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Control problems in nonlinear systems

VERIFICATION OF STRUCTURAL STABILITY [1]

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1 Introduction

The structural stability was originated by A.A.Andronov and L.S.Pontryagin 1937 [2].

Let $f : M \to M$ be a diffeomorphism of a compact Remannian manifold M, and $\rho(*, *)$ be the distance on M. We suppose that the manifold M is in R^n. The space of diffeomorphisms has the following topology. The C^0-distance between f and g is $\rho(f, g) = \max_{x\in M} \rho(f(x), g(x))$. The differential $Df : TM \to TM$ has a norm

$$\|Df\| = \max_{x\in M} |Df(x)| = \max_{x\in M} \max_{|v|=1} |Df(x)v| .$$

The C^1-distance $\rho_1(f, g) = \rho(f, g) + \|Df - Dg\|$.

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Definition 1 *The diffeomorphism f is structurally stable if for each $\varepsilon > 0$ there is $\delta > 0$ that from the inequality $\rho_1(f, g) < \delta$ it follows the existence a homeomorphism $h : M \to M$ such that $fh = hg$ and C^0-distance $\rho(h, id) < \varepsilon$, where id is the identity mapping on M.*

Let x be a point of M and $TM(x)$ be the tangent space at x. The stable and unstable subspaces at x is defined as

$$\begin{aligned} S(x) &= \{v \in TM(x) : \ |Df^n(x)v| \to 0 \ as \ n \to +\infty\}, \\ U(x) &= \{v \in TM(x) : \ |Df^n(x)v| \to 0 \ as \ n \to -\infty\}, \end{aligned}$$

respectively.

Definition 2 *The transversality condition holds on the manifold M if*

$$TM(x) = S(x) + U(x)$$

at each point $x \in M$.

By summing results by J.Robin [26], C.Robinson [27] and R.Mane [16, 17] we can formulate the following theorem.

Theorem 1 *In order to the diffeomorphism f be structurally stable it is necessary and sufficient that the transversality condition holds on the manifold M.*

Thus Theorem 1 solves the problem of structural stability. However application of this result is limited by difficulties of verification of the transversality condition [19]. Even an application of the classical result by A.A.Andronov and L.S.Pontryagin requires to recognize a structure of dynamics governed by the system of differential equations. In this paper we describe equivalent necessary and sufficient conditions for the structural stability and prove that these conditions can be test by a finite algorithm.

2 Needed results

2.1 Linear extension.

Let (E, M, π) be a vector bundle over M, E be a total space, and π be a projector from E onto the base M. Assume that for each $x \in M$ a fiber $E(x) = \pi^{-1}(x)$ is d-dimension linear space isomorphic to R^d.

Definition 3 *A homeomorphism F of the total space E is said to be a linear extension of f, if F takes fibers to fibers: $f \circ \pi = \pi \circ F$ and the restriction $F|_{E(x)} : E(x) \to E(f(x))$ on each fiber $E(x)$ is a linear isomorphism.*

As an example of a linear extension we will keep in mind a diffeomorphism $f : M \to M$ and its differential $Df = F$ on the tangent bundle $TM = E$. The vector bundle E is associated with a projective bundle P, in this process the linear extension $F : E \to E$ induces a mapping $PF : P \to P$. For a nonzero vector v we denote by $[v] = y$ a point from P corresponding to the space spanned over v. Let $(P, M, P\pi)$ be a bundle over M such that each fiber $(P\pi)^{-1}(x)$ is a projective manifold $P^{d-1}(x)$ associated with the fiber $E(x)$ of E. The bundle $(P, M, P\pi)$ is called the projective bundle associated with the linear bundle (E, M, π). We use the following coordinates: (x, v) or (x, y, l) on E and (x, y) on P, where $x \in M$, $v \in E(x)$, $y \in P^{d-1}(x)$, $l \in L(y)$. In this coordinates the linear extension $F : E \to E$ of f is of the form $F(x, v) = (f(x), A(x)v)$. The mapping F induces the mapping $PF : P \to P$ on the projective bundle that the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ \downarrow^{L} & & \downarrow^{L} \\ P & \xrightarrow{PF} & P \\ \downarrow^{P\pi} & & \downarrow^{P\pi} \\ M & \xrightarrow{f} & M \end{array}$$

commutes.

2.2 Dual differential.

The adjoint of an operator $T : E \to E$ is the operator $T^* : E \to E$ defined by the equality $\langle Tv, u\rangle = \langle v, T^*u\rangle$, where $\langle v, u\rangle$ is an inner produc. As mention above the differential $Df : TM \to TM$ of diffeomorphism f is an example of linear extension of f. The differential Df takes the tangent space $TM(x)$ at point x onto $TM(f(x))$ at point $f(x)$. Let us consider a cascade conjugate to the differential, which defined as

$$\widehat{D}f(x) = ((Df(x))^*)^{-1} : TM(x) \to TM(f(x)).$$

By the definition, the conjugate cascade covers the diffeomorphism f. We can consider the mapping $\widehat{D}f : TM \to TM$ as the dual differential. The main property of the conjugate cascade is

$$\left\langle Dfv, \widehat{D}fu \right\rangle = \langle v, u \rangle ,$$

i.e. the inner product does not change under the actions of the differential and the dual differential [6]. Let $Z = \{... - 2, -1, 0, 1, 2, ...\}$ be the set of integers.

Definition 4 *[6, 32] The dual differential is said to have only trivial bounded trajectory, if any trajectory* $\{(x_{n+1}, v_{n+1}) = (f(x_n), \widehat{D}f(x_n)v_n, \ n \in Z\}$ *that is bounded, is the zero trajectory, i.e.,* $v_n = 0$.

Theorem 2 *[5] The transversality condition holds if and only if the dual differential has only trivial bounded trajectory.*

2.3 Chain recurrent set

Definition 5 *[3] Let* $\varepsilon > 0$ *be given. An infinite in both direction sequence* $\{x_k, \ k \in Z\}$ *is named an* ε*-trajectory if for any* k *the distance*

$$\rho(f(x_k), x_{k+1}) < \varepsilon. \tag{2.1}$$

If the squence $\{x_k\}$ *is periodic, the points are called* ε*-periodic.*

Denote the set of ε-periodic points by $Q(\varepsilon)$.

Definition 6 *[9] A point* x *is called chain recurrent if* x *is* ε*-periodic for each positive* ε*, i.e., there exists a periodic* ε*-trajectory passing through* x*. A chain recurrent set, denoted* Q*, is the set of all the chain recurrent points.*

It should be remarked that if a chain recurrent point is not periodic then there exists as small as one likes perturbation of f in C^0-topology for which this point is periodic [25]. One may say that a chain recurrent point generates periodic trajectory by C^0-perturbation [33].

From the definition of the chain recurrent set it follows immediately that

$$Q = \lim_{\varepsilon \to 0} Q(\varepsilon) = \bigcap_{\varepsilon > 0} Q(\varepsilon).$$

Thus the family of open sets $\{Q(\varepsilon), \ \varepsilon > 0\}$ forms a fundamental system of neighborhoods of the chain recurrent set.

2.4 Hyperbolicity.

The diffeomorphism f is said to be hyperbolic on $\Lambda \subset M$ if there are invariant subspaces $E^s,\ E^u \subset TM|_\Lambda$ and positive d and α such that

$$\begin{aligned} TM|_\Lambda &= E^s \oplus E^u, \\ |Df^n(x)v| &\le d\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^s(x)\ n > 0, \\ \left|Df^{-n}(x)v\right| &\le d\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^u(x)\ n > 0. \end{aligned}$$

The condition $TM|_\Lambda = E^s \oplus E^u$ is equivalent $TM|_\Lambda = E^s + E^u$ and $E^s \cap E^u = \emptyset$. The invariance of subbundle E^* means $Df(x)E^*(x) = E^*(f(x))$.

Theorem 3 *[6, 16] If the transversality condition holds, then*

i) the set $\Lambda = \{x \in M : TM(x) = S(x) \oplus U(x)\}$ is closed and invariant,

ii) the diffeomorphism f is hyperbolic on Λ, and $S(x) = E^s(x),\ U(x) = E^u(x),\ x \in \Lambda$,

iii) the chain recurrent set $Q \subset \Lambda$.

2.5 Morse spectrum.

Let us consider the linear extension $F : E \to E$ and the mapping $PF : P \to P$ on the projective bundle associated with F. If $\xi = \{(x_0, y_0), ..., (x_p, y_p) = (x_0, y_0)\}$ is a periodic ε-trajectory of a period p, then the Lyapunov exponent of ξ

$$\lambda(\xi) = \frac{1}{p} \sum_{i=0}^{p-1} \ln |A(x_i) e(y_i)|,$$

where $|e(y_i)| = 1$.

Definition 7 *The Morse spectrum of F is*

$$\Sigma(F) = \{\lambda \in R : \textit{ there are } \varepsilon_k \to 0 \textit{ and periodic}$$

$$\varepsilon_k - \textit{trajectories } \xi_k \textit{ with } \lambda(\xi_k) \to \lambda \textit{ as } k \to \infty\}.$$

Thus, the Morse spectrum is the limit set of Lyapunov exponents of ε-periodic trajectories under $\varepsilon \to 0$.

2.6 Attractor and repeller.

A subset Ω is called a component of the chain-recurrent set if Ω is a maximal set such that each two points from Ω can be connected by a periodic ε-trajectory for any $\varepsilon > 0$. Recall that a closed invariant asymptotically stable by Lyapunov set is called an attractor. An attractor of the inverse mapping f^{-1} is called a repeller of f. An intersection of an attractor and a repeller is called a Morse set.

Proposition 1 *An invariant set Λ is an attractor of f if and only if there is a neighborhood U of Λ such that*

$$f(cl\ U) \subset U,\ \Lambda = \bigcap_{n>0} f^n(U),$$

where $cl\cdot$ stands for the closure.

The described set U is called a fundamental neighborhood of an attractor Λ, and the set $W^s(\Lambda) = \bigcup_{n<0} f^n(U)$ is called an attraction domain of Λ. The closed set $\Lambda^* = M \backslash W^s(\Lambda)$ is a repeller which is named the dual repeller to Λ.

3 Theorems

Theorem 4 *The following conditions are equivalent*

(i) f is hyperbolic on the chain recurrent set Q,

(ii) the dual differential is hyperbolic on the chain recurrent set Q,

(iii) the Morse spectrum of the differential does not contain 0,

(iv) the Morse spectrum of the dual differential does not contain 0,

Suppose that one of the conditions of the previous Theorem holds. Then the Morse spectrum of the dual differential consists of two separated parts positive Σ^+ and negative Σ^-. The chain recurrent set $CR \subset P$ is divided on two separated parts CR^+ and CR^- such that $CR^+ + CR^- = CR$ and the Morse spectrum $\Sigma(\widehat{D}f|_{CR^+}) = \Sigma^+$ of the restriction $\widehat{D}f|_{CR^+}$ is positive and the Morse spectrum $\Sigma(\widehat{D}f|_{CR^-}) = \Sigma^-$ is negative. Let $W^u(CR^+)$ be the unstable manifold of CR^+ and $W^s(CR^-)$ be the stable manifold of CR^- for the projective mapping $P\widehat{D}f$.

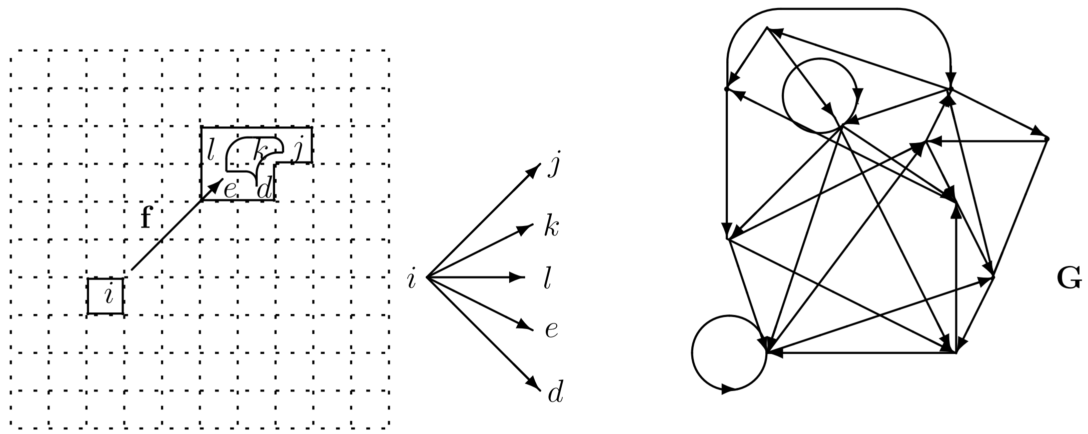

Figure 1: **Construction of a Symbolic Image.**

Theorem 5 *The diffeomorphism f is structurally stable if and only if the Morse spectrum of the dual differential does not contain* 0, *the unstable manifold* $W^u(CR^+)$ *is an attractor and the stable manifold* $W^s(CR^-)$ *is its dual repeller.*

4 Symbolic Image

In this section we describe a finite algorithm for verification of Theorem 5 conditions by applied symbolic dynamics methods. The verification consists of two main parts: a calculation of Morse spectrum of $\widehat{D}f$ and a construction of the attractor $W^u(CR^+)$ and its dual repeller $W^s(CR^-)$. A finite algorithm for calculation of Morse spectrum is available in [24], and a finite algorithm for construction of an attractor, its domain of attraction and dual repeller is given in [23]. Let $f : M \to M$ be a homeomorphism of manifold M and $C = \{M(1), \cdots, M(s)\}$ be a finite covering of M by closed sets. The sets $M(i)$ are called cells of the covering.

Definition 8 *[18] Let G be a directed graph having s vertices where each vertex i corresponds to the cell $M(i)$. The vertices i and j are connected by a directed edge $i \to j$ if and only if $M(j) \cap f(M(i)) \neq \emptyset$. The graph G is called a symbolic image of f with respect to the covering C.*

Denote by Ver the set of vertices of G. The graph G can be considered as a correspondence $G : Ver \to Ver$ between the vertices. Graph G is uniquely determined by its $s \times s$ matrix of transitions $\Pi = (\pi_{ij})$: $\pi_{ij} = 1$ if and only if there is the directed edge $i \to j$, otherwise $\pi_{ij} = 0$. Let d be the largest diameter of the cells $M(i)$ of the covering C.

Definition 9 *A sequence $\{z_k\}$ of vertices of the graph G is called an admissible path or simply a path if for each k the graph G contains the edge $z_k \to z_{k+1}$. If the sequence $\{z_k\}$ is periodic, then $\{z_k\}$ is called a periodic (admissible) path.*

There is a natural connection between admissible paths on the symbolic image G and ε-trajectories of the homeomorphism f.

Definition 10 *A vertex of the symbolic image is called recurrent if there is a periodic path passing through it. The set of recurrent vertices is denoted by RV. A pair of recurrent vertices i, j are called equivalent if there is a periodic path through i and j.*

By Definition 10, the set of recurrent vertices RV decomposes into several classes $\{H_k\}$ of equivalent recurrent vertices.

Consider a linear bundle $(E, M.\pi)$ and a linear extension $F(x, v) = (f(x), A(x)v)$. Let us apply the symbolic image construction to the projective mapping PF. Let $G(f)$ be a symbolic image of f to a covering $C(M) = \{m(1), \cdots, m(q)\}$. To construct a symbolic image of the induced mapping $PF : P \to P$ it is convenient to choose a covering $C(P) = \{M(z)\}$ of the projective bundle P agreed with the covering $C(M)$ such that the projection of each cell is a cell: $P\pi(M(z)) = m(j)$. The agreed covering generates a natural mapping h from $G(PF)$ onto $G(f)$ taking the vertices z on the vertex j: $h(z) = j$. Since $PF(M(z_1)) \cap M(z_2) \neq \emptyset$ and $P\pi(M(z_{1,2})) = m(j_{1,2})$ implies $f(m(j_1)) \cap m(j_2) \neq \emptyset$, the directed edge $z_1 \to z_2$ on $G(PF)$ is mapped by h on the directed edge $j_1 \to j_2$ on $G(f)$. Hence, the mapping h takes the directed graph $G(PF)$ on the directed graph $G(f)$ so that the diagram

$$\begin{array}{ccc} Ver & \stackrel{G(PF)}{\to} & Ver \\ \downarrow^{h} & & \downarrow^{h} \\ ver & \stackrel{G(f)}{\to} & ver \end{array}$$

commutes, where Ver and ver are the vertices of $G(PF)$ and $G(f)$, respectively.

4.1 Spectrum of symbolic image.

Let $F : E \to E$ be a linear extension of f, $PF : P \to P$ be the associated projective mapping and $G(PF)$ be the symbolic image of PF. The existence of an edge $i \to j$ on $G(PF)$ guarantees the existence of a point (x, y) in the cell

$M(i)$ such that the image $PF(x, y)$ is in the cell $M(j)$. Obviously, such point is not unique. By setting $a[ji] = |A(x)e(y)|$, $|e| = 1$ we fix a framing of the edge $i \to j$. The value $a[ji]$ is a change coefficient of a vector length. The structure consisting of the symbolic image $G(PF)$ and the values $\{a[ji],\ i \to j\}$ is said to be the framed symbolic image $G(PF)$.

Each periodic path $\omega = \{z_0, z_1, ..., z_p = z_0\}$ on $G(PF)$ of period p induces the Lyapunov exponent

$$\lambda(\omega) = \frac{1}{p}\sum_{k=1}^{p} \ln a[z_k z_{k-1}].$$

Definition 11 *The spectrum of the symbolic image on the set of recurrent vertices RV is*

$$\Sigma(G(PF)) = \{\lambda \in R : \text{ there are periodic paths } \omega_k \text{ with } \lambda(\omega_k) \to \lambda \text{ as } k \to \infty\}.$$

4.2 Computation of spectrum.

Let us consider some class H of equivalent recurrent vertices. A periodic path $\omega = \{z_1, ..., z_p = z_0\}$ is called simple if the vertices $z_1, ..., z_p$ are different, i.e., $z_i \neq z_j$ as $i \neq j$; $i, j = 1, ..., p$. Since a symbolic image has a finite number of vertices, the number of simple period paths is finite. For a class H let $\phi_1, ..., \phi_q$ be the all simple periodic paths of periods $p_1, ..., p_q$, respectively. Let

$$\lambda(\phi_j) = \frac{1}{p_j}\sum_{k=1}^{p_j} \ln a[z_k^j z_{k-1}^j]$$

be the characteristic exponent of periodic path $\phi_j = \{z_1^j, ..., z_{p_j}^j\}$. Let

$$\begin{aligned} \lambda_{\min}(H) &= \min\{\lambda(\phi_j),\ j = 1, ..., q\}, \\ \lambda_{\max}(H) &= \max\{\lambda(\phi_j),\ j = 1, ..., q\} \end{aligned}$$

be the minimum and the maximum of characteristic exponents of simple periodic paths of the class H.

Theorem 6 *[24] The spectrum of the symbolic image consists of the intervals $[\lambda_{\min}(H_k), \lambda_{\max}(H_k)]$, where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices of the symbolic image $G(PF)$.*

Since M is compact, the mapping $A(x)$ has a modulus of continuity $\eta_A(\rho)$ on x. Set

$$\eta(\rho) = \eta_A(\rho) + \max_{x\in M} |A(x)|\rho,$$

$$\theta = (\min_{x\in M,\ |e|=1} |A(x)e|)^{-1} = \max_{x\in M} |A^{-1}(x)|.$$

Theorem 7 *[24] The Morse spectrum $\Sigma(F)$ of the linear extension F is in the union*

$$\bigcup_k [\lambda_{\min}(H_k) - \theta\eta(d), \lambda_{\max}(H_k) + \theta\eta(d)],$$

where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices on symbolic image $G(PF)$, d is a maximal diameter of cells of covering $C(P)$.

Thus Theorem 7 lets to estimate the Morse spectrum.

4.3 Localization of the chain recurrent set.

Denote by $RV(d)$ the union of cells $M(i)$ for which the vertices i are recurrent

$$RV(d) = \{\bigcup M(i) : i \textit{ is recurrent}\},$$

where d is the largest diameter of the cells $M(i)$. It should be noted that in fact the constructed set $RV(d)$ depends on the covering $C(P)$. However, in what follows we need only to consider the dependence of $RV(d)$ on d.

Theorem 8 ***[20, 21]***

1. The set $RV(d)$ is a closed neighborhood of the chain recurrent set CR, i.e.,

$$CR \subset RV(d).$$

2. The chain recurrent set CR coincides with the intersection of the sets $RV(d)$ for all positive d:

$$CR = \bigcap_{d>0} RV(d).$$

Theorem 8 makes possible to localize the chain recurrent set without preliminary information on a dynamical system. The subdivision is a main step of the localizing algorithm.

4.4 Subdivision process.

Let $C = \{M(i)\}$ be a covering of M and G be the symbolic image to C. Suppose a new covering NC is a subdivision of C. It is a convenience to designate the cells of the new covering as $m(i,k)$. This means that each cell $M(i)$ is subdivided such that new cells $m(i,k)$, $k = 1, 2, \ldots$, form a subdivision of the cell $M(i)$

$$\bigcup_k m(i,k) = M(i).$$

Denote by NG the symbolic image to the new covering NC. In this case the vertices of the new symbolic image are designed as (i,k). The described subdivision generates a natural mapping h^* from NG onto G which takes the vertices (i,k) onto the vertex i. Since from $f(m(i,k)) \cap m(j,l) \neq \emptyset$ it follows that $f(M(i)) \cap M(j) \neq \emptyset$, the directed edge $(i,k) \to (j,l)$ is mapped onto the directed edge $i \to j$. Hence, the mapping h^* takes the directed graph NG on the directed graph G. From this it follows that every path on the new graph NG is transformed on some path on the graph G.

4.5 Localizing algorithm.

We apply the process of adoptive subdivision. Here the adaptive subdivision means that some cells are excluded from consideration, but other are subdivided. An algorithm localizing the chain recurrent set CR is available in [20, 21]. It consists of the following steps:

1. Starting with an initial covering C, the symbolic image G of the map PF is found. The cells of the initial covering may have arbitrary diameter d_0 .
2. The recurrent vertices $\{i_k\}$ of the graph G are recognized. Using the recurrent vertices, a closed neighborhood $V = \{\cup M(i_k) : \ i_k \textit{ is recurrent}\}$ of the chain recurrent set CR is found.
3. The classes $\{H_m\}$ of equivalent recurrent vertices are found, and the family of simple periodic paths $\{\phi_j^m\}$ is recognized for each class H_m.
4. The intervals $I_m = [\lambda_{\min}(H_m) - \theta\eta(d), \lambda_{\max}(H_m) + \theta\eta(d)]$ are determined by the families $\{\phi_j^m\}$.
5. The cells corresponding to recurrent vertices $\{M(i_k) : i_k \textit{ is recurrent}\}$ are subdivided. The new covering is defined.

6. The symbolic image G is constructed for the new covering. It should be noted that the new symbolic image may be constructed on the set $V = \{\cup M(i_k) : \ i_k \ is \ recurrent\}$. In other words, the cells corresponding to non recurrent vertices do not participate in the construction of the new covering and the new symbolic image.

7. Then one goes back to the second step.

Repeating this subdivision process we obtain we obtain

1) a sequence of neighborhoods $\{V_k\}$ of the chain recurrent set,

2) a sequence of the largest diameters of cells $\{d_k\}$,

3) a sequence of families of intervals $\Sigma^k = \{I_m^k\}$. The following theorem substantiates the described algorithm.

Theorem 9 *[20, 21] The sequence of sets $V_0, \ V_1, \ V_2, \ ...$ offers the following properties:*
(i) the neighborhoods V_k are imbedded one inside the other, i.e.,

$$V_0 \supset V_1 \supset V_2 \supset ... \supset CR,$$

(ii) each set $\Sigma^k = \cup_m I_m^k$ contains the Morse spectrum of F,

(ii) if $d_k \to 0$ as k becomes infinite then

$$\lim_{k\to\infty} V_k = \bigcap_k V_k = CR.$$

$$\lim_{k\to\infty} \Sigma^k = \Sigma(F).$$

Corollary 1 *If the diffeomorphism is hyperbolic on the chain recurrent set then a finite steps is enough to verify that 0 is not in the Morse spectrum.*

4.6 Attractor on symbolic image.

Consider a symbolic image G of the homeomorphism f. A set of vertices L is invariant if for each vertex $i \in L$ there exist the edges $j \to i$ and $i \to k$ such that $j, \ k \in L$. Let L be an invariant set of vertices on the symbolic image G. The set of vertices

$$En(L) = \{j \in L : \ there \ exists \ an \ edge \ i \to j, \ i \notin L\}$$

is called the entrance of L. The set of vertices

$$Ex(L) = \{i \in L,\ there\ exists\ an\ edge\ i \to j,\ j \notin L\}$$

is called the exit of L.

Definition 12 • *An invariant set $L \subset Ver$ is an attractor if $Ex(L) = \emptyset$.*

• *An invariant set $L \subset Ver$ is a repeller if $En(L) = \emptyset$.*

Let L be an attractor. A basin or domain of attraction is the set of vertices

$$D(L) = \{j :\ each\ path\ through\ j\ finishes\ in\ L\},$$

i.e., for each admissible path $\{..., j, ..., i_k, ...\}$ there exists a number k^* such that the vertices i_k with $k > k^*$, belongs to L.

If $L \subset Ver$ is an attractor, the repeller $L^* = Ver \setminus D(L)$ is named as dual for the attractor L. As one would expect, there is a natural correlation between the attractors of a dynamical system and the attractors of its symbolic image.

Theorem 10 *[23] If L and $D(L)$ are an attractor and its domain of attraction on a symbolic image G, then there are an attractor Λ of the homeomorphism f and its basin $W^s(\Lambda)$ such that*

(i) the set

$$U = int\{\bigcup M(i),\ i \in L\},$$

where $int\cdot$ denotes the interior, is a fundamental neighborhood of Λ,

(ii) the set $W = \{\bigcup M(j),\ j \in D(L)\}$ is in the basin $W^s(\Lambda)$,

(iii) the set $V = int\{\bigcup M(k),\ k \in L^*\}$ is a neighborhood of the dual repeller Λ^*.

The following theorem shows that an attractor of a dynamical system and its domain of attraction can be defined as precisely as one likes by employing a symbolic image with covering cells of small enough diameter.

Theorem 11 *[23] Let $\Lambda \subset M$ be an attractor, V_1 be its arbitrarily small neighborhood, and V_2 be an arbitrarily large neighborhood such that*

$$\Lambda \subset V_1 \subset V_2 \subset clV_2 \subset W^s(\Lambda).$$

Then there exists $d_0 > 0$ such that each symbolic image G, with the maximal diameter of covering cells $d < d_0$, has an attractor L and its domain of attraction $D(L)$ such that

$$\Lambda \subset \{\cup M(i),\ i \in L\} \subset V_1 \subset V_2 \subset \{\cup M(j),\ j \in D(L)\} \subset W^s(\Lambda).$$

Thus the construction of a dynamic system attractor and its domain of attraction is reduced by Theorem 11 to the same task on symbolic image.

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