 3.4611 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 10.7188 & 0.2262 \\ 0.2262 & 12.2987 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 6.7947 & -0.1060 \\ -0.1060 & 4.5724 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 8.4561 & -0.1254 \\ -0.1254 & 3.4611 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 10.7188 & 0.2262 \\ 0.2262 & 12.2987 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

By theorem 3.1, system (2.1) is exponentially stable with decay rate $\alpha = 0.5$. Moreover, every solution $x(t, \phi)$ satisfies the estimation

$$\|x(t, \phi)\| \le 7.6359 \|\phi\| e^{-0.5t}, \quad t \ge 0.$$

4 Conclusion

This paper has proposed new sufficient conditions for exponential stability of linear uncertain polytopic systems with distributed time varying delays. Based on an improved Lyapunov-Krasovskii parameter dependent functional, delay

dependent exponential stability conditions of the system are derived in terms of the Mondié-Kharitonov type's LMIs, which allows to compute simultaneously the two bounds that characterize the exponential stability of the solution. A numerical example illustrate the effectiveness of the obtained result is given.

Acknowledgements

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