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Control problems in nonlinear systems

On the stability of nonlinear systems with the monotonic differentiable nonlinear characteristics

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Abstract.

Some new frequency criteria of stability of pulse systems with the monotonic differentiable static characteristics of pulse element are obtained.

1 Problem setting

Suppose, nonlinear operator $\mathcal{M}$, mapping a continuous signal $\sigma(t)$ on the modulator input into a signal $f(t)$ on its output, has the following properties.

a) For any $\sigma(t) \in \mathbf{C}[0, +\infty)$ there exists a sequence t_n $(n = 0, 1, ...; \ t_0 = 0)$ such that

$$\delta_0 T \leq t_{n+1} - t_n \leq T \quad (0 < \delta_0 < 1, \ T > 0) \tag{1}$$

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and the function $f(t)$ is piecewise continuous and does not change its sign on the interval $[t_n, t_{n+1})$;
b) t_n depends only on $\sigma(t)$ for $t \le t_n$, $f(t)$ depends only on $\sigma(\tau)$ for $\tau \le t$;
c) for any n there exists $\widetilde{t}_n \in [t_n, t_{n+1})$ such that the mean value of the n-th pulse

$$v_n = \frac{1}{t_{n+1} - t_n} \int\limits_{t_n}^{t_{n+1}} f(t)\, dt$$

is related with $\sigma(\widetilde{t}_n)$ by formula

$$v_n = \varphi(\sigma(\widetilde{t}_n)),$$

where $\varphi(\sigma)$ is a continuously differentiable function (static characteristics of pulse element) such that : $\varphi(0) = 0$,

$$0 < \frac{\varphi(\sigma)}{\sigma} < \frac{1}{\sigma_*} \quad for \;\; \sigma \ne 0 \tag{2}$$

$$0 \le \frac{d\varphi}{d\sigma} \le l. \tag{3}$$

$$\frac{\varphi(\sigma)}{|\sigma|} \to 0 \quad for |\sigma| \to \infty. \tag{4}$$

Properties a), b), c) turn out to be ordinary for the most of modulators used in technology.

Consider a pulse system described by the following functional differential equation

$$\dot{x} = Ax + bf, \qquad \sigma = c'x, \qquad f = \mathcal{M}\sigma, \tag{5}$$

where A is a constant Hurwitz $m \times m$-matrix, b and c are constant m-dimensional columns

The problem is to define the properties of the transfer functions $W(p) = c'(A - pI_m)^{-1}b$, which assure the asymptotics $x(t) \to 0$ as $t \to +\infty$ for any $x(0)$.

2 The formulation of result

Consider system (5) and suppose that the following conditions

$$\lim_{p \to \infty} pW(p) = \lim_{p \to \infty} p^2 W(p) = 0. \tag{6}$$

are satisfied.

Theorem *Suppose that the transfer function W(p) is nondegenerate, relations (1)-(4),(6) are valid, and there exist positive constants* $\tau, \tau_1, \varepsilon_1, \varepsilon_2$ *and* $\kappa \geq 0$ *such that for all* $\omega \in [0, +\infty]$ *the frequency condition holds*

$$\alpha(\omega)\beta(\omega) - |\delta(\omega)|^2 > 0, \tag{7}$$

where

$$\alpha(\omega) = \tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3} - \varepsilon_3\omega^2|W(i\omega)|^2 + (\tau + \kappa\omega^2)ReW(i\omega) - \kappa\varepsilon_1\omega^4|W(i\omega)|^2,$$

$$\beta(\omega) = \tau_1 - \varepsilon_3\omega^4|W(i\omega)|^2 - \kappa\varepsilon_1[\kappa_1 + i\omega^3 W(i\omega)]^2,$$

$$\delta(\omega) = \kappa\kappa_1\frac{i\omega}{2}(\tau+\kappa\omega^2)W(i\omega) - \varepsilon_3 i\omega^3|W(i\omega)|^2 - \kappa\varepsilon_1[(\kappa_1 + i\omega^3 W(i\omega))\omega^2\overline{(W(i\omega))}]$$

$$\nu = \tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3}\ \ , \varepsilon_3 = \frac{T^2}{\pi^2}(\frac{\kappa l^2}{\varepsilon_1} + \frac{\tau}{\varepsilon_2}),$$

$$\kappa_1 = \lim_{p\to\infty} p^3 W(p).$$

Then solutions of system (5) have asymptotics $x(t) \to 0$ *as* $t \to +\infty$ *for any* $x(0)$.

This theorem extends result, obtained in [1] under condition $\kappa_1 = 0$, to case $\kappa_1 \neq 0$.

3 The proof of theorem

We introduce, following [2], the functions $v(t) = v_n$ for $t_n \leq t < t_{n+1}$, $u(t) = \int_0^t [f(t) - v(t)]dt$, and $y = x - bu$ and transform system (5) to the form

$$\dot{y} = Ay + bv + Abu. \tag{8}$$

The objective of such a transformation is in finding the system such that the function f is excluded, the function v is a "frozen" function $\varphi(\sigma(t))$, and the function u is small in a certain sense.

By (6) we have

$$c'b = c'Ab = 0. \tag{9}$$

Consider now the Lyapunov function [3]

$$V = y^* Hy - \kappa c' Ay\varphi(\sigma), \tag{10}$$

where $H \in \mathbf{R}^{k\times k}$ is a constant positively definite matrix, which will be given below. Differentiating (10) by using system (8) and taking into account the following equality

$$\dot{\sigma} = c'Ay, \tag{11}$$

which is resulted from (9), we obtain

$$\dot{V} = W_1 - \kappa c' A^2 y\varphi - \kappa(\dot{\sigma})^2 \frac{d\varphi}{d\sigma},$$

where $W_1 = 2y'H(Ay{+}bv{+}Abu)$. Applying the S-procedure with the coefficients τ and τ_1, we transform the above relation into

$$\dot{V} = W_1 - \kappa c' A^2 y\varphi + W_2 + \tau(\overline{\sigma} - \sigma_* v)v +$$
$$+\tau_1(\frac{T^2}{3}v^2 - u^2), \tag{12}$$

where $\overline{\sigma}(t) = \sigma(\widetilde{t_n})$ for $t_n \le t < t_{n+1}$,

$$W_2 = -\kappa\dot{\sigma}^2\frac{d\varphi}{d\sigma} - \tau(\overline{\sigma} - \sigma_* v)v - \tau_1(\frac{T^2}{3}v^2 - u^2)$$

Using (2), (3) and the property, stated in [2],

$$\int_{t_n}^{t_{n+1}} u^2 dt \le \frac{(t_{n+1} - t_n)^2}{3} \int_{t_n}^{t_{n+1}} v^2 dt$$

the following estimate holds

$$\int_{t_n}^{t_{n+1}} W_2 dt \le 0. \tag{13}$$

Having performed the changes of variables in (12), namely, $\varphi = v{+}(\varphi{-}v)$, $\overline{\sigma} = c'y + (\overline{\sigma} - \sigma)$, we obtain

$$\dot{V} = W_1 + W_2 - \kappa c' A^2 y v + \tau(c'y - \sigma_* v)v + W_3 + \tau_1(\frac{T^2}{3}v^2 - u^2), \tag{14}$$

where $W_3 = \kappa c' A^2 y(v - \varphi) + \tau(\overline{\sigma} - \sigma)v$. By (3), estimate W_3 takes the form

$$W_3 \le \kappa[\varepsilon_1(c'A^2y)^2 + \frac{l^2(\overline{\sigma} - \sigma)^2}{4\varepsilon_1})] + \tau[\varepsilon_2 v^2 + \frac{(\overline{\sigma} - \sigma)^2}{4\varepsilon_2}] \tag{15}$$

According to the inequality of Virtinger [2] and property (1), the following estimate

$$\int_{t_n}^{t_{n+1}} (\overline{\sigma} - \sigma)^2 dt \le \frac{4T^2}{\pi^2} \int_{t_n}^{t_{n+1}} \dot{\sigma}^2 dt$$

is valid. Therefore by (15) and (11)

$$\int_{t_n}^{t_{n+1}} W_3 dt \le \tau\varepsilon_2 \int_{t_n}^{t_{n+1}} v^2 dt + \varepsilon_3 \int_{t_n}^{t_{n+1}} (c'Ay)^2 dt + \kappa\varepsilon_1 \int_{t_n}^{t_{n+1}} (c'A^2y) dt. \quad (16)$$

By (13), (14), (16) we have

$$\int_{t_n}^{t_{n+1}} \dot{V} dt \le \int_{t_n}^{t_{n+1}} (W_1 - G) dt, \quad (17)$$

where G is a quadratic form with the real coefficients

$$G(y,v,u) = [\tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3}]v^2 +$$

$$+\tau_1 u^2 - \varepsilon_3(c'Ay)^2 + (\kappa c'A^2y - \tau c'y)v - \tau_1|u|^2 - \kappa\varepsilon_1|c'A^2y|^2.$$

Extending it to the Hermitean one, we obtain

$$G(y,v,u) = [\tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3}]|v|^2 + \tau_1|u|^2 - \varepsilon_3|c'Ay|^2 +$$

$$+Re[(\kappa c'A^2y - \tau c'y)\overline{v}], -\tau_1|u|^2 - \kappa\varepsilon_1|c'A^2y|^2, \quad (18)$$

where v, u are complex numbers, $\overline{v}$ is a complex number, associated with v, $y \in \mathbf{C}^k$. Having performed the Laplace transformation with the zero initial conditions in (8) and saving a notation of variables, we arrive at a formula

$$y = -(A - pI_m)^{-1}bv - (A - pI_m)^{-1}Abu.$$

Using the representation $A = (A - pI_m) + pI_m$ and properties (9), we find

$$c'y = -W(p)v - pW(p)u,$$

$$c'Ay = -pW(p)v - p^2W(p)u,$$

$$c'A^2y = \kappa_1 u - p^2W(p)v - p^3W(p)u.$$

Substituting these expressions into (18) and putting $p = i\omega$, we obtain

$$G\,|_{p=i\omega} = \alpha(\omega)|v|^2 + \beta(\omega)|u|^2 + 2Re[\delta(\omega)u\overline{v}].$$

If the Hermitean matrix

$$\begin{pmatrix} \alpha & \delta \\ \overline{\delta} & \beta \end{pmatrix}$$

is positively definite, then by the frequency theorem of V.A. Yakubovich for the nondegenerate case [4] there exist $\mu > 0$ and a positively definite matrix H such that the term under integral sign in (17) may be estimated as follows

$$W_1 - G < -\mu(u^2 + v^2 + |y|^2). \tag{19}$$

The Silvester criterion implies that for the positive definiteness of this matrix it is necessary and sufficient that for all $\omega \in [0, +\infty]$ the inequalities

$$\alpha(\omega) > 0, \;\; \alpha(\omega)\beta(\omega) - |\delta(\omega)|^2 > 0$$

were satisfied. The second inequality coincides with frequency condition (7) and therefore is also satisfied. The first inequality follows directly from the second one. Really, as $\omega \to \infty$ $\beta \to \tau_1$, $\delta \to 0$, and, consequently, $\alpha(\infty) > 0$ and $\alpha(\omega)$ cannot be zero for any ω.

Thus estimate (19) is proved.

Relations (17) and (19) resulted in the inequality

$$V \mid_{t=t_n} + \mu \int_0^{t_n} (v^2 + u^2 + |y|^2) dt \leq V \mid_{t=0} . \tag{20}$$

By (4) and by the positive definiteness of the matrix H $V \to +\infty$ as $|y| \to \infty$. Therefore (20) implies that $u, v \in L_2[0, +\infty)$. Since the matrix A is the Hurwitzian one, it follows from (8) that $y(t) \to 0$ as $t \to +\infty$. From that $v \in L_2[0, +\infty)$ and from (1) the asymptotics $v(t) \to 0$ as $t \to +\infty$ follows. In [2] it is stated that $|u| \leq T|v|$ and therefore $u \to 0$ as $t \to +\infty$. Finally, from the relation $x = y - bu$ it follows that $x \to 0$ as $t \to +\infty$. The proof of theorem is completed

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