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# Transient Motion of Electrically Conducting Fluids of Different Prandtl Numbers: Flow Near a Vertical Boundary

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## 1 Introduction

Boundary layer flow of electrically conducting fluids finds applications in several industrial and technological fields such as, for example, chemical, aerospace and nuclear engineering. As a result, several theoretical investigations have been carried out in the last four decades or so to primarily analyze the influence of externally applied magnetic fields on a variety of flows fields. In particular, a number of hydromagnetic flow problems have been reported in the literature to consider the individual as well as the combined effects of magnetic, buoyancy and viscous forces under certain simplifying assumptions.

It is well known that the partial differential equations which govern the flow of electrically conducting fluids under the action of the magnetic and buoyancy forces are highly nonlinear and coupled. However, when the flow takes place near flat plates, the quadratic convection term in the momentum equation can drop out, and the system thus becomes linear. It is therefore possible to obtain exact closed form solutions of the linearized problem in a number of practically important situations – for instance, when a stationary or moving bounding plate is subject to heating, cooling or heat flux conditions [1–5]. While the majority of the relevant works reported in literature has emphasized the effects of magnetic field and buoyancy force on specifically chosen liquid and gaseous media, the detailed analysis of the underlying real fluid properties characterized by the Prandtl numbers of the fluids has not received much attention. Our main objective in this study is thus to present exact analytical solutions for the transient hydromagnetic impulsive flow of fluids of different Prandtl numbers when the flow takes place near a moving infinite vertical plate subject to uniform heat flux. We have shown in our analysis that one needs to consider two separate solutions – one for $\mathrm{Pr} \neq 1$ and the other for $\mathrm{Pr} = 1$, Pr being the Prandtl number of the fluid under consideration. The explicit forms of these solutions have been presented in a unified form for both the cases of the magnetic field being fixed to the fluid and to the plate. The variation of the boundary layer velocity profiles with the Prandtl number has been shown for a combination of Grashoff and Hartmann numbers.

## 2 Governing Equations

Consider the unsteady flow of an infinite extent of an electrically conducting incompressible fluid past a moving infinite vertical flat plate. With respect to the rectangular cartesian coordinate system $Oxyz$, the axis $Oz$ is taken along the wall in the upward direction and the axis $Oy$ is taken perpendicular to it into the fluid. The flow is assumed to take place under the influence of an external magnetic field. The usual equations of motion of the fluid, neglecting viscous dissipation and Ohmic heating, are

$$\begin{aligned} \nabla \cdot \mathbf{V} &= 0 && (1)\\ \rho \frac{d\mathbf{V}}{dt} &= -\nabla p + \mu \nabla^2 \mathbf{V} + g\beta(T - T_\infty)\mathbf{k} + \mathbf{J} \times \mathbf{B} && (2)\\ \rho c_p \frac{dT}{dt} &= k\nabla^2 T && (3)\\ \mathbf{J} &= \sigma(\mathbf{V} - \mathbf{U}) \times \mathbf{B} && (4) \end{aligned}$$

where $\mathbf{V} = (V_x, V_y, V_z)$ is the fluid velocity, $\mathbf{U}$ is the bounding plate velocity, $\rho$ the density, $p$ the pressure, $T$ the temperature, $\mathbf{B}$ the magnetic field, $\mathbf{J}$ the current density, $\mu$ the fluid viscosity, $g$ the acceleration due to gravity, $\beta$ the volumetric coefficient of thermal expansion, $k$ the thermal conductivity of the fluid, $\sigma$ the electrical conductivity and $c_p$ is the specific heat of the fluid at constant pressure. In equations (2) and (3), $d/dt$ is the convective derivative operator.

For the two-dimensional motion considered here, at times $t \leq 0$, the plate and the fluid medium are assumed to be at rest and at the constant temperature $T_\infty$. At time $t > 0$, the plate is set into motion in its own plane with a velocity proportional to $t^n$, and simultaneously, heat is also supplied to the plate at a constant rate. A uniform magnetic field of strength $B_y$ is applied in the $y$ direction. We consider two different flow situations with respect to the magnetic field: (i) the magnetic lines of force are fixed relative to the fluid, and (ii) the magnetic lines of force are fixed relative to the boundary. These two cases will, however, be incorporated into a single momentum equation so as to obtain a unified solution. As is common in Stokes problems of flow near flat boundaries, we assume that the effects of the convective and pressure gradient terms in the momentum and energy equations are negligible. The density of the liquid is assumed to be constant; however, in the case of free convection flow, it is considered variable in forming the buoyancy force. We also assume that the magnetic Reynolds number is very small so that the induced magnetic field produced by the motion of the electrically conducting fluid is negligible in comparison with the applied one. As a result of the boundary layer approximations, the physical variables become functions of the time variable $t$ and the space variable $y$ only. Moreover, the only non-zero component of velocity occurring in the analysis will be $V_z$ which we shall hereafter denote by $u(y, t)$. Under these assumptions, the boundary layer momentum equation can be written in the form [6]

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) - \frac{\sigma B_y^2}{\rho}(u - K\lambda t^n) \tag{5}$$

where $\nu$ is the kinematic viscosity, $\lambda$ a constant and

$$K = \begin{cases} 0, & \text{if } B_y \text{ is fixed relative to the fluid} \\ 1, & \text{if } B_y \text{ is fixed relative to the plate.} \end{cases}$$

The boundary layer energy equation is

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \tag{6}$$

The initial and boundary conditions relevant to the fluid flow subject to a power-law velocity of the bounding plate then become

$$\begin{aligned} u = 0, \quad T = T_\infty, \quad & \text{for} \quad y \geq 0 \quad \text{and} \quad t \leq 0 \\ u = \lambda t^n, \quad \frac{\partial T}{\partial y} = -\frac{q}{k} \quad & \text{at} \quad y = 0 \quad \text{for} \quad t > 0 \\ u \to 0, \quad T \to T_\infty \quad & \text{as} \quad y \to \infty \quad \text{for} \quad t > 0 \end{aligned} \tag{7}$$

where $q$ is the heat flux per unit area at the plate and $\lambda$ is a non-zero constant.

## 3 Impulsive Motion

For impulsively moving vertical plate, we take the plate velocity as $\lambda$, which corresponds to $n = 0$. We shall solve the boundary layer equations in dimensionless forms. Using a characteristic length scale $L = \nu/\lambda$, we introduce the non-dimensional quantities

$$\begin{aligned} \bar{y} &= y/L, \quad \bar{t} = \lambda t/L, \quad \bar{u} = u/\lambda, \quad \bar{T} = k(T - T_\infty)/(qL) \\ \mathrm{Pr} &= \rho\nu c_p/k, \quad G = qg\beta L^2/(k\lambda^2), \quad m = \sigma L B_y^2/(\rho\lambda) \end{aligned} \tag{8}$$

In the above, the dimensionless parameters Pr, $G$, and $m$ denote the Prandtl number of the fluid, Grashof number and the square of Hartmann number, respectively.

Using equation (8), equations (5) and (6) can be expressed in the dimensionless forms (dropping the *bar*, for convenience)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m(u-K) + GT \tag{9}$$

$$\frac{\partial T}{\partial t} = \frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{10}$$

The initial and boundary conditions become

$$\begin{aligned} u=0,\quad T=0 \quad &\text{for} \quad y\ge 0 \quad \text{and} \quad t\le 0\\ u=1,\quad \partial T/\partial y=-1 \quad &\text{at} \quad y=0 \quad \text{for} \quad t>0\\ u\to 0,\quad T\to 0 \quad &\text{as} \quad y\to\infty \quad \text{for} \quad t>0 \end{aligned} \tag{11}$$

We observe that the energy equation (10) is uncoupled from the momentum equation (9). We can therefore solve for the temperature variable $T(y,t)$ whereupon $u(y,t)$ can be obtained by solving equation (9). Taking Laplace transforms of equations (9) and (10) with respect to the $t$-variable will result in a set of (ordinary) differential equations for the transformed functions in the independent variable $y$. The resulting solutions of temperature and velocity variables in the $ys$-plane can be inverted using standard inverses combined with convolution [7, 8]. The solutions in the physical $yt$-plane can be written in the form

$$T(y,t) = 2\sqrt{\frac{t}{\pi \Pr}}\,\exp\left(-\frac{\Pr y^2}{4t}\right) - y\,\mathrm{erfc}\left(\frac{\sqrt{\Pr}\,y}{2\sqrt{t}}\right) \tag{12}$$

$$\begin{aligned} u(y,t) = {} & K\left[1-\exp(-mt)+\exp(-mt)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right]\\ & + \frac{1-K}{2}\left[\exp(-y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{mt}\right)\right.\\ & \left. + \exp(y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{mt}\right)\right]\\ & - \alpha\left[\int_0^t \{\varphi_1(y,\xi)+\varphi_2(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right.\\ & \left. - \int_0^t \{\varphi_3(y,\xi)+\varphi_4(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right] \end{aligned} \tag{13}$$

where

$$\begin{aligned}
\varphi_{1,2}(y,t) &= \frac{1}{2}\exp(-bt \mp iy\sqrt{b\,\mathrm{Pr}})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} \mp i\sqrt{b\,\mathrm{Pr}\,t}\right)\\
\varphi_{3,4}(y,t) &= \frac{1}{2}\exp(-bt \mp iy\sqrt{b\,\mathrm{Pr}})\,\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Pr}}}{2\sqrt{t}} \mp i\sqrt{bt}\right)\\
\alpha &= \frac{2G}{\sqrt{\pi\,\mathrm{Pr}}(1-\mathrm{Pr})}, \quad b = \frac{m}{1-\mathrm{Pr}}
\end{aligned}$$

and $\mathrm{erfc}(x)$ is the complementary error function defined by

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x), \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-\eta^2\right)\mathrm{d}\eta$$

In the above definitions of $\varphi_{i,j}(y,t)$, $(i = 1,3;\ j = 2,4)$, the upper sign goes with $i$ and the lower sign with $j$.

In the non-magnetic case, $m = 0$, it can be shown that

$$\begin{aligned}
u(y,t) &= \mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) + \frac{\alpha}{12}\left[\sqrt{\pi}y(y^2+6t)\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right.\\
&\quad - \sqrt{\pi\,\mathrm{Pr}}y(\mathrm{Pr}\,y^2+6t)\mathrm{erfc}\left(\frac{y}{2}\sqrt{\frac{\mathrm{Pr}}{t}}\right)\\
&\quad - \sqrt{t}(y^2+8t)\exp\left(-\frac{y^2}{4t}\right)\\
&\quad \left. + \sqrt{t}(\mathrm{Pr}\,y^2+8t)\exp\left(-\frac{\mathrm{Pr}\,y^2}{4t}\right)\right] \qquad (14)
\end{aligned}$$

It may be noted that the general velocity expression for the hydromagnetic flow, as given in equation (13), is not valid when $\mathrm{Pr} = 1$. The solution for this case is [9]

$$
\begin{aligned}
u(y,t) &= K\left[1-\exp(-mt)+\exp(-mt)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\
&+ \frac{1-K}{2}\left[\exp(-y\sqrt{m})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{mt}\right)\right. \\
&+ \left.\exp(y\sqrt{m})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{mt}\right)\right] \\
&+ \frac{2G}{m}\left[\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right)-y\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\
&- \frac{G}{2\sqrt{\pi}m}\int_0^t\left[\exp(-y\sqrt{m})\operatorname{erfc}\left(\frac{y}{2\sqrt{\xi}}-\sqrt{m\xi}\right)\right. \\
&+ \left.\exp(y\sqrt{m})\operatorname{erfc}\left(\frac{y}{2\sqrt{\xi}}+\sqrt{m\xi}\right)\right](t-\xi)^{-1/2}\,\mathrm{d}\xi \qquad (15)
\end{aligned}
$$

The above solutions correspond to the special case $n = 0$ of the power law velocity assumption $\lambda t^n$. However, the flow problems involving non-uniformly accelerated motion of the vertical plate, $(n \neq 0, 1)$, may also become important in certain applications. Solutions of equation (5) for such cases will be investigated separately.

## 4 Discussion of Results

As mentioned earlier, our main objective in this work is to analyze the variation of the boundary layer velocity $u(y,t)$ with the Prandtl number. In Figures 1–3, we have shown the velocity profiles for a set of combinations of the Grashoff number $G$ and the magnetic parameter $m$ for $0.1 \leq \mathrm{Pr} \leq 10.0$ in the early stages of the onset of motion $(t = 0.1)$. The Figure 1 shows the velocity profiles close to the boundary $(y = 0.1)$ while the Figure 2 depicts the profiles further away from the bounding plate $(y = 0.5)$. The velocity profiles have been plotted for the case of the applied magnetic field, $B_y$, being fixed relative to the fluid. In general, velocity in the boundary layer is seen to decrease with Pr. The decrease in the velocity is, however, more pronounced for those fluids whose Prandtl numbers are less than unity (e.g., air). It is also to be noted that velocity variations are far more sensitive to the changes in the parameters $G$ or $m$ when Pr is less than unity, as can be seen from the profiles in Figures 1 and 2. In other words, one may conclude that the effects of the applied magnetic field or the buoyancy forces are dominant mainly for fluids whose Prandtl numbers are less than unity. Furthermore, in order to analyze the response of the boundary layer velocity to the mode of application $(K)$ of the magnetic field, we have plotted in Figure 3 the velocity profiles for a fixed value of the magnetic parameter $m$. It can be seen that the fluid velocity variations indeed depend upon whether the applied magnetic field is fixed relative to the plate $(K = 1)$ or to the fluid $(K = 0)$; the velocity in the former case is higher than the one in the latter case for all values of the Prandtl number considered in this study. This observation holds good for both low and moderate values of the buoyancy parameter $G$.

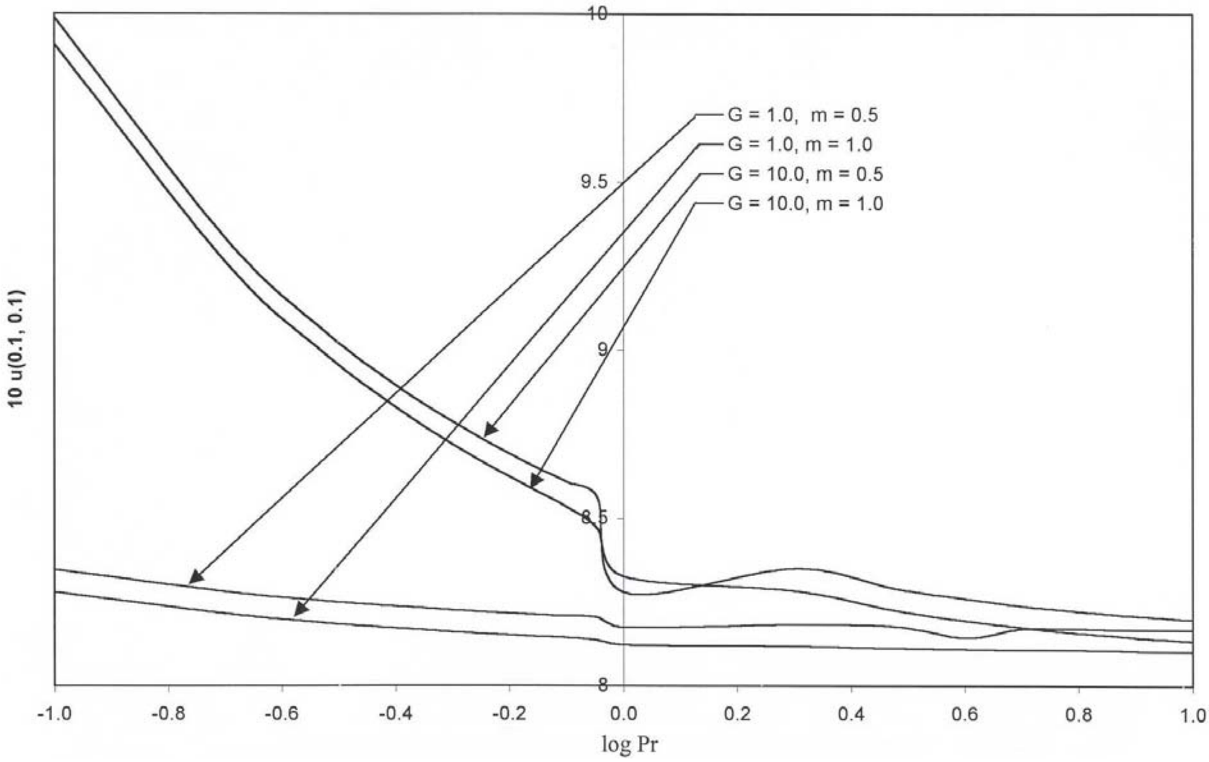


Figure 1: Variation of velocity with Prandtl number. $K = 0, y = 0.1$

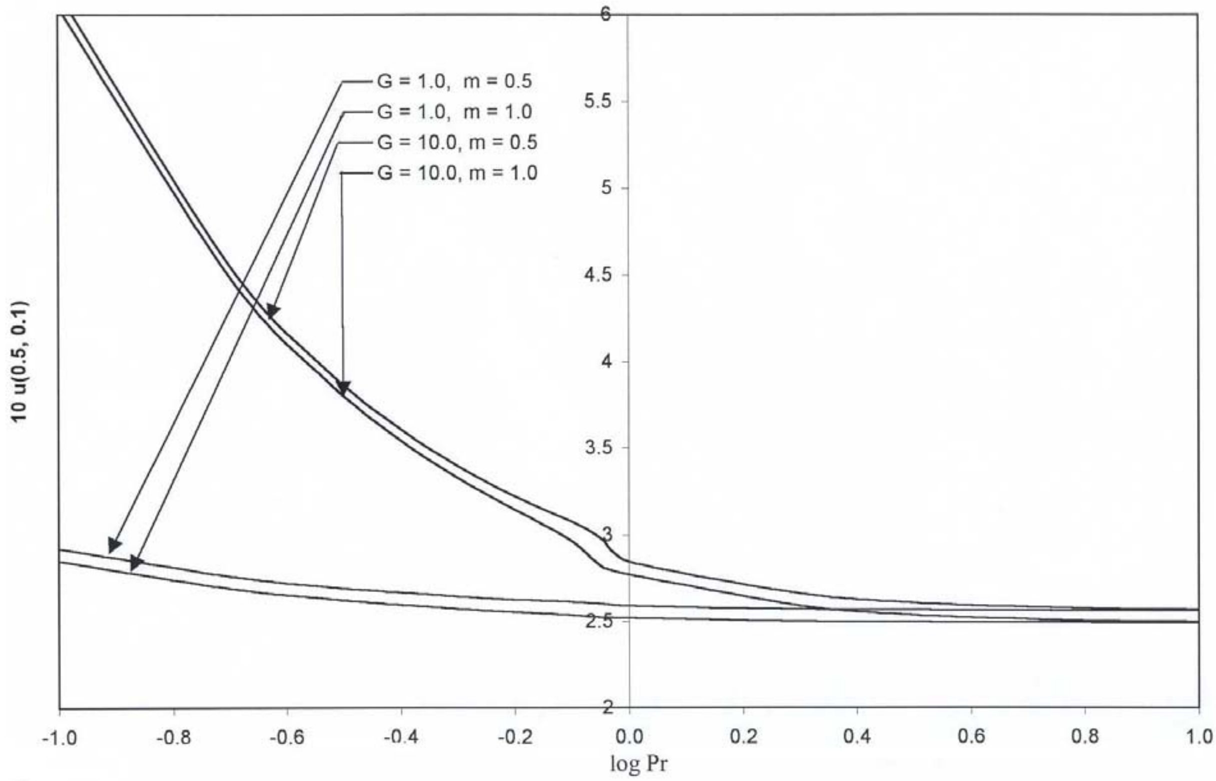


Figure 2: Variation of velocity with Prandtl number. $K = 0, y = 0.5$

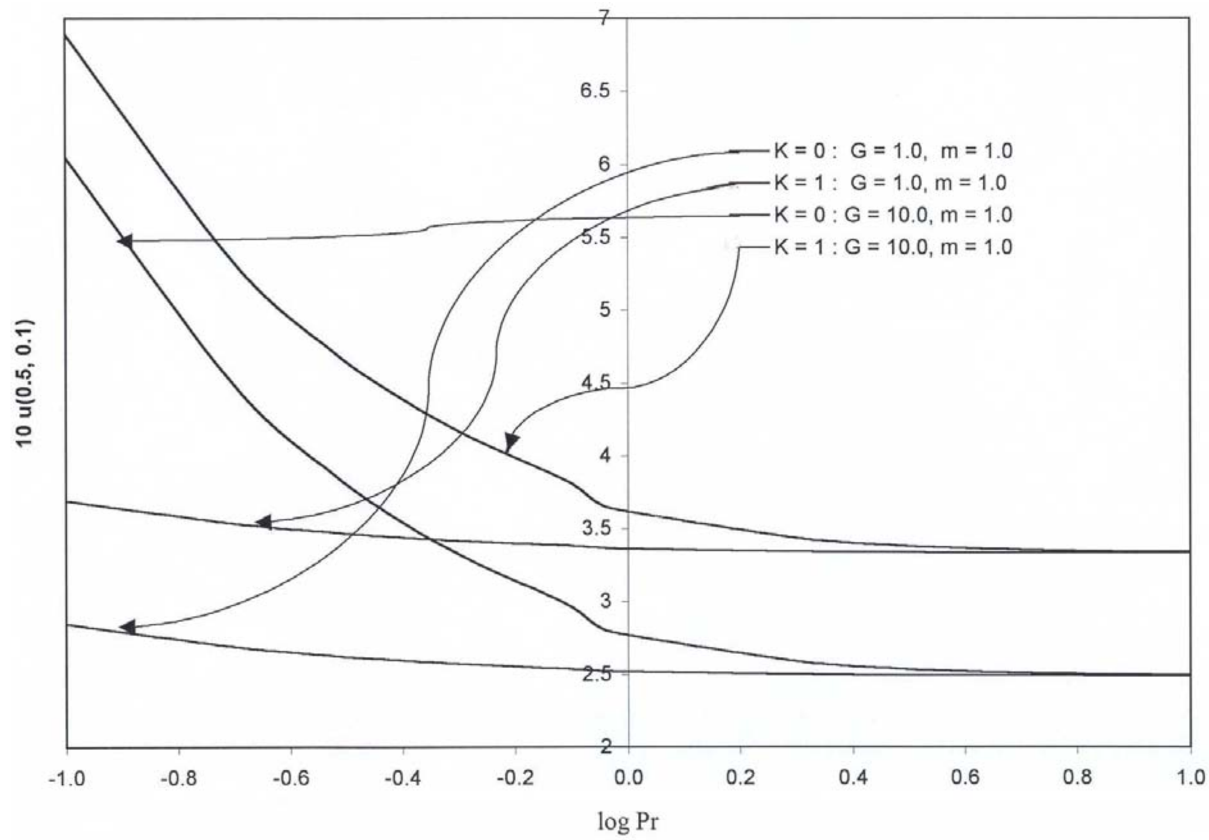


Figure 3: Variation of velocity with Prandtl number. $K = 0.1, y = 0.5$

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