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Dynamical systems on manifolds

ATTRACTORS OF THE DYNAMICAL SYSTEMS CONNECTED TO THE PARABOLIC EQUATION

Abstract of the PhD thesis

LEBEDEV A.

Faculty of mathematics and mechanics
Department of Differential Equations
St.Petersburg State University
Bibliotechnaya pl., 2., Petrodrovets,
St.Petersburg, 198504, Russia
e-mail: Andrey.Lebedev@pobox.spbu.ru

Abstract

There are investigated the qualitative properties of the global attractors of two classes of the finite dimensional dynamical systems connected to the parabolic equation. For the semidynamical system generated by the discretization of the parabolic equation there are obtained the sufficient conditions under which the dynamical system appears, the size and the Hausdorff dimension of the attractor are estimated. For the dynamical system generated by the restriction of Chafee-Infante system in the critical case on the inertial manifold there is proved the polynomial estimate of the rate of attraction of the trajectories to the attractor in the terms of the starting approximation. Also there is obtained the estimate of the deviation of the attractor of the perturbed system

from the attractor of the source system in the terms of the value of the system perturbation.

Basic characteristics of the study

The actuality of the topic. The investigation of the global structure of the infinite dimensional dynamical and semidynamical systems is the rapidly developing area of the modern mathematics [11], [19]. So called evolutionary systems generated by the partially differential equations are especially important and interesting for the investigation. The special attention is paied to the structure of global attractors of such systems [1], [16], [9]. Except the structure of the attractors the question concerning the dependence of the attractors from the perturbations of the system has also received the intensive study [10], [11]. Along with the exploration of the infinite dimensional evolutionary systems there are studied two classes of the finite dimensional dynamical and semidynamical systems generated either by the discretization of the appropriate partial differential equation or by the restriction of the evolutionary equation on its finite dimensional positive invariant manifold (so called inertial manifold) [8], [18]. The most developed area of the infinite dimensional evolutionary systems theory is the theory of systems generated by the parabolic partial differential equation [12].

The basic theory of the discretization of the parabolic equations started to develop after the publication of the paper by Ladyzhenskaya O.A. [4]. The paper was devoted to the global stability of the difference schemes for such equations. For some of the schemes there was proved the existence of the global attractors and their sizes and Hausdorff dimension were estimated. The investigation of the dynamical system generated by full discretizations of the parabolic equations was continued in the papers by different authors. Let's remark the paper by Eirola T. and Pilyugin S.Yu. [7]. In all mentioned investigations the time discretization used the simpliest method – Euler method.

One of the most important characteristics of the global attractor of the dynamical system is the rate of attraction of the trajectories. In recent years a lot of researchers [5] are interested in the exploration of the qualitative properties of the trajectories and the attractors of the systems of the differential equations arising from the restriction of the evolutionary system corresponding to so called Chafee-Infante problem [6]. One-dimensional Chafee-Infante problem studies a nonlinear semigroup generated by the Dirichlet boundary-value problem for the parabolic equation

$$u_t = u_{xx} + bu - f(u),$$

where $b > 0$ is a parameter and the nonlinearity f belongs to the class of functions which contains a function $f(u) = u^3$ as a typical representative. It is known [12] that all the fixed points of the generated semigroup $S(t)$ are hyperbolic while the system itself is structurally stable if and only if $b \neq m^2$, $m = 1, 2, \dots$ That's why the construction of the complete theory of Chafee-Infante problem requires the investigation of the critical case $b = m^2$. Such research was started in the papers by Kostin I.N. [14], [13], where there are explored the perturbations of the global attractor in the critical case. Hence, by the example of Chafee-Infante problem there were obtained the first estimations of the rate of attraction for the infinite dimensional semigroup which is not structurally stable. It was shown that under certain assumptions the rate of attraction is polynomial.

The research of the infinite dimensional systems like those considered by Kostin I.N. was continued in the papers by Kornev A.A. [2], [3]. In all mentioned papers the polynomial rate of attraction to the global attractor of generated system was proved for the case of one non-hyperbolic rest point. Also there were obtained the results concerning the continuous parameter-dependence of the attractor of the perturbed system. But the method of the papers didn't allow to obtain the sufficient information about the rate of attraction of the individual points of phase space, because all the estimations obtained deal with the neighbourhoods of the attractor of the system.

Another approach that appeared in the paper by Kostin I.N. and Pilyugin S.Yu [15] allowed to prove the exponential estimation of the rate of attraction in the terms of the starting approximation and to estimate the distance of the global attractors of the perturbed and nonperturbed systems for gradient-like systems generated by Chafee-Infante problem in non-critical case.

The aim of the study. To study the qualitative properties of the global attractors for two classes of finite dimensional dynamical systems generated by the discretization of parabolic equations and by the restriction of the appropriate evolutionary systems on the inertial manifolds.

For the semidynamical system generated by the discretization of the parabolic equation to obtain the sufficient conditions under which the dynamical system appears, to obtain the conditions of the dissipativity and to estimate the size and Hausdorff dimension of the attractor.

For the dynamical system generated by the restriction of Chafee-Infante system in the critical case on its inertial manifold to prove the polynomial estimation of the rate of attraction of the trajectories to the attractor in the terms of the starting approximation, to obtain the estimation of the deviation of the attractor of the perturbed system from the attractor of the source system in the terms of the value of the system perturbation.

Basic method of the research. The research uses the methods of the qualitative theory of the dynamical systems and differential equations, methods of the functional analysis and linear algebra.

Scientific newness. All the basic results of the study are new.

1. For the first time there are investigated the qualitative properties of the dynamical and semidynamical systems generated by the discretization of the Dirichlet problem for the parabolic equation by using Adams method of the arbitrary degree.
   (a) The sufficient conditions of the dynamical system construction are obtained.
   (b) The sufficient condition is given under which the system is dissipative and the upper estimation of the diameter of the global attractor of the system is obtained.
   (c) The upper estimations of the Hausdorff dimension of the global attractor of the system are obtained for both cases of small and large Lipschitz constants of the nonlinearity. The obtained estimate of the Hausdorff dimension does not depend on the parameters of the approximation method.
2. New results concerning the behaviour of the trajectories of gradient-like system of the differential equations generated by the restriction of Chafee-Infante system on its inertial manifold in the case of critical parameter value are obtained.
   (a) The global polynomial estimation of the rate of attraction of the trajectories to the attractor in the terms of the starting approximation is proved.
   (b) The logarithmic approximation of the deviation of the attractor of the perturbed system from the attractor of the source system in the terms of the value of system perturbation is obtained.

Theoretical and practical value of the study. The results that were obtained may be used in the construction of the dissipativity conditions, in the construction of the estimations of the quantitive characteristics of the global attractors of semidynamical systems generated by the discretizations of the partial differential equations. The methods of the proofs used in the study may be used in the construction of both local and global estimations of the rate of attraction of the trajectories of the dynamical systems to the global attractor.

Approbation of the study. Basic results were expounded in the following conferences and seminars.

1. Third International Conference "Differential Equations and Applications" (St.Petersburg, 2000).
2. International Seminar "Patterns and waves: theory and applications" (St.Petersburg, 2002).
3. St.Petersburg differential equations seminar (supervised by Pliss V.A.), St.Petersburg, 2002.

Publications. Basic results of the PhD thesis were published in author's papers [1 - 4] listed in the end of the abstract.

Structure of the PhD thesis. PhD thesis contains 92 pages of typewritten text and consists of the introduction, two chapters divided into 11 paragraphs and the references page of 38 items.

Brief contents of the study.

Chapter 1 is devoted to the research of the semidynamical system generated by the discretization of the parabolic equation. Consider the parabolic equation

$$u_t = u_{xx} + f(u), \quad t > 0, \quad x \in (0, 1), \tag{1}$$

with boundary-value problem

$$u(t, 0) = u(t, 1) = 0, \quad u(0, x) = u_0(x), \quad t > 0, \quad x \in (0, 1). \tag{2}$$

It is assumed that the nonlinearity f satisfies Lipschitz condition with constant δ and the following condition

$$xf(x) \leq a_0 + a_1 x^2 \text{ where } a_0 > 0, \quad 0 < a_1 < \pi^2.$$

Consider the following difference approximation of the parabolic equation (1). Take $h > 0$, natural number N and $d = \frac{1}{N+1}$. Let's call h the *time step* and d the *space step*. We'll approximate the values $u(t, x)$ in the points $t = nh$, $x = md$ $(0 \le m \le N + 1,\ n \ge 0)$ by numbers v_m^n that satisfy the system of difference equations that will be described later. Boundary conditions (2) of the source problem will naturally generate the boundary conditions of the discrete problem:

$$v_0^n = v_{N+1}^n = 0, \quad n \ge 0,$$
$$v_m^0 = u_0(md), \quad 1 \le m \le N.$$

Vector

$$v^n = \begin{pmatrix} v_1^n \\ \cdots \\ v_N^n \end{pmatrix}$$

will be called a *time layer*.

Space discretization (by the space variable x) will be obtained from the standard approximation of the second derivative

$$u_{xx}(nh, md) \approx \frac{1}{d^2}\left(v_{m-1}^n - 2v_m^n + v_{m+1}^n\right),$$

and time discretization will use the interpolational (implicit) Adams method of arbitrary degree $p \ge 2$. The described discretization yields the infinite system of difference equations

$$\frac{v^{n+1} - v^n}{h} = \sum_{k=1}^{p} c_{pk}(Av^{n+2-k} + f(v^{n+2-k})), \quad n \ge p, \quad 1 \le m \le N, \qquad (3)$$

where c_{pk} are the coefficients of Adams method of degree p, the mapping f is a natural spreading of the nonlinearity f into the $\mathbf{R}^N$ space, and the square matrix A arises from the approximation of the second derivative. The first member v^0 of the initial data for system (3) arises from the initial data of the boundary-value problem for the parabolic equation, while other $p - 2$ members are probably obtained from another approximation method.

In § 2 there is studied a problem of the solvability of the recurrent equation (3) regarding to v^{n+1}. The following theorem is proved.

Theorem 2.1. *For each natural number N and for all h satisfying the inequality*

$$h < \frac{1}{c_{p1}\left(|A| + \delta\right)}, \qquad (4)$$

the recurrent equation (3) is solvable regarding to v^{n+1} *and defines a Lipschitz discrete semidynamical system.*

Let's denote by φ the semidynamical system arising from the described discretization.

In § 3 there are found the sufficient conditions under which the recurrent equation (3) generates the dynamical system. These conditions appear fundamentally different for $p = 2$ and $p \geq 3$ cases.

Lemma 3.1. *Let* $p = 2$*. If the condition (4) holds, the recurrent equation (3) defines a Lipschitz discrete dynamical system.*

Lemma 3.2. *Let* $p \geq 3$*. If the conditions (4) and*

$$\delta < \frac{|c_{pp}|\,\eta}{\sum\limits_{k=3}^{p} |c_{pk}|},$$

where $\eta = \eta(N)$ *is defined by formula*

$$\eta = \left|A^{-1}\right|^{-1} = \frac{4}{d^2}\sin^2\frac{\pi d}{2},$$

hold, then the recurrent equation (3) defines a Lipschitz discrete dynamical system.

In § 4 the dissipativity conditions for the semidynamical system φ are studied, and the upper estimate of the diameter of the global attractor of this system is constructed.

Theorem 4.1. *If the natural number* N *satisfies the condition*

$$a_1 < \eta(N), \tag{5}$$

then there exists such $h_0 = h_0(N) > 0$ *that for all* $0 < h < h_0$ *the semidynamical system* φ *is dissipative in the sense of Levinson.*

It is known [17] that any dissipative semidynamical system has a global attractor. In § 4 it is proved that the attractor $\mathcal{A}$ of the semidynamical system φ is contained in the closed bounded set W^{p-1} which is a Descartes power of the closed ball

$$W(h) = \left\{ x \in \mathbf{R}^N : \quad |x|^2 \leq \frac{a_0}{\eta - a_1} + k(h)h \right\},$$

where $k(h)$ is a function with the property $\lim\limits_{h\to 0} k(h) > 0$.

§ 5 is devoted to the construction of the estimation of Hausdorff dimension of attractor $\mathcal{A}$, the estimation that is uniform regarding all small enough steps of the discretization. The following theorem is proved.

Theorem 5.1. *The following statements hold.*

1) If the Lipschitz constant of the nonlinearity f satisfies the condition

$$\delta < \pi^2,$$

then for any natural number N, satisfying the condition (5), there exists such $h_1 = h_1(N) > 0$, that for any $0 < h < h_1$ the semidynamical system φ possesses the global attractor $\mathcal{A}$ with the Hausdorff dimension not exceeding 1*.*

2) If the inequality

$$\delta \geq \pi^2$$

holds then for any natural number N, satisfying the condition (5), there exists such $h_1 = h_1(N) > 0$, that for any $0 < h < h_1$ the semidynamical system φ possesses the global attractor $\mathcal{A}$ with the Hausdorff dimension not exceeding the minimum natural number k satisfying the inequality

$$\frac{2}{3}\,(k+1)\,(2k+1) > \delta + 1.$$

Chapter 2 is devoted to the study of the dynamical system generated by the restriction of the infinite dimensional semigroup corresponding to Chafee-Infante problem in the critical case on its inertial manifold. To do this there is described and investigated the class of gradient-like systems of the differential equations

$$\dot{x} = F(x), \;\; F \in C^1\left(\mathbf{R}^n\right),$$

and the dynamical systems generated by these equations, which satisfy seven conditions (A)-(G), listed in § 1 and § 2 of chapter 2.

(A) There exists the open bounded domain $G \subset \mathbf{R}^n$ in the closure of which the dynamical system φ has a finite number of rest points.

(B) For each $t > 0$ the inclusion $\varphi(t, \overline{G}) \subset G$ holds.

(C) System φ has a smooth Lyapunov functional $V : \overline{G} \to \mathbf{R}$, whose derivative regarding the system φ

$$\dot{V}(x) = \frac{\partial}{\partial t} V\left(\varphi(t,x)\right)\Big|_{t=0},$$

has the property
$$\dot{V}(x) < 0$$
for any $x \in \overline{G}$ that is not a rest point. For any rest point p the equality $\dot{V}(p) = 0$ holds.

(D) All except one rest points of the dynamical system φ in G domain are hyperbolic. For the nonhyperbolic rest point p_0 the matrix
$$\frac{\partial F}{\partial x}(p_0)$$
has one zero eigenvalue. Other eigenvalues have nonzero real parts. It is known [19] that in this case the rest point p_0 has one dimensional central manifold $W^c(p_0)$. We assume that
$$\varphi(t,x) \to p_0 \text{ as } t \to +\infty$$
for any $x \in W^c(p_0)$.

(E) The value of Lyapunov functional for rest points satisfies the condition
$$V(p_0) > V(p_i), \quad i = 1, \ldots, m,$$
where p_0 is a nonhyperbolic point and $(p_i)_{i=1,\ldots,m}$ are all other rest points.

(F) Unstable manifold of the nonhyperbolic rest point, stable and unstable manifolds of the hyperbolic rest points of the dynamical system φ are transverse.

Let's denote by $\mathcal{A}$ the global attractor of the dynamical system φ.

(G) For the nonhyperbolic rest point p_0 of system φ there exist such neighbourhood U and numbers $C > 0$ and $l > 0$, that for any point $x \in U$ for which $\varphi(t,x) \in U$ for $0 < t < \tau$ the following estimate
$$\operatorname{dist}(\varphi(t,x), \mathcal{A}) \le \left(Ct + \operatorname{dist}^{-l}(x, \mathcal{A})\right)^{-1/l}$$
holds for $0 < t < \tau$.

In § 6 it is proved that all these conditions are satisfied for the restriction of the infinite dimensional evolutionary system generated by Chafee-Infante problem in the critical case on its inertial manifold.

In § 1 it is proved a set of statements concerning the basic qualitative properties of the trajectories of the dynamical system φ, satisfying three conditions (A)-(C).

In § 2 and later it is assumed that the system φ satisfies seven conditions (A)-(G) described earlier. It is defined the "arrow" relation for the pairs of the rest points of the dynamical system φ (similarly to the one defined for Kupka-Smale systems) and several auxilary statements concerning the behaviour of the trajectories of the system φ in the neighbourhood of the rest points are proved.

In § 3 the shift mapping for the φ system is defined

$$S = \varphi(1, \cdot).$$

This mapping generates the discrete dynamical system S. For this discrete system in § 3 the local exponential estimation of the rate of attraction in the neighbourhood of the hyperbolic fixed point is proved.

In § 4 the main statement of chapter 2 is proved – the global polynomial estimation of the rate of attraction of the trajectories of the system φ to the attractor in the terms of the value of the initial approximation. Let's denote by $B_r(\mathcal{A})$ open r-neighbourhood of attractor $\mathcal{A}$.

Theorem 4.1. *There exist such numbers $r > 0$, $K \geq 1$, $C > 0$ and $l > 0$ that for any point $x \in B_r(\mathcal{A})$ the estimate*

$$\operatorname{dist}(\varphi(t, x), \mathcal{A}) \leq K \left(Ct + \operatorname{dist}^{-l}(x, \mathcal{A})\right)^{-1/l}$$

holds for $t \geq 0$.

Also in § 4 is shown that the transversality condition ((F) condition) is essential for the polynomial estimate of the rate of attraction to exist. An example of the two-dimensional system not satisfying the transversality condition for which the polynomial estimate of the rate of attraction doesn't hold is described.

In § 5 the attractors of the family of the perturbed systems S_τ, $\tau \in T$ are studied and the estimate of the deviation of the attractor of the perturbed system from the attractor of the source system S is obtained:

$$\operatorname{dist}(\mathcal{A}_\tau, \mathcal{A}) \leq \left(\frac{1}{\alpha} \ln\left(1 + \left(\frac{L-1}{2Ld(\tau)}\alpha^{1/l}\right)^{\frac{l}{l+1}}\right)\right)^{-1/l},$$

where A_τ is an attractor of the perturbed system S_τ, $\mathcal{A}$ is an attractor of the source system S,

$$d(\tau) = \sup_{u \in G} \|S_\tau(u) - S(u)\|$$

is the value of the system perturbation, and $\alpha > 0$, $l > 0$ and $L > 1$ are some numbers which do not depend on τ.

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