DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 3, 1998
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

LIE GROUP OF TRANSFORMATIONS FOR MAGMA EQUATIONS

E.V.Krishnan and Q.J.A.Khan

Department of Mathematics and Statistics
Sultan Qaboos University
P.O.Box 36, Al-Khod 123
Muscat, Sultanate of Oman

Abstract.

Group invariant solutions of the magma equation have been investigated for different values of two physical parameters.

1 Introduction

The flow of melt in the earth's mantle which is like a porous flow is effected by the buoyant force caused by the density difference between melt and matrix. The model equation describing this motion and the phase transition have been investigated by many authors [1,2]. Neglecting the phase transition and allowing only vertical motions, Scott and Stevenson [3] proposed an equation assuming that the melt and matrix are fully connected and incompressible, which was given by

$$u_t = [u^n \{(u^{-m}\, u_t)_x - 1\}]_x. \tag{1.1}$$

[0]This work was supported by the Sultan Qaboos University, College of Science research grant no. MS/98/04.

x and t denote vertical space co-ordinate and time respectively, and $u = u(x,t)$ is the mean volume fraction of the liquid phase which is non-negative for any x and t. The parameters n and m denote the dependency of permeability and effective viscosity. Scott and Stevenson suggested that the reasonable values of n and m are $2 \sim 5$ and $0 \sim 1$ respectively. Equation (1.1) is called the magma equation.

We assume a travelling wave solution $u = u(z)$, $z = x - ct$ for equation (1.1). Integrating equation (1.1) twice, we get the ordinary differential equation

$$\frac{c}{2}\,u_z^2 + \frac{1}{1-m}\,u^{m+1} + \frac{2A}{1-m-n}\,u^{m-n+1} - \frac{c}{2-m-n}\,u^{m-n+2} - B\,u^{2m} = 0, \quad (1.2)$$

where c is the wave velocity, A and B are integration constants. Equation (1.2) is not valid for $m = 1$ or $m + n = 1, 2$.

Takahashi and Satsuma[4)] obtained the explicit form of travelling wave solutions of equation (1.2) by using the transformation from z to ζ given by

$$\zeta = \int^z u^\alpha \, dz,$$

where α is determined by the values of n and m. They found solitary wave solutions and periodic wave solutions in terms of ζ for $n = 3$, $m = 0$; $n = 4$, $m = 0$ and $n = \frac{5}{2}$, $m = \frac{1}{2}$.

An n^{th} order ordinary differential equation which admits an r-parameter Lie group of transformations, $2 \le r \le n$, can be reduced to an $(n-r)^{th}$ order ODE if the Lie group is solvable. The reduced $(n-r)^{th}$ order ODE can be obtained directly from the given n^{th} order ODE without the need to determine any intermediate ODEs of orders $n - r + 1$ to $n - 1$. Solvable Lie groups play an important role in the study of group invariance of ordinary and partial differential equations[5−7)]. In this paper, we shall investigate the group invariant solutions of the magma equation for different values of the parameters n and m.

2 Group Invariant solutions for $n = 3$ and $m = 0$

When $n = 3$ and $m = 0$, the equation (1.1) becomes

$$u_t = [u^3(u_{xt} - 1)]_x \quad (2.1)$$

and (1.2) reduces to

$$\frac{c}{2} u_z^2 \;=\; -u + B - \frac{c}{u} + \frac{A}{u^2}. \tag{2.2}$$

We consider the wave velocity $c > 0$.

Introducing the independent variable ζ defined by

$$\zeta \;=\; \int^z u^{-1}\, dz, \tag{2.3}$$

we get,

$$\frac{c}{2} u_\zeta^2 \;=\; -u^3 + Bu^2 - cu + A. \tag{2.4}$$

Differentiating equation (2.4) twice with respect to ζ, we have

$$u_{\zeta\zeta\zeta} \;=\; -\frac{6}{c}\, u\, u_\zeta \;+\; \frac{2B}{c}\, u_\zeta. \tag{2.5}$$

We write equation (2.5) in the form

$$u_3 \;=\; (\alpha\, u \;+\; \beta)\, u_1 \tag{2.6}$$

where,

$$u_1 \;=\; u_\zeta,\;\; u_3 \;=\; u_{\zeta\zeta\zeta},\;\; \alpha \;=\; -\frac{6}{c},\;\; \beta \;=\; \frac{2B}{c} \tag{2.7}$$

The invariance criterion for (2.6) is,

$$\eta^{(3)} \;=\; \alpha\, u\, \eta^{(1)} \;+\; \alpha\, u_1\, \eta \;+\; \beta\, \eta^{(1)}, \tag{2.8}$$

where, $X \;=\; \xi(\zeta, u)\dfrac{\partial}{\partial \zeta} + \eta(\zeta, u)\dfrac{\partial}{\partial u}$ is the infinitesimal generator of the one parameter Lie group of transformations and $\eta^{(1)}$ and $\eta^{(3)}$ are given by

$$\eta^{(1)} \;=\; \eta_\zeta \;+\; (\eta_u \;-\xi_\zeta) u_1 - \xi_u\, (u_1)^2, \tag{2.9a}$$

$$\eta^{(3)} \;=\; \eta_{\zeta\zeta\zeta} \;+\; (3\eta_{\zeta\zeta u} - \xi_{\zeta\zeta\zeta}) u_1 \;+\; +3(\eta_{\zeta uu} - \xi_{\zeta\zeta u})(u_1)^2$$

$$+(\eta_{uuu} \;-\; \xi_{\zeta uu})(u_1)^3 \;-\; \xi_{uuu}(u_1)^4 \;+\; 3(\eta_{\zeta u} \;-\; \xi_{\zeta\zeta}) u_2$$

$$+3(\eta_{uu} - 3\xi_{\zeta u})u_1u_2 - 6\xi_{uu}(u_1)^2u_2$$

$$-3\xi_u(u_2)^2 + (\eta_u - 3\xi_\zeta)u_3 - 4\xi_u\, u_1u_3. \qquad (2.9b)$$

The resulting set of determining equations for $\xi(\zeta, u)$ and $\eta(\zeta, u)$ are:

$$\eta_{\zeta\zeta\zeta} - (\alpha\, u + \beta)\eta_\zeta = 0 \qquad (2.10a)$$

$$3\eta_{\zeta\zeta u} - \xi_{\zeta\zeta\zeta} - 2\xi_\zeta\,(\alpha u + \beta) - \alpha\eta = 0 \qquad (2.10b)$$

$$3\eta_{\zeta uu} - 3\xi_{\zeta\zeta u} - 3\,\xi_u\,(\alpha u + \beta) = 0 \qquad (2.10c)$$

$$\eta_{uuu} - 3\,\xi_{\zeta uu} = 0 \qquad (2.10d)$$

$$\xi_{uuu} = 0 \qquad (2.10e)$$

$$3\,\eta_{\zeta u} - 3\,\xi_{\zeta\zeta} = 0 \qquad (2.10f)$$

$$3\,\eta_{uu} - 9\,\xi_{\zeta u} = 0 \qquad (2.10g)$$

$$\xi_{uu} = 0 \qquad (2.10h)$$

$$\xi_u = 0 \qquad (2.10i)$$

ξ is a function of ζ alone from (2.10i) and (2.10e) and (2.10h) are automatically satisfied. From (2.10g), we have $\eta_{uu} = 0$. Differentiating (2.10b) with respect to u, we have,

$$-2\,\alpha\,\xi_\zeta - \alpha\,\eta_u = 0 \qquad (2.11)$$

which leads to

$$\eta_u = -2\,\xi_\zeta \qquad (2.12)$$

Substituting (2.12) in $(2.10f)$, we get

$$\xi_{\zeta\zeta} = 0 \tag{2.13}$$

giving rise to

$$\xi = K_1 + K_2\,\zeta \tag{2.14}$$

Now, using (2.12) and (2.14) in $(2.10b)$, we obtain

$$\eta = -2K_2\,u - \frac{2\beta}{\alpha}K_2 = -2\,K_2\,u + K_3 \tag{2.15}$$

where K_1 , K_2 and K_3 are arbitrary constants.

(2.14) and (2.15) give a nontrivial three-parameter Lie group of transformations acting on (ζ, u)-space with infinitesimal generators given by

$$X_1 = \frac{\partial}{\partial\zeta},\ X_2 = \zeta\frac{\partial}{\partial\zeta} - 2u\frac{\partial}{\partial u},\ X_3 = \frac{\partial}{\partial u}. \tag{2.16}$$

It can be easily verified that the differential equation (2.6) has a 3-dimensional solvable Lie algebra.

Now, we reduce the third order ordinary differential equation to a second order from invariance under translations (K_1). Obvious invariants of the first extension of $\zeta^* = \zeta + \epsilon,\ u^* = u$, are

$$U(\zeta, u) = u\,,\ V(\zeta, u, u_1) = u_1. \tag{2.17}$$

So,

$$u_2 = V\frac{dV}{dU},\ \ u_3 = V^2\frac{d^2V}{dU^2} + V(\frac{dV}{dU})^2. \tag{2.18}$$

Hence the ordinary differential equation (2.6) reduces to

$$V\frac{d^2V}{dU^2} + (\frac{dV}{dU})^2 = \alpha\,U + \beta. \tag{2.19}$$

In particular, if

$$V = \psi(U; C_1, C_2) \tag{2.20}$$

is the general solution of (2.19), then the first order ODE

$$V \;=\; u_1 \;=\; \psi(u; C_1, C_2) \tag{2.21}$$

admits $\zeta^* \;=\; \zeta + \epsilon,\;\; u^* \;=\; u$. Consequently, the general solution of equation (2.6) is given by

$$\int \frac{dz}{\psi(z; C_1, C_2)} \;=\; \zeta \;+\; C_3, \tag{2.22}$$

where C_1 , C_2 , C_3 are arbitrary constants.

3 Group Invariant solutions for $n = 4$ and $m = 0$

When $n = 4$ and $m = 0$, the equation(1.1) becomes

$$u_t \;=\; [u^4\,(u_{xt} \;-\; 1)]_x, \tag{3.1}$$

and (1.2) reduces to

$$\frac{c}{2}\,u_z^2 \;=\; -\,u \;+\; B \;-\; \frac{c}{2\,u^2} \;+\; \frac{2A}{3u^3}. \tag{3.2}$$

Introducing the independent variable ζ defined by

$$\zeta \;=\; \int^z u^{-3/2}\,dz, \tag{3.3}$$

we get

$$\frac{c}{2}\,u_\zeta^2 \;=\; -\,u^4 \;+\; Bu^3 \;-\; \frac{c}{2}u \;+\; \frac{2A}{3}. \tag{3.4}$$

Differentiating (3.4) w.r.t. ζ twice, we have,

$$u_{\zeta\zeta\zeta} \;=\; -\,\frac{12}{c}u^2 u_\zeta \;+\; \frac{6B}{c}uu_\zeta. \tag{3.5}$$

We write equation (3.5) in the form

$$u_3 \;=\; (\alpha u^2 \;+\; \beta u)\,u_1, \tag{3.6}$$

where,

$$u_1 = u_\zeta, \;\; u_3 = u_{\zeta\zeta\zeta}, \;\; \alpha = -\frac{12}{c}, \;\; \beta = \frac{6B}{c}. \tag{3.7}$$

The invariance criterion for (3.6) is,

$$\eta^{(3)} = (\alpha\, u^2 \; + \; \beta\, u)\, \eta^{(1)} \; + \; \beta\, u_1\, \eta \; + \; 2\alpha u u_1 \eta, \tag{3.8}$$

where, $X = \xi(\zeta, u)\,\dfrac{\partial}{\partial\zeta} + \eta(\zeta, u)\,\dfrac{\partial}{\partial u}$ is the infinitesimal generator of the one-parameter Lie group of transformations and $\eta^{(1)}$ and $\eta^{(3)}$ are given by (2.9a) and (2.9b).

The resulting set of determining equations for $\xi(\zeta, u)$ and $\eta(\zeta, u)$ are:

$$\eta_{\zeta\zeta\zeta} - (\alpha u^2 + \beta\, u)\eta_\zeta = 0 \tag{3.9a}$$

$$3\eta_{\zeta\zeta u} - \xi_{\zeta\zeta\zeta} - 2\xi_\zeta\,(\alpha\, u^2 + \beta\, u) - (2\alpha\, u + \beta)\,\eta = 0 \tag{3.9b}$$

$$3\eta_{\zeta uu} - 3\xi_{\zeta\zeta u} - 3\xi_u\,(\alpha\, u^2 + \beta\, u) = 0 \tag{3.9c}$$

$$\eta_{uuu} - 3\xi_{\zeta uu} = 0 \tag{3.9d}$$

$$\xi uuu = 0 \tag{3.9e}$$

$$3\eta_{\zeta u} - 3\xi_{\zeta\zeta} = 0 \tag{3.9f}$$

$$3\eta_{uu} - 9\xi_{\zeta u} = 0 \tag{3.9g}$$

$$\xi uu = 0 \tag{3.9h}$$

$$\xi_u = 0 \tag{3.9i}$$

From (3.9i), ξ is a function of ζ alone and (3.9e) and (3.9h) are automatically satisfied. From (3.9g), $\eta_{uu} = 0$. Differentiating (3.9b) with respect to u twice, we get $\eta_u = -\xi_\zeta$. Using this in (3.9f), it leads to $\xi_{\zeta\zeta} = 0$. So, $\xi = L_1 + L_2\,\zeta$.

From $(3.9f)$ again, $\eta_{\zeta u} = 0$ which gives rise to $\eta = -L_2 u + L_3$. Here, L_1, L_2 and L_3 are arbitrary constants. Thus,

$$\xi = L_1 + L_2 \zeta \tag{3.10}$$

$$\eta = -L_2 u + L_3 \tag{3.11}$$

give a nontrivial three-parameter Lie group of transformations acting on (ζ, u) - space with infinitesimal generators given by

$$X_1 = \frac{\partial}{\partial \zeta}, \;\; X_2 = \zeta \frac{\partial}{\partial \zeta} - u \frac{\partial}{\partial u}, \;\; X_3 = \frac{\partial}{\partial u}. \tag{3.12}$$

One can easily verify that the differential equation (3.6) has a 3-dimensional solvable Lie algebra.

Now, we reduce the third order ordinary differential equation to a second order ODE from invariance under translations (L_1). Obvious invariants of the first extension $\zeta^* = \zeta + \epsilon$, $u^* = u$ are

$$U(\zeta, u) = u, \;\; V(\zeta, u, u_1) = u_1. \tag{3.13}$$

So, using (2.18), the ODE (3.6) reduces to

$$V \frac{d^2 V}{dU^2} + (\frac{dV}{dU})^2 = \alpha U^2 + \beta U. \tag{3.14}$$

In particular, if $V = \psi(U; C_1 C_2)$ is the general solution of (3.14), then the first order ODE

$$V = u_1 = \psi(u; C_1 C_2) \tag{3.15}$$

admits $\zeta^* = \zeta + \epsilon$, $u^* = u$. Thus the general solution of (3.6) is given by

$$\int \frac{dz}{\psi(z; C_1, C_2)} = \zeta + C_3, \tag{3.16}$$

where, C_1, C_2, C_3 are arbitrary constants.

4 Group Invariant solutions for $n = \frac{5}{2}$ and $m = 0$

When $n = \frac{5}{2}$ and $m = \frac{1}{2}$, the equation (1.2) reduces to

$$\frac{c}{2} u_z^2 + 2\, u^{3/2} - A\, u^{-1} + c - B\, u = 0. \tag{4.1}$$

Transforming the dependent variable u inti v by

$$u = v^2, \tag{4.2}$$

(4.1) can be written as

$$2cv_z^2 + 2v - Av^{-4} + cv^{-2} - B = 0. \tag{4.3}$$

Introducing the independent variable ζ defined by

$$\zeta = \int^z v^{-1}\, dz, \tag{4.4}$$

and assuming that the integration constant A is zero, equation (4.2) will be reduced to

$$2cv_\zeta^2 = -2v^3 + Bv^2 - c. \tag{4.5}$$

Differentiating equation (4.5) twice with respect to ζ, we get

$$v_{\zeta\zeta\zeta} = -\frac{3}{c} v v_\zeta + \frac{B}{2c} v_\zeta. \tag{4.6}$$

We write equation (4.6) in the form

$$v_3 = (\frac{\alpha}{2} v + \frac{\beta}{4})\, v_1, \tag{4.7}$$

where,

$v_1 = v_\zeta,\ v_3 = v_{\zeta\zeta\zeta}$ and α and β are as defined in (2.7).

Therefore, using the same notations as in section 2, the ODE (4.7) will reduce to

$$V \frac{d^2V}{dU^2} + (\frac{dV}{dU})^2 = \frac{\alpha}{2} U + \frac{\beta}{4}, \tag{4.8}$$

where,

$$U(\zeta, v) = v, \; V(\zeta, v, v_1) = v_1. \tag{4.9}$$

The general solution of (4.7) can be got as in previous sections.

References

1. D. J. Stevenson: Trans. Amer. Geophys. Un. EOS. **61** (1980) 1021.
2. A. C. Fowler: Geophys. Astrophys. Fluid Dynamics **28** (1984) 99.
3. D. R. Scott and D. J. Stevenson: Geophys. Res. Lett. **11** (1984) 1161.
4. D. Takahashi and J. Satsuma: J. Phys. Soc. Japan **57** (1988) 417
5. P. J. Olver: Applications of Lie Groups to Differential Equations, Springer Verlag, New York, **107** (1986).
6. G. W. Bluman and S. Kumei: Symmetries and Differential Equations, Springer Verlag, New York, **81** (1989).
7. B. S. Bhatt and E. V. Krishnan: Advances in Modelling and Analysis, **29** (1995) 1.