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Group analysis of differential equations

DISCRETE-GROUP ANALYSIS AND THE CONSTRUCTION OF EXACT MODELS

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Abstract.

At the end of XIX Century S.Lie offered group approach of integration of ordinary differential equations (ODE) which forced out algorithms of first generation (more exactly, set of the empirical methods which based known E.Kamke's reference-book) at the middle of XX Century. Now, set of algorithms of second generation were formed fully and it is able to find symmetries of ODE (Lie groups, discrete groups and first integrals) and to integrate of equation in analytic form in event that quantity of symmetries is sufficiently.

Nevertheless, a number of the equations describing real processes and phenomena don't enjoy "rich" of symmetries because experimental results are absolutly exact never. These results are modeled as series, moreover, transition to more exact approximation "kills" even "poor" symmetries which simplified model had.

Offering algorithms of third generation permit to show necessary (sometimes sufficient too) criterious of existence of the model with a priori symmetry (algorithms prognostic) or to discribe all models of the given form with required symmetry (algorithms of inverse problem). As problem of modelling is many-valied the above set of equation is more effective for description of real processes and phenomena then looking over all variants and making more precise of concrete models by algorithms of second generation.

The generalized problem of two motionless centres is a good example of algorithm of third generation for modelling of orbit of artificial Earth satellite.

It's well known that differential equations theory takes a central place among possible instruments for the modelling of different processes and phenomena. All the existing methods of exact solution of ordinary differential equations (ODE) can be conditionally divided in two groups:

A) a search for transformation of the original ODE D to some other ODE D_1 which belongs to one of the "standard" classes of ODE having known solutions;

B) a search for transformation leaving original ODE D invariant, i.e. transformation into "itself", that gives independent information about solution.

Practically all of the classic methods of exact solution of ODE use the approach A based on a rather restricted number of "standard" soluble ODE, therefore transformations often are "artificial". Thus the known soluble equations generate a set without visible logic, its classification is difficult, and their search was essentially a matter of art until the nearest past, the fact that can be perfectly illustrated with the help of [1]. To develop the approach A one can apply different transformations to "standard" soluble equations infinitely extending the set of such equations. However the probability that the ODE chosen for the investigation belongs to the extended set is very small.

The approach B suggested by S.Lie in the end of the last century is based on a search of continuous transformation groups leaving the tested equation invariant. Specifically, for example, a knowledge of a two-parmetric group admitted by a second order ODE gives a general solution of this equation (if there is a possibility to find universal invariants of the group). However, as S.Lie has shown the search for a group admitted by a first order ODE is as difficult as the search for its general solution; the Lie method for higher order ODE gives rather non-numerous partial cases of solubility which simply had slipped from the attention of the researchers who were using the approach A. Nevertheless, the approach B allows to classify the cases of order reduction of the ODE; this approach has been very effective for partial differential equations (see, for ex., [2]).

A wish to unite the advantages and to possibly free oneself from the flaws of the approaches A and B in the case of ODE has led to the creating of a discrete-group analysis. It may seem paradoxical, but the decreed goal can be attained by giving up two basic principles – continuity and integrability. Giving up continuity involves non-applicability of the infinitesimal approach and as a consequence nonlinearity of the defining system. Giving up integrability one would think that we deprive ourselves of the property which is the basic goal of our research. Yet practical applications show that on the contrary a realization of the idea pointed out above turns out to be very fruitful and allows one to describe discrete symmetries of the classes of equations which have no integrable cases at all as well as to discover some new solvable cases on the basis of the known ones, which couldn't be usually found by another regular methods (classical and group-theoretical ones).

The discrete-group analysis (DGA) does not operate with a single equation as in applications of Lie method, but operates with a class of equations D , depending on a vector $\vec{a}$ of parameters, containing the investigated equation; but contrary to the approach A one consider the transformations of the given class P , which are closed in itself on a chosen class of ODE, i.e. the transformations, which bring an element of the class D $D(\vec{a}) \in D_1 \subset D$ again to an element of the class D : $D(\vec{a}) \to D(\vec{b})$ (elements $\vec{a}$ and $\vec{b}$ may either coincide, or not coincide).

A set **G** of transformations, which possess the pointed out property, is named a discrete transformation group (DTG), admissible by a class D . The invariance of a class of ODE with respect to DTG does not directly imply the integrability (as in the case of approach B) due to giving up the continuity condition (an admissible DTG is as a rule essentially larger than a continuous equivalence group [2] and it can't be found by Lie method). Nevertheless a knowledge of a DTG allows one to get a wide-range information about the properties of a class of ODE:

1. One determines a discrete symmetry of a class D with a use of classification parameter – vector $\vec{a}$ (a group classification of a class D).

2. A class D splits into non-intersecting equivalence subclasses - orbits, which can not be connected by DTG **G** transformations of elements of D .

3. Invariants of DTG are determined – these are equations and subclasses of equations from D , which are invariant with respect to either all or some transformations from **G** (such discrete invariants, as a rule, can't be found by Lie method).

4. In some cases one can find classes of particular solutions of ODE for a class D or some subclass $D_1 \subset D$.

It is well known [2], that the knowledge of a one-parametric Lie group admissible by ordinary differential equation gives us a first integral of this equation, i.e. conservation law. In this connection one has a right to ask oneself what is the physical meaning of a discrete transformation group. Firstly it is obvious that any orbit of the group can be represented by some polyhedron in the space of parameters and any generator of the group specifies a discrete symmetry of this polyhedron similar to a crystallographic group. Secondly a discrete transformation group can be treated either as a similarity law for the models or, taking into account a discreteness of symmetry (i.e. an absence of a connectivity of the orbit), as a resonance or a commensurability of the models . Finally, in some cases (for instance, for many linear equations) a discrete group admits a unique interpretation as selection rules , moreover that group elements often act as transformations of translation of the spectrum $\mathbf{a} : \lambda_k \to \lambda_{k+1}$.

The discrete-group method of integration is based on the next two principles.

1. Any orbit of DTG $\mathbf{G}$, which contains a solvable element, is solvable as a whole.

2. A substitution of a concomitant [2] of a DTG $\mathbf{G}$ to the invariant of a DTG $\mathbf{G}$ brings it to the integrable or canonical form.

Therefore a set of integrable cases of a class D – "a mesh" – has a structure $D_0 \cup \mathbf{G}D_0 \cup D^*$, where D_0, $D^* \subset D$, where D_0 is a subset of the elements of D, which are integrable by classical methods (the approaches A and B), $\mathbf{G}D_0$ is a set of elements of all the orbits of subset D_0 , D^* is a set of solvable invariants of a class D . Let us note that the extension of a "classical" set D_0 by means of DGA will be the more essential the more elements of $\mathbf{G}$ will belong to those classes of transformations the search of which is difficult by classical methods (we mean the point transformations for ODE of the $1-st$ order, the Backlund transformations for ODE of the $2-nd$ order and nonlocal transformations for ODE of the $3-rd$ and of the higher orger).

For example the generalized Emden-Fowler equation

$$y'' = Ax^n y^m (y')^l, \ \ \vec{a} = (n, m, l) \qquad (1)$$

admits the basic (i.e. valid for arbitrary n , m , l) group $\mathbf{D}_3$ of the order 6 [3] but only the subgroup of the $2-nd$ order consists of point transformations. Therefore the application of only the basic group allows us to increase the number of integrable cases of the equation (1) in three times as compared with the "classical" set. A great advantage of DGA before the approach A is the algorithmicity of its principal methods which are basically analogous to Lie method (the approach B). The most general (direct) method consists in the following procedure. An arbitrary equation of a class D

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}, \vec{a})$$

is transformed by a substitution of a general form, which belongs to a chosen class of transformations, for example, to the class of Backlund transformations of the order k

$$\begin{aligned} y &= f(t, u, \dot{u}, \ldots, u^{(k)}), \\ x &= g(t, u, \dot{u}, \ldots, u^{(k)}) \end{aligned} \tag{2}$$

with the following condition of closure – the exclusion of the highest derivative $u^{(n)}$ with the help of the transformed equation, belonging to the same class D

$$u^{(n)} = F(t, u, \dot{u}, \ldots, u^{(n-1)}, \vec{b}). \tag{3}$$

Let us note that if we change the closure condition (3) by the condition to belong to another class D_1

$$u^{(n)} = F_1(t, u, \dot{u}, \ldots, u^{(n-1)}, \vec{\alpha}),$$

then we'll get the mapping $H : D \longrightarrow D_1$ instead of DTG, admissible by a class D .

In any case the obtained differential expression splits with respect to different degrees of independent variables $u^{(s)}$, where $k < s < n$, which gives an overdetermined set of partial differential equations where the unknowns are the functions f and g from the transformation (2). In the general case the determining system is nonlinear as contrary to analogous linear system of Lie algorithm. A successive specifying of a transformation class leads successively to the determining system of ordinary differential equations and to the algebraical system.

All the following operations with the determined DTG may be fulfilled not with the equations and transformations but with their algebraical representations – the vectors $\vec{a}$ and $\vec{b}$ and the algebraical relations induced by the action of DTG $\mathbf{G}$ generators

$$\mathbf{g}_i : D(\vec{a}) \longrightarrow D(\vec{b}_i), \Longrightarrow \vec{b}_i = G_i(\vec{a}).$$

A great advantage of the discrete-group analysis is also a possibility of a prognosis of the solution structure, i.e. of its representation by specific classes of functions: either elementary, or special functions. For the purpose of illustration of this statement we shall use the differential algebra technique.

Let $\mathrm{C}_p[\tau, h_1(\tau), \ldots, h_{p-1}(\tau)]$ be a differential ring [4], generated by functions $h_1, \ldots, h_{p-1}$,with a differentiation defined as the ordinary differentiation with respect to the argument τ . The number p of functionally independent generators will named the rank of the ring C_p . Let us consider a subset $\tilde{\mathrm{C}}_p \subset \mathrm{C}_p$ which consists of products of degrees (not necessarily integer) of polynomials of the following form

$$P_\sigma = \sum_{(k)} \tau^{\alpha_k} h_1^{\beta_{1k}} \ldots h_{k-1}^{\beta_{p-1,k}}, \tag{4}$$

where $\alpha_k, \ldots, \beta_{p-1,k}$, are integers. Apparently $\tilde{\mathrm{C}}_p$ is also a differential ring of the rank p . The following statement takes place.

THEOREM. The subring $\tilde{\mathrm{C}}_p$ is invariant with respect to any DTG $\mathbf{G}$, which is generated by the Backlund transformations (2) if the functions f and g are rational.

CONSEQUENCE. Let a class of ODE D admits the DTG $\mathbf{G}$, which meets the conditions of the theorem and the element $D(\vec{a})$ has a general solution, given in a parametric form $(x(\tau), y(\tau)) \in \tilde{\mathrm{C}}_p$. Then a general solution of any element of the orbit $\mathbf{G}D(\vec{a})$ belongs to $\tilde{\mathrm{C}}_p$.

A *REMARK.* The limitation, that the class of Backlund transformations (2) must be the class of rational differential substitutions, can be omitted for the appropriate extension of the original differential ring C_p .

EXAMPLE. The orbit of the element $(0, -1/2, 0)$ of the pseudogroup, admissible by the class (1) has an order 16. The transformations $\mathbf{r}, \mathbf{g}, \mathbf{s}$ are specified by the rational expressoins of new variables:

$$\mathbf{r} : (n, m, l) \longrightarrow (m, n, 3 - l), \;\; x = u, y = t;$$

$$\mathbf{g} : (n, m, l) \longrightarrow (1/(1 - l), -n/(n + 1), (2m + 1)/m), \;\; x = u^{1/(n+1)}, y = \dot{u}^{-1/m};$$

$$\mathbf{s} : (n, m, 0) \longrightarrow (-n - m - 3, m, 0), \;\; x = t^{-1}, y = t^{-1}u.$$

Taking into account that the solution of the original equation $(0, -1/2, 0)$ can be expressed through the polynomial functions of τ : $x(\tau) = C_1^3(\tau^3 - 3\tau + C_2), y(\tau) = C_1^4(\tau^2 - 1)^2$, we reveal easily, that all the solutions of the equations taked out from the orbit of DTG are elements of the differential ring $\mathrm{C}_1[\tau]$ of rank 1; more precisely they are the elements of the subring $\tilde{\mathrm{C}}_1$ consisting of the products of degrees of polynomial functions of τ of the form (4). In this case a system of polynomials P_σ has the form

$$P_0 = 1,$$

$$P_1 = \tau,$$

$$P_2 = \tau^2 - 1,$$

$$P_3 = \tau^3 - 3\tau + C_2,$$

$$P_4 = \tau^4 - 6\tau^2 + 4C_2\tau - 3,$$

$$P_5 = \tau^5 - 10\tau^3 + 10C_2\tau^2 - 15\tau + 2C_2,$$

$$P_6 = \tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau - 8C_2^2 + 27.$$

It is obvious, that $\dot{P}_\sigma = \sigma P_{\sigma-1}$. A "closure" of the differential ring, generated by this system of polynomials, takes place as a consequence of the relations

$$4P_1P_3 - 3P_2^2 = P_4,$$

$$9P_2P_4 - 8P_3^2 = P_6,$$

$$4P_3P_5 - 3P_4^2 = P_8 \equiv P_2P_6.$$

note that further growth of polynomial degree breaks off as a consequence of the reduction $P_8 = P_2P_6$. A solution of any equation of the orbit can be represented in the form

$$x(\tau) = aC_1^{r_1}P_\alpha^{r_2}P_\beta^{r_3}, \;\; y(\tau) = bC_1^{s_1}P_\gamma^{s_2}P_\delta^{s_3}.$$

For example, for the equation $(1, -7/4, 7/5)$ $r_1 = 27, r_2 = -1/2, r_3 = 1, \alpha = 3, \beta = 6; s_1 = 32, s_2 = 4/3, s_3 = 0, \gamma = 4.$

Apparently a prognosis of solvability and of the solution structure allows one to investigate easily analytical properties of solutions of the equations which belong to solvable orbits and also to classify them according to their analytical features.

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