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Group analysis of differential equations

CREATION OF NEW METHODS OF MATHEMATICAL PHYSICS, SEARCH OF THE EXACT SOLUTIONS AND FIRST INTEGRALS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract.

The symmetry is funndamental property of any phenomenon or process. In an equal degree it concerns and model - equation, describing this phenomenon or process. Moreover, the model as mathematical abstraction, as a rule, more is idealized, than the first copy, and by virtue of this circumstance has symmetry of the higher order. The methods symmetry of research are effective practically for all types of equations - from algebraic up to integro-differential.

1 Symmetry and Invariant

At a level of unformal concepts symmetry define as property to remain constant under action of any transformations. Constant or invariant can be the separate equation or class of equations, and in the latter case each element of a class will be not without fail transformed " to self ", and can go in other element that of a class; transformation of equivalence on a class is thus defined.

Main problems of the practical group analysis of differential equations are:

1) Development regular (algorithm) methods of search of all kinds of symmetry of equations;

2) Decision of return problems - search of classes of equations (or, that most - classes of models), having by a priori symmetry of a given kind and other by a priori properties;

3) Establishment of general principles of use found symmetry in practical problems.

Large advantage of the group analysis is the fact, that his methods permit to receive the solutions in a closed kind – so-called exact solutions, not containing of limiting transitions, iterations and norepresentation in a kind of the final formulas of series. The exact solutions of differential equations always played and continue to play a huge role in formation of correct understanding of qualitative features of many phenomena and processes in various areas of science.

The equations of applied and theoretical physics frequently contain parameters or functions, which are experimentally and consequently are not strictly fixed. At the same time the equations, simulating the real phenomena and processes, should be enough simple that they can successfully be be to be analysed and decide. Naturally as one of possible criteria of simplicity to take the requirement, that the modelly equation admitted the decision in the closed form.

It should note, that even the partial exact solutions of nonlinear equations (including that, which have not clear physical sense and do not correspond to the real phenomena and processes) play an important role of "testy" problems at check of correctness and valuation of accuracy various numerical, asymptotical and approached methods. Besides the admitting exact solutions an modelly equation and the problems form the basis for development new numerical, asymptotical and approached methods, which, in turn, permit to investigate already more complex problems, not having of the exact analytical solution.

2 Group analysis

The group analysis has arisen in at the end of XIX century as result of fundamental researches of the norvegian mathematician S.Lie. In the elementary case the search of symmetry of an equation is reduced to the search of continuous group of transformations

$$\begin{cases} \overline{x} &= \phi(x,y,a), \quad \overline{x}\mid_{a=0} = x, \\ \overline{y} &= \psi(x,y,a), \quad \overline{y}\mid_{a=0} = y \end{cases} \tag{1}$$

continuously dependent from parameter a and leaving an equation invariant. The final transformations (1) are unequivocally defined the infinitesimal operator

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y, \tag{2}$$

coordinates which satisfy to Lie equations

$$\begin{cases} \frac{d\phi}{da} &= \xi(\phi,\psi), \quad \phi(0) = x, \\ \frac{d\psi}{da} &= \eta(\phi,\psi), \quad \psi(0) = y \end{cases},$$

If group (1), admissible of a differential equation, and her invariant, i.e. function, satisfying to a parity is known

$$F(x,y) = F(\overline{x},\overline{y}),$$

about the order of this equation it is possible to lower per unit of (and if an equation of the first order - integrate him in squaring).

S.Lie considered and contact transformation

$$\overline{x} = \phi(x,y,y',a),$$

$$\overline{y} = \psi(x,y,y',a),$$
$$\overline{y'} = \chi(x,y,y',a),$$

in the formulas for which, as against dot transformations (1), enter and derivative, but has appeared, that, for example, for equations of the second order it is impossible to find admissible group, not entering some of the a priori assumptions of a structure of dependence of coordinates of the operator (2) from derivative. At the same time on prolongation of a XX century number of equations not having dot symmetry grew, but integrate in squaring or in the terms of known special functions, and more and more urgent became the problem of the description of such equations on the basis of uniform of symmetry principles. Not less important was and development regular (algorithm) methods of the decision of such equations and construction of models with a priori properties.

3 Discrete-group analysis

In 1976 was the first publication under the discrete - group analysis, in which were described of discretic symmetry of a generalized equation Emden-Fowler

$$y'' = Ax^n y^m (y')^l, \ \ \vec{a} = (n, m, l) \tag{3}$$

which for brevity is designated by a vector of essential parameters. The group of equivalence of a class of equations (3) can be given of the planar graph or noplanar, in which one of forming has the order 2 and is set by dot transformation, and other has the order 3 and is set by Backlund transformation

$$x = \overline{y}^{\frac{1}{n+1}}, \quad y = \overline{y'}^{\frac{1}{m}}, \quad y' = \overline{x}^{\frac{1}{1-l}}, \tag{4}$$

which is tangent, but not in all continued space (x, y, y'), and only on manifold of the solutions of an initial equation (3). Such transformations can translate dot continuous groups in Lie-Backlund groups, which it is impossible to find by classical Lie algorithm.

For example, the equation $(-m-3, m, 0)$ admits two dot operators - scaling

$$X_1 = (m-1)x\partial_x + (m+1)y\partial_y$$

and projective group

$$X_2 = x^2\partial_x + xy\partial_y.$$

Transformation

$$\overline{x} = y', \quad \overline{y} = x^{-m-2}, \quad \overline{y'} = -\frac{m+2}{A}y^{-m} \tag{5}$$

translates an initial equation in an equation $\left(1, -\frac{m+3}{m+2}, \frac{2m+1}{m}\right)$, admitting the scaling operator

$$\overline{X}_1 = 2\overline{x}\partial_{\overline{x}} - (m-1)(m-2)\overline{y}\partial_{\overline{y}}.$$

The other dot operators at this equation is not present, and second admissible operator (the operator of Lie-Backlund group) can be received only by transformation of the operator X_2 with the help of the formulas (5):

$$\overline{X}_2 = (\alpha\overline{y}'^{-\frac{1}{m}} - \overline{xy}^{-\frac{1}{m+2}})\partial_{\overline{x}} - (m+2)\overline{y}^{\frac{m+1}{m+2}}\partial_{\overline{y}},$$

where

$$\alpha = \Big(-\frac{A}{m+2}\Big)^{-1/m}.$$

Moreover it is obvious, that if one of equations in the graph can be integrated, the other equations are too integrated, as the discretic group (in this case diedre group D_3) is formed by already found transformations. Thus all decisions of elements of a determined orbit belong some (rather narrow) to a differential ring of functions, that permits to predict representation of the decision in the given terms.

Therefore it is important not only to find main discretic group admissible by a class of equations, but also to determine all expansions of this group on various specializations (i.e. subclasse of equations): each expansion increases the order of admissible group, hence, "makes multiple copies" quantity of the integrated representatives of a class. So, at $n = 1, \quad l = 0$ the graph of group D_3 up to the graph of group $C_2 \times D_6$ of the order 24 is expanded.

4 Equations of mathematical physics

A generalized equation Emden-Fowler a number of classes of equations 2-th and 3-rd order, having wide application in appendices, in particular, equation of a boundary layer for degree liquids is reduced

$$y''' = yy''^{2-n}, \tag{6}$$

for which until recently was known only one case be solved $n = 2$. Found discretic groups, admissible by an equation (3), have given much 7 of admissible cases:

1. $n = \frac{3}{5}$

$$\begin{cases} y(\tau) & = A(\tau^2 - C_1)(\tau^3 - 3C_1\tau + C_2)^{-1/2}, \\ x(\tau) & = B\int(\tau^3 - 3C_1\tau + C_2)^{-3/2}\,d\tau + C_3 \end{cases},$$

between A and B there is a pariti $\frac{15}{2AB}\left(\frac{B^2}{2A}\right)^{2/5} = -1.$

2. $n = \frac{5}{7}$

$$\begin{cases} y(\tau) & = AE^{-1/6}S_1^{-1}S_2, \\ x(\tau) & = B\int E^{1/2}S_1^{-3/2}\,d\tau + C_3 \end{cases},$$

where $\frac{7\sqrt{3}}{6AB}\Big(\frac{3B^2}{8A}\Big)^{2/7} = -1, \quad E = C_1 exp(\sqrt{3}\tau), \quad S_1 = C_1 exp(\sqrt{3}\tau) + C_2\sin\tau$, $S_2 = 2C_1 exp(\sqrt{3}\tau) - C_2\sin\tau + \sqrt{3}C_2\cos\tau$.

3. $n = \frac{1}{7}$

$$\begin{cases} y(\tau) & = AU^{-3/2}V, \\ x(\tau) & = B\int P^{-1}U^{-5/2}\,d\tau + C_3 \end{cases},$$

4. $n = \frac{1}{4}$

$$\begin{cases} y(\tau) & = AV, \\ x(\tau) & = B\int P^{-1}U^{2}\,d\tau + C_3 \end{cases},$$

where $P = (4\tau^3 - C_1)^{1/2}$,

$$U = \tau\Big(\frac{2}{C_1}\int\frac{\tau d\tau}{P} + C_2\Big) - \frac{P}{C_1}, \quad V = P\Big(\frac{2}{C_1}\int\frac{\tau d\tau}{P} + C_2\Big) - \frac{4\tau^2}{C_1}.$$

5. $n = -1$

$$\begin{cases} y(\tau) & = A\tau^{2/3}Z_{2/3}, \\ x(\tau) & = B\Big(Z_{2/3} + \frac{2}{3}\int\tau^{-1}Z_{2/3}\,d\tau\Big) + C_3 \end{cases},$$

6. $n = 1/2$

$$\begin{cases} y(\tau) & = A\tau^{-2/3}\Big[\tau(\ln Z_{1/3})'_\tau + 1/3\Big], \\ x(\tau) & = B\int\tau^{-1}(Z_{1/3})^{-2}\,d\tau + C_3 \end{cases},$$

where Bessel functions J_ν and Y_ν first and second sort of the order ν correspondingly for a designation $Z_\nu = C_1J_\nu(\tau) + C_2Y_\nu(\tau)$ are used.

7. $n = 1/5$

$$\begin{cases} y(\tau) & = Ae^{\tau}, \\ x(\tau) & = B\int[u(\tau)]^{-1/2}e^{3\tau}\,d\tau + C_3 \end{cases},$$

where the function $u(\tau)$ is determined by the inversion of integral $\int\frac{du}{u\omega(u)} = \tau + C_2$, and the function $\omega(u)$ is the decision of an algebraic equation:

$$\frac{\omega - 5}{(\omega - 4)^{4/5}} = u^{-1/5}\Big(C_1 - \frac{2}{5}u^{1/2}\Big).$$

We shall notice, that some of a regional problem for an equation (6) have the ununique decision, therefore in this case the search new of integrated specializations of an equation (6) is especially urgent.

5 Groups, graphs and handbooks

The linear equations admit, as a rule, infinite discretic groups, and infinite subgroup is present till any component of a vector of essential parameters. A good illustration can also serve of the graph of discretic group of transformations of an hypergeometric Gauss equation representing infinite 3-d a lattice (with slowly, equal to unit on each of parameters of hypergeometric Gauss function a, b, c), in each of units of which is truncated cubooctaedr (graph of group $C_2 \times S_4$ of the order 48 - maximum final subgroup).

Discretic groups of transformations of linear equations contain subgroup, which induced of transformation of shift of a spectrum $T : \lambda \longrightarrow \lambda + 1$ of Sturm's-Liouville problems, and the elements of such transformations are the decisions of nonlinear equations (discretic analogue of a inverse dispersion problem).

The statement of inverse problems for continuous groups has allowed to find a general kind of equations of any order, admitting some operator of a kind (2) or dot algebra of greater dimension, than unit. If the inverse problem is put for tangent groups or Lie-Backlund operators, a kind of an auxiliary equation (as a rule, partial differential equation) - his order and degree of nonlinearity is predicted. The decision of this auxiliary equation gives a general kind of a right part of an ordinary differential equation, admitting given operator, that permits to choose among set of possible models that, which have required a priori symmetry.

The opportunities of the modern group analysis are well illustrated by the help literature, the creation of which has become possible only in result of wide application of symmetry methods.

We shall notice, that, as against the handbook Kamke E. [1], in the new handbook [2] are practically away of an trivial equation, and majority of equations including in his practically classes of equations (dependent from any functions or, as a minimum, from any parameter) represent.

At the present stage of development of mathematical modeling a role of the computer analysis of complex nonlinear multiparametrical problems has increased. The application of algorithms of analytical calculations (computer algebra) on modern computer systems (Mathematica, Maple, Reduce) is the unique working tool of the researcher.

The polynomial algebra, reduction a polinoms, the algorithms of formal integration make main algorithms of computer algebra. The polynomial calculation and the algebraic analysis of basic functions for differential rings have a high degree of standardization and easily will be realized on the computer.

Thanking advanced opportunities of systems of analytical calculations, main algorithms of the group analysis will be realized: Lie algorithm, algorithm of construction and partial decision of determining systems, algorithms of search of discretic groups of transformations. Such algorithms make a nucleus of software, which much accelerate process of scientific researches (so, for example, calculations on discretic group 24-th order in a system Reduce takes read out minutes of real time). Working with library files and programs, frequently it should to create new sections and software packages, to modernize old, thus to derivate, in a final result, own tool of scientific researches. With increase of variety of soluble problems are enriched and methods of achievement of results. Here the already used system of analytical calculations begins to play a role of base of knowledge and to serve filling for systems of artificial intelligence.

Is possible and computer support of the group analysis with use of modern technologies of the animation graphics, which is carried out for an illustration of group structures, but also for the analysis of their dynamics. Unique results gives the animation graphic for researches of an hypergeometric Gauss equation.

Developed computer bank of models on the basis of the directory [2] permits to allocate model from set allowable with the account of the a priori information: the conservation laws, principle of mobile balance, qualitative characteristics of behaviour of a system and other. The modelly classes are united on group principles of an equation of linear and nonlinear mechanics and mathematical physics, but also their new exact solutions. The additional information on the first integrals, symmetry, invariant, group structures and other provides diverse search and analysis of mathematical models in an analytical kind.

References

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[2] Zaitsev V. F., Polyanin A. D. *Handbook on ordinary differential equations. Exact Solutions*. Nauka Publishers Moscow, 1995.