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General control problems

ALMOST PERIODIC SOLUTIONS IN CONTROL SYSTEMS WITH MONOTONE NONLINEARITIES

Y.N. Kalinin and V. Reitmann[1]
Department of Applied Cybernetics,
Saint-Petersburg State University,
Saint-Petersburg

Abstract.

We investigate control systems as variational equations in non-standard chains of rigged Hilbert spaces. Monotonicity properties of nonlinearities are introduced with respect to such riggings generated by Lyapunov operators and invariant cones. Sufficient frequency domain conditions for boundedness and the existence of Bohr and Stepanov almost periodic solutions are derived. As an example we consider equations with Duffing-type nonlinearities and almost periodic forcing terms.

1 Introduction

Let us introduce some function spaces. Suppose $(E, \|\cdot\|_E)$ is a Banach space.

If $J \subset \mathbb{R}$ is an interval, denote by $C(J; E)$ the space of all continuous functions from J to E, endowed with the topology of uniform convergence on

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compact sets. If $G = \mathbb{R}$ or $G = \mathbb{R}_+$ the space $C_b(G;E)$ is the subspace of $C(G;E)$ of bounded functions equipped with the norm

$$\|f\|_{C_b} := \sup_{u\in G} \|f(u)\|_E \ .$$

The Banach space of *Stepanov bounded* on $J = \mathbb{R}$ or $J = \mathbb{R}_+$ *functions* (of exponent $p = 2$) is the space $BS^2(J;E)$ which consists of all functions $f \in L^2_{\mathrm{loc}}(J;E)$ having finite norm

$$\|f\|^2_{S^2} := \sup_{t\in J} \int_t^{t+1} \|f(\tau)\|^2_E \, d\tau \ .$$

A subset $\mathcal{S} \subset \mathbb{R}$ is *relatively dense* if there is a compact interval $\mathcal{K} \subset \mathbb{R}$ such that $(s+\mathcal{K}) \cap \mathcal{S} \neq \varnothing$ for all $s \in \mathbb{R}$. A function $f \in C_b(\mathbb{R}\,;E)$ is said to be *Bohr almost periodic* if for any $\varepsilon > 0$ the set

$$\{\tau \in \mathbb{R} \,|\, \sup_{s\in\mathbb{R}} \|f(s+\tau) - f(s)\|_E \leq \varepsilon\}$$

of ε-almost periods is relatively dense in $\mathbb{R}$.

For a function $f \in L^2_{\mathrm{loc}}(\mathbb{R}, E)$, put

$$f^b(t) := f(t+w), w \in [0,1], t \in \mathbb{R}.$$

Function $f^b(t)$ is regarded with values in the space $L^2(0,1;E)$. Then

$$BS^2(\mathbb{R};E) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R};E) | f^b \in L^\infty(\mathbb{R}; L^2(0,1;E))\}$$

and, moreover, $\|f\|_{S^2} = \|f^b\|_{L^\infty}$. A function $f \in BS^2(\mathbb{R};E)$ is called an *almost periodic function in the sense of Stepanov and of exponent 2* (abbreviated S^2-a.p.) if $f^b \in CAP(\mathbb{R}; L^2(0,1;E))$. In this case the ε -almost periods of f^b are called the ε-almost periods of f. The space of S^2-a.p. functions with values in E is denoted by $S^2(\mathbb{R};E)$. Obviously, $CAP(\mathbb{R};E) \subset S^2(\mathbb{R};E)$.

2 Control systems with monotone nonlinearities

Consider the Gelfand rigging of a real Hilbert space Y_0, i.e. a chain

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{1}$$

in which Y_1 ("positive" space) and Y_{-1} ("negative" space) are further real Hilbert spaces and the inclusions are dense and continuous. Let $(\cdot,\cdot)_i$ and

$\| \cdot \|_i, i = 1, 0, -1$, denote the scalar product and the norm in Y_i, respectively. Continuity of the inclusions means that there are constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|y\|_0 \le k_1 \|y\|_1 , \quad \forall\, y \in Y_1 \tag{2}$$

and

$$k_2 \|y\|_{-1} \le \|y\|_0 , \quad \forall\, y \in Y_0 . \tag{3}$$

Suppose that the rigging (1) – (3) is realized in the following sense ([1, 15]). Assume that from the inclusion chain (1) only $Y_1 \subset Y_0$ is given and (2) is satisfied, for simplicity, with $k_1 = 1$. We introduce on Y_0 a second norm by

$$\|y\|_{-1} := \sup_{0 \ne \eta \in Y_1} \frac{|(y, \eta)_0|}{\|\eta\|_1} \tag{4}$$

and denote by Y_{-1} the completion of Y_0 with respect to this norm. Then Y_{-1} can be taken as third space in the Gelfand rigging (1) (see [1, 15]). This space can be considered as dual to Y_1 with respect to Y_0, i.e. when the duality of Y_1 and Y_{-1} is written in terms of Y_0. Extending by continuity the function $(u, v)_0$ onto $Y_{-1} \times Y_1$, we get the pairing between Y_{-1} and Y_1, i.e. the bilinear form $(\cdot, \cdot)_{-1,1}$ on $Y_{-1} \times Y_1$ which coincides with $(\cdot, \cdot)_0$ on $Y_0 \times Y_1$ and which satisfies the inequality

$$|(h, y)_{-1,1}| \le \|h\|_{-1} \|y\|_1 , \quad \forall\, h \in Y_{-1},\ \forall\, y \in Y_1 . \tag{5}$$

With respect to the chain (1) we consider the three linear operators

$$A \in \mathcal{L}(Y_1, Y_{-1}) , \quad B \in \mathcal{L}(\mathbb{R}, Y_{-1}) , \quad C \in \mathcal{L}(Y_0, \mathbb{R}) . \tag{6}$$

Together with the operator $A \in \mathcal{L}(Y_1, Y_{-1})$ we also need the *adjoint with respect to* Y_0 *operator* $A^+ \in \mathcal{L}(Y_1, Y_{-1})$ which is given by the relation ([1])

$$(Ay, \eta)_{-1,1} = (A^+ \eta, y)_{-1,1} , \quad \forall\, y, \eta \in Y_1 . \tag{7}$$

If $A^+ = A$ the operator A is called *self-adjoint with respect to* Y_0*.* The adjointness with respect to Y_0 can be introduced similarly for linear operators acting between other spaces in the chain (1).

The construction of some auxilary evolutionary variational equation is based on the following function spaces which we shortly introduce.

If $-\infty \le T_1 < T_2 \le +\infty$ are two arbitrary numbers, we define the norm for Bochner measurable functions ([15]) in $L^2(T_1, T_2; Y_j)$, $\quad j = 1, 0, -1$, by

$$\|y\|_{2,j} := \left(\int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt \right)^{1/2} . \tag{8}$$

Let $\mathcal{W}(T_1, T_2)$ denote the space of functions y such that $y \in L^2(T_1, T_2; Y_1)$ and $\dot{y} \in L^2(T_1, T_2; Y_{-1})$ equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2)} := \left(\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2\right)^{1/2} . \tag{9}$$

By an embedding theorem ([10, 15]) one can assume that any function from $\mathcal{W}(T_1, T_2)$ belongs to $C(T_1, T_2; Y_0)$.

Throughout the paper we use the following assumptions about the operators A, B, C.

Denote for $-\infty \leq T_1 < T_2 \leq +\infty$ by $L^2(T_1, T_2; Y_j)$ with $j = 0, P$ and $j = -1, P$ the Bochner measurable functions for which the norm $\|\cdot\|_{2,j}$, defined by (8), is finite. Let $\mathcal{W}_P(T_1, T_2)$ be the space of functions such that

$$y \in L^2(T_1, T_2; Y_1) \text{ and } \dot{y} \in L^2(T_1, T_2; Y_{-1,P}) ,$$

equipped with the norm

$$\|y\|_{\mathcal{W}_P(T_1,T_2)} := (\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1,P}^2)^{1/2} .$$

(H1) For any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = Ay + f(t) , \quad y(0) = y_0 \tag{10}$$

is well-posed, i.e. for arbitrary $y_0 \in Y_0, f \in L^2(0, T; Y_{-1})$ there exists a unique solution $y \in \mathcal{W}(0, T)$ satisfying (10) in a variational sense and depending continuously on the initial data, i.e.

$$\|y(\cdot)\|_{\mathcal{W}(0,T)}^2 \leq k_3\|y_0\|_0^2 + k_4\|f(\cdot)\|_{2,-1}^2 , \tag{11}$$

where $k_3 > 0$ and $k_4 > 0$ are some constants.

(H2) The operator A is *Hurwitz*, i.e. any solution of

$$\dot{y} = Ay , \quad y(0) \in Y_0 , \tag{12}$$

is exponentially decreasing for $t \to +\infty$.

(H3) The operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular* ([8, 9]), i.e. for any $T > 0$, $y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_1)$ the solution of the direct problem (10) and the solution of the adjoint problem (understood in the above sense)

$$\dot{z} = -A^+z + f(t) , \quad z(T) = z_T , \tag{13}$$

are strongly continuous in t in the norm of Y_1.

(H4) The pair (A, B) is L^2-*controllable*, i.e. for arbitrary $y_0 \in Y_0$ there exists a control $\alpha(\cdot) \in L^2(0, \infty; \mathbb{R})$ such that the problem

$$\dot{y} = Ay + B\alpha \ , \quad y(0) = y_0 \tag{14}$$

is well-posed in the variational sense on $(0, +\infty)$.

Let us denote by H^c and L^c the complexification of a linear space H and a linear operator L, respectively, and introduce by

$$\chi(p) = C^c(A^c - p\, I^c)^{-1} B^c \ , \quad p \in \rho\,(A^c) \tag{15}$$

the *transfer operator function* of the triple (A^c, B^c, C^c).

(H5) There exist numbers $\kappa_0 > 0$ and $\epsilon > 0$ such that

$$\frac{1}{\kappa_0} + \operatorname{Re}\chi(i\omega) > \epsilon \ , \quad \forall\, \omega \in \mathbb{R} \ . \tag{16}$$

Theorem 1 *Assume for the linear operators* $A \in \mathcal{L}(Y_1, Y_{-1}), B \in \mathcal{L}(\mathbb{R}, Y_{-1})$ *and* $C \in \mathcal{L}(Y_0, \mathbb{R})$ *that the assumptions* **(H1)** – **(H5)** *are satisfied. Then there exists an operator* $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$*, self-adjoint and positive in* Y_0*, and a number* $\varepsilon > 0$ *such that*

$$(Ay + B\xi, Py)_{-1,1} + \xi(Cy - \xi\kappa_0^{-1}) \le -\varepsilon(\|y\|_1^2 + \xi^2), \quad \forall\,(y, \xi) \in Y_1 \times \mathbb{R} \ . \tag{17}$$

Proof. Consider in $Y_1 \times \mathbb{R}$ the quadratic form $F(y, \xi) = \xi(Cy - \xi\kappa_0^{-1})$ and their Hermitian extension $F^c(y, \xi) = \operatorname{Re}(\xi^* C^c y) - |\xi|^2\kappa_0^{-1}$ in $Y_1^c \times \mathbb{C}$. From the Likhtarnikov-Yakubovich theorem ([8]) it follows that under the conditions **(H1)**, **(H3)**, **(H4)** and the frequency-domain condition

$$\begin{aligned} &\operatorname{Re}(\xi^* C^c y) - |\xi|^2\kappa_0^{-1} < -\epsilon|\xi|^2 \ , \\ &\forall\, \xi \in \mathbb{C} \setminus \{0\} \quad \forall\, \omega \in \mathbb{R} \quad \forall\, y \in Y_1^c : i\omega y = A^c y + B^c \xi \ , \end{aligned} \tag{18}$$

there exists a number $\varepsilon > 0$ and an operator $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ self-adjoint in Y_0, such that (17) is satisfied. As it is easy to see, inequality (18) is equivalent to (16).

Let us show that $P \ge 0$. Introduce on Y_0 the Lyapunov functional $V(y) := (y, Py)_0$. Putting in (17) $\xi = 0$ we get the inequality

$$(Ay, Py)_{-1,1} \le -\varepsilon\|y\|_1^2 \ , \quad \forall\, y \in Y_1 \ . \tag{19}$$

Thus we have along an arbitrary solution $y(\cdot)$ of $\dot{y} = Ay$ with $y(0) = y_0 \in Y_0$ on an interval $[0, t]$ the inequality

$$V(y(t)) \leq V(y_0) - \varepsilon \int_0^t \|y(\tau)\|_1^2 d\tau \ . \tag{20}$$

From **(H2)** and (20) it follows for $t \to +\infty$ that

$$0 \leq V(y_0) - \varepsilon \int_0^\infty \|y(\tau)\|_1^2 d\tau \ .$$

But this implies that $V(y_0) > 0$ if $y_0 \neq 0$. ■

In the following we suppose the properties **(H1)** – **(H5)**. Thus we can assume that there exists an operator P and a number $\varepsilon > 0$ satisfying (17). Note that the operator P can be explicitly determined as solution of a Hamiltonian system of equations ([8]). The number $\varepsilon > 0$ can be estimated with the knowledge of ϵ. Our aim is to derive with the help of P a new Gelfand chain from (1) which is better adapted to the nonlinear system which will be investigated.

Consider in Y_0 the new scalar product $(\cdot, \cdot)_{0,P}$ given by

$$(y, \eta)_{0,P} := (y, P\eta)_0 \ , \quad \forall\, y, \eta \in Y_0 \ .$$

The associated norm is denoted by $\| \cdot \|_{0,P}$. The completion of Y_0 w.r.t. the scalar product $(\cdot)_{0,P}$ gives the Hilbert space $Y_{0,P}$. The space Y_1 is dense in $Y_{0,P}$ since Y_1 is dense in Y_0 and Y_0 is dense in $Y_{0,P}$. By (2) and the boundedness of P it follows that for all $y \in Y_1$

$$\|y\|_{0,P} = (y, Py)_0^{1/2} \leq \|P\|^{1/2} \|y\|_0 \leq \|P\|^{1/2} k_1 \|y\|_1 \ . \tag{21}$$

But this means that the inclusion $Y_1 \subset Y_{0,P}$ is continuous. Thus we can continue the inclusion $Y_1 \subset Y_{0,P}$ to a Gelfand rigged chain

$$Y_1 \subset Y_{0,P} \subset Y_{-1,P} \tag{22}$$

of Hilbert spaces. In order to define the negative space in this chain explicitly we introduce ([1]) on $Y_{0,P}$ the negative norm $\| \cdot \|_{-1,P}$ given on $Y_{0,P}$ by

$$\|y\|_{-1,P} := \sup_{0 \neq \eta \in Y_1} \frac{|(y, \eta)_{0,P}|}{\|\eta\|_1} \ . \tag{23}$$

The completion of $Y_{0,P}$ in this norm gives the negative space $Y_{-1,P}$ in the chain (22).

Let us denote the pairing between $Y_{-1,P}$ and Y_1 by $(\cdot,\cdot)_{-1,P;1}$. We extend by continuity the operators A, B and C from (6) to operators

$$A_P \in \mathcal{L}(Y_1, Y_{-1,P}) \ , \ B_P \in \mathcal{L}(\mathbb{R}, Y_{-1,P}) \ , \ C_P \in \mathcal{L}(Y_{0,P}, \mathbb{R}) \ . \tag{24}$$

Now we introduce a class of nonlinearities which will be considered in the sequel.

(H6) The function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi(t,0) = 0, \ \forall \, t \in \mathbb{R}$, and

$$0 \le (\varphi(t,w_1) - \varphi(t,w_2))(w_1 - w_2) \le \kappa_0 (w_1 - w_2)^2 \, , \quad \forall \, t \in \mathbb{R}, \, \forall \, w_1, w_2 \in \mathbb{R} \, , \tag{25}$$

where $\kappa_0 > 0$ is the constant from assumption **(H5)**.

Note that for $w_1 = w$ and $w_2 = 0$ we have from (25) and $\varphi(t,0) = 0$ the inequality

$$0 \le \varphi(t,w)w \le \kappa_0 w^2 \, , \quad \forall \, t \in \mathbb{R}, \, \forall \, w \in \mathbb{R} \, . \tag{26}$$

Let us consider the family of nonlinear operators $\mathcal{A}_P(t) : Y_1 \to Y_{-1,P}$, given by

$$\mathcal{A}_P(t)\eta := -A_P \eta - B_P \varphi(t, C_P \eta) \ , \quad \forall \, t \in \mathbb{R}, \, \forall \, \eta \in Y_1 \tag{27}$$

and the family $\mathcal{A}(t) : Y_1 \to Y_{-1}$ given by

$$\mathcal{A}(t)\eta := -A\eta - B\varphi(t, C\eta) \ , \quad \forall \, t \in \mathbb{R}, \, \forall \eta \in Y_1 \, . \tag{28}$$

Theorem 2 *Under the hypotheses* **(H1)** – **(H6)** *the operator family* $\{\mathcal{A}_P(t)\}_{t \in \mathbb{R}}$ *has the following properties:*

(P1) *For each* $t \in \mathbb{R}$ *the operator* $\mathcal{A}_P(t)$ *is* monotone*, i.e.,*

$$(\mathcal{A}_P(t)\eta - \mathcal{A}_P(t)\vartheta, \eta - \vartheta)_{-1,P;1} \ge 0 \ , \quad \forall \, \eta, \vartheta \in Y_1 \, ; \tag{29}$$

(P2) *For each* $t \in \mathbb{R}$ *the operator* $\mathcal{A}_P(t)$ *is* semicontinuous*, i.e., for any* $\eta, y, \vartheta \in Y_1$ *the scalar-valued function* $\xi \mapsto (\mathcal{A}_P(t)(\eta + \xi y, \vartheta)_{-1,P;1}$ *is continuous;*

(P3) *For any* $\vartheta \in Y_1$ *and any bounded set* $\mathcal{S} \subset Y_1$ *the family of functions* $\{(\mathcal{A}_P(t)\eta, \vartheta)_{-1,P;1} \,|\, \eta \in \mathcal{S}\}$ *is equicontinuous on any compact subinterval* $J \subset \mathbb{R}$*;*

(P4) *The family* $\{\mathcal{A}_P(t)\}_{t \in \mathbb{R}}$ *is uniformly bounded, i.e., there is a constant* $k_5 > 0$*, which is independent on* $t \in \mathbb{R}$*, such that*

$$\|\mathcal{A}_P(t)\eta\|_{-1,P} \le k_5 \|\eta\|_1 \ , \quad \forall \, \eta \in Y_1 \, ; \tag{30}$$

(P5) *There is a constant $k_6 > 0$, which does not depend on $t \in \mathbb{R}$, such that*

$$(\mathcal{A}_P(t)\eta, \eta)_{-1,P;1} \geq k_6 \|\eta\|_1^2 \,, \quad \forall\, \eta \in Y_1 \,; \tag{31}$$

(P6) *There is a constant $k_7 > 0$, which does not depend on $t \in \mathbb{R}$, such that*

$$(\mathcal{A}_P(t)\eta - \mathcal{A}_P(t)\vartheta, \eta - \vartheta)_{-1,P;1} \geq k_7 \|\eta - \vartheta\|_0^2 \,, \quad \forall\, \eta, \vartheta \in Y_1 \,. \tag{32}$$

Proof.

(P1): Inequality (29) is satisfied if for each $t \in \mathbb{R}$,

$$(A(\eta - \vartheta) + B[\varphi(t, C\eta) - \varphi(t, C\vartheta)]\,,\ P\,(\eta - \vartheta))_{-1,1} \leq 0\,, \quad \forall\, \eta, \vartheta \in Y_1 \,. \tag{33}$$

If we put in (17) $y = \eta - \vartheta$ and $\xi = \varphi(t, C\eta) - \varphi(t, C\vartheta)$ we receive the inequality

$$\begin{aligned} &(A(\eta - \vartheta) + B[\varphi(t, C\eta) - \varphi(t, C\vartheta)]\,,\ P\,(\eta - \vartheta))_{-1,1} \\ &+ (\varphi(t, C\eta) - \varphi(t, C\vartheta))\,((C\eta - C\vartheta) - (\varphi(t, C\eta) - \varphi(t, C\vartheta))\kappa_0^{-1}) \\ &\leq\ -\varepsilon(\|\eta - \vartheta\|_1^2 + (\varphi(t, C\eta) - \varphi(t, C\vartheta))^2)\,. \end{aligned} \tag{34}$$

From (25) it follows that the second term on the left-hand side of (34) is non-negative. Thus (34) implies (33).

(P2): This is equivalent to the property that for each $t \in \mathbb{R}$

$$\xi \mapsto (A(\eta + \xi y) + B\varphi(t, C(\eta + \xi y)),\ P\vartheta)_{-1,1}$$

is continuous. The last property is satisfied because of (6) the boundedness of P and the continuity of φ.

(P3): We have to show that for any $\vartheta \in Y_1$ and any bounded set $\mathcal{S} \subset Y_1$ the family of functions $\{(A\eta + B\varphi(t, C\eta),\ P\vartheta)_{-1,1} \,|\, \eta \in \mathcal{S}\}$ is equicontinuous on any compact subinterval $J \subset \mathbb{R}$. But this follows from (6), the boundedness of P, and the equicontinuity of $\varphi(\cdot, \cdot)$ on any compact set $\mathcal{K}_1 \times \mathcal{K}_2 \subset \mathbb{R} \times \mathbb{R}$.

(P4): This is true because the family $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is uniformly bounded, i.e., there is a constant k_5' such that

$$\|\mathcal{A}(t)\eta\|_{-1} \leq k_5' \|\eta\|_1 \,, \quad \forall\, \eta \in Y_1 \,. \tag{35}$$

Really, we have for each $\eta \in Y_1$ and $t \in \mathbb{R}$

$$\|\mathcal{A}(t)\eta\|_{-1} = \|A\eta + B\varphi(t, C\eta)\|_{-1} \leq \|A\|_{\mathcal{L}(Y_1, Y_{-1})} \|\eta\|_1 + \|B\|_{\mathcal{L}(\mathbb{R}, Y_{-1})} |\varphi(t, C\eta)| \,. \tag{36}$$

The value $|\varphi(t,C\eta)|$ in (36) can be estimated from (26) and (6) by

$$|\varphi(t,C\eta)| \le \kappa_0|C\eta| \le \kappa_0\|C\|_{\mathcal{L}(Y_0,\mathbb{R})}\|\eta\|_0\,. \qquad (37)$$

Using now (2), we see that (36) and (37) imply (35).

(P5): This property is shown if there is a constant $k_6' > 0$, which does not depend on t, such that

$$(A\eta + B\varphi(t,C\eta)\,,\ P\eta)_{-1,1} \le -k_6'\|\eta\|_1^2\,, \quad \forall\,\eta\in Y_1\,. \qquad (38)$$

In order to show (38) we put in (17) $y=\eta$ and $\xi=\varphi(t,C\eta)$ and receive the inequality

$$(A\eta + B\varphi(t,C\eta)\,,\ P\eta)_{-1,1} + \varphi(t,C\eta)(C\eta - \varphi(t,C\eta)\kappa_0^{-1}) \le -\varepsilon\|\eta\|_1^2\,. \qquad (39)$$

The second term in the left-hand side of (39) is non-negative because of (26). Thus (39) implies (38).

(P6): Again it is sufficient to show that there is a constant $k_7' > 0$, which does not depend on $t\in\mathbb{R}$, such that

$$(A(\eta-\vartheta) + B[\varphi(t,C\eta)-\varphi(t,C\vartheta)]\,,\ P(\eta-\vartheta))_{-1,1} \le -k_7'\|\eta-\vartheta\|_0^2\,, \quad \forall\,\eta,\vartheta\in Y_1\,. \qquad (40)$$

If we put $y=\eta-\vartheta$ and $\xi=\varphi(t,C\eta)-\varphi(t,C\vartheta)$ in (17) we get the inequality (34). Using (25) we receive from (34) the inequality

$$(A(\eta-\vartheta) + B[\varphi(t,C\eta)-\varphi(t,C\vartheta)]\,,\ P(\eta-\vartheta))_{-1,1} \le -\varepsilon|\eta-\vartheta\|_1^2\,, \quad \forall\,\eta,\vartheta\in Y_1,\ \forall\,t\in\mathbb{R}\,. \qquad (41)$$

From (2) we have the estimate

$$-\varepsilon\|\eta-\vartheta\|_1^2 \le -\frac{\varepsilon}{k_1}\|\eta-\vartheta\|_0^2\,, \quad \forall\,\eta,\vartheta\in Y_1\,. \qquad (42)$$

Clearly, that (41) and (42) imply (40). ∎

Let us consider w.r.t. the Gelfand triple $Y_1\subset Y_{0,P}\subset Y_{-1,P}$ on the interval $J\subset\mathbb{R}$ the equation

$$\dot y = A_P y + B_P\varphi(t,C_P y) + f(t)\,, \qquad (43)$$

where $f\in L^2_{\rm loc}(J;Y_{-1,P})\,.$

A solution of (43) is a function $y\in L^2_{\rm loc}(J;Y_1)\cap C(J;Y_{0,P})$ such that $\dot y\in L^2_{\rm loc}(J;Y_{-1,P})$ and (43) is satisfied in a variational sense, i.e. for a.a. $t\in J$

$$(\dot y(t) - A_P y(t) - B_P\varphi(t,C_P y(t)) - f(t)\,,\ \eta - y(t))_{-1,P;1} = 0\,, \quad \forall\,\eta\in Y_1\,. \qquad (44)$$

In this situation we have the following existence and uniqueness result ([3]).

Theorem 3 *Assume that the hypotheses* **(H1)** – **(H6)** *are satisfied. Then for any* $f \in L^2_{\rm loc}(\mathbb{R}_+; Y_{-1})$ *and any* $y_0 \in Y_{0,P}$ *there exists a unique solution* $y \in L^2_{\rm loc}(\mathbb{R}_+; Y_1) \cap C(\mathbb{R}_+; Y_{0,P})$ *of (44) such that* $y(0) = y_0$*. Moreover, we have for any* $T > 0$

$$\|y\|_{L^2(0,T;Y_1)} \le K_1(\|f\|_{L^2(0,T;Y_{-1,P})}, \|y_0\|_{0,P}) \tag{45}$$

and

$$\|y\|_{C([0,T];Y_{0,P})} \le K_2(\|f\|_{L^2(0,T;Y_{-1,P})}, \|y_0\|_{0,P}), \tag{46}$$

where $K_1(\cdot,\cdot)$ *and* $K_2(\cdot,\cdot)$ *are continuous non-decreasing to each variable functions.*

Proof. Since P is bounded and $f \in L^2_{\rm loc}(\mathbb{R}_+; Y_1)$ we have $f \in L^2_{\rm loc}(\mathbb{R}_+; Y_{-1,P})$. According to Theorem 2 we have for the family $\{\mathcal{A}_P(t)\}_{t\in\mathbb{R}_+}$ of operators (among others) the properties **(P1)**, **(P2)** and **(P5)**. Thus w.r.t. the rigging $Y_1 \subset Y_{0,P} \subset Y_{-1,P}$ the assumptions of the existence and uniqueness theorem from [3] are satisfied and the assertion follows immediately. ■

Lemma 1 *Assume that* **(H1)** – **(H6)** *are satisfied.*

a) Suppose that $y_i (i = 1, 2)$ *are two solutions of (44) with* $f = f_i \in L^2_{\rm loc}(J; Y_{-1})$ $(i = 1, 2)$*. Then for any* $s, t \in J, s \le t$*, the following estimate is valid:*

$$\frac{1}{2}\|y_1(\tau) - y_2(\tau)\|^2_{0,P}\Big|_s^t + \varepsilon \int_s^t \|y_1(\tau) - y_2(\tau)\|_1^2 \, d\tau$$
$$\le \int_s^t (f_1(\tau) - f_2(\tau),\ P(y_1(\tau) - y_2(\tau))_{-1,1} d\tau\,. \tag{47}$$

b) Suppose that $y_i (i = 1, 2)$ *are two solutions of (44) with common* $f_1 = f_2 = f$ *on* J*. Then for any* $t_0 \in J$ *and all* $t \ge t_0, t \in J$*, we have the inequality*

$$\|y_1(t) - y_2(t)\|_{0,P} \le e^{-k_9(t-t_0)} \|y_1(t_0) - y_2(t_0)\|_{0,P}\,, \tag{48}$$

where the constant $k_9 > 0$ *depends only on the constants* ε *from (17),* k_1 *from (2) and from the operator norm of* P*.*

Proof. a) For the difference $y_1 - y_2$ we derive from (44) for a.a. $\tau \in J$ the equation

$$\dot y_1(\tau) - \dot y_2(\tau) = A\,[y_1(\tau) - y_2(\tau)] + B\,[\varphi(\tau, Cy_1(\tau))$$
$$- \varphi(\tau, Cy_2(\tau))] + f_1(\tau) - f_2(\tau)\,. \tag{49}$$

Multiplying (49) with $y_1(\tau) - y_2(\tau)$ w.r.t. the scalar product $(\cdot,\cdot)_{0,P}$ and integrating over $[s,t]$, we receive the equation

$$\tfrac{1}{2}\|y_1(\tau) - y_2(\tau)\|^2_{0,P}\Big|_s^t =$$
$$\int_s^t (A[y_1(\tau) - y_2(\tau)] + B[\varphi(\tau, Cy_1(\tau)) - \varphi(\tau, Cy_2(\tau))], P(y_1(\tau) - y_2(\tau)))_{-1,1}\, d\tau$$
$$+ \int_s^t (f_1(\tau) - f_2(\tau),\, P(y_1(\tau) - y_2(\tau)))_{-1,1}\, d\tau\ . \tag{50}$$

If we put in (17) $y = y_1(\tau) - y_2(\tau), \xi = \varphi(\tau, Cy_1(\tau)) - \varphi(\tau, Cy_2(\tau))$, and integrate over $[s,t]$ we get the inequality

$$\int_s^t (A[y_1(\tau) - y_2(\tau)] + B[\varphi(\tau, Cy_1(\tau)) - \varphi(\tau, Cy_2(\tau))], P(y_1(\tau) - y_2(\tau)))_{-1,1} d\tau$$
$$+ \int_s^t [\varphi(\tau, Cy_1(\tau)) - \varphi(\tau, Cy_2(\tau))]\, (C[y_1(\tau) - y_2(\tau)]$$
$$- [\varphi(\tau, Cy_2(\tau)) - \varphi(\tau, Cy_2(\tau))]\kappa_0^{-1}) d\tau \le -\varepsilon \int_2^t \|y_1(\tau) - y_2(\tau)\|_1^2\, d\tau\ . \tag{51}$$

The second integral on the left-hand side of (51) is by **(H6)** non-negative. Thus we receive from (50) and (51) the inequality (47).

b) For $f_1 = f_2$ we get from (47) the inequality

$$\frac{1}{2}\|y_1(\tau) - y_2(\tau)\|^2_{0,P} + \varepsilon \int_s^t \|y_1(\tau) - y_2(\tau)\|_1^2\, d\tau \le 0\ . \tag{52}$$

From (1) it follows that on $[s,t]$

$$\varepsilon'\|y_1(\tau) - y_2(\tau)\|^2_{0,P} \le \varepsilon\, \|y_1(\tau) - y_2(\tau)\|_1^2\,, \tag{53}$$

where $\varepsilon' = \frac{\varepsilon}{\|P\|_{\mathcal{L}(Y_0,Y_0)k_1}}$. Thus with $m(\tau) := \frac{1}{2}\|y_1(\tau) - y_2(\tau)\|^2_{0,P}$ we get from (52) and (53) the estimate

$$m(\tau)\,|_s^t + 2\,\varepsilon' \int_s^t m(\tau)\, d\tau \le 0\,. \tag{54}$$

Now Gronwall's inequality gives the estimate (48) with $k_9 = \varepsilon'$. ■

Lemma 2 *Let* **(H1)** – **(H6)** *be satisfied and let $y_n \in L^2(J;Y_1) \cap C\,(J;Y_{0,P})$ be solutions of (44) with perturbations $f_n \in L^2(J;Y_{-1})$. Assume that $\lim_{n\to\infty} f_n = f$ in $L^2(J;Y_{-1}), \lim_{n\to\infty} y_n = y$ in $C(J;Y_{0,P})$. Then y is a solution of (44) with forcing function f.*

Proof. The result follows immediately from [11], Proposition 1.6, Ch. 3 and Lemma 1.7, Ch. 4 , if we note that $f_n \to f$ in $L^2(J; Y_{-1})$ implies that $f_n \to f$ in $L^2(J; Y_{-1,P})$. The last property results from the fact that for a.a. $t \in J$ we have

$$\|f - f_n\|^2_{-1,P} = \sup_{0 \neq \eta \in Y_1} \frac{|(f(t) - f_n(t), \eta)_{-1,P:1}|}{\|\eta\|_1}$$
$$\leq \sup_{0 \neq \eta \in Y_1} \frac{\|f(t) - f_n(t)\|_{-1,P} \|P\eta\|_1}{\|\eta\|_1} \leq \|f(t) - f_n(t)\|_{-1,P} \,.$$

■

Theorem 4 *Assume that the hypotheses* **(H1)** – **(H6)** *are satisfied. Then for any* $f \in BS^2(\mathbb{R}\,; Y_{-1})$ *there exists a solution* $y_* \in BS^2(\mathbb{R}\,; Y_1) \cap C_b(\mathbb{R}\,; Y_{0,P})$ *of equation (44) and such a solution is unique. Moreover, the solution* y_* *is exponentially stable in the whole in the norm of* $Y_{0,P}$*, i.e. there exist numbers* $k_{10} > 0, k_{11} > 0$ *such that for any other solution* y *of (44) on* $[t_0, \infty)$ *and any* $t \geq t_0$ *we have*

$$\|y(t) - y_*(t)\|_{0,P} \leq k_{10}\, e^{-k_{11}(t-t_0)} \|y(t_0) - y_*(t_0)\|_{0,P} \tag{55}$$

Proof. To prove the existence of at least one solution y_* on $\mathbb{R}$ we consider as in [11] sequences $\{y_n\}$ of solutions. Define a solution $y_n \in L^2_{\text{loc}}([-n, \infty); Y_1) \cap C([-n, +\infty); Y_{0,P})$ of equation (44) such that $y_n(-n) = 0$. By Theorem 1 such a solution is uniquely defined. Put

$$f_n(t) := \begin{cases} f(t) & , \quad t \geq -n \\ 0 & , \quad t < n\,, \end{cases}$$

and extend y_n by zero to the whole $\mathbb{R}$. Then y_n is a solution of (44) with forcing term f_n. Estimate (47) gives on $[s, t]$ the inequality

$$\frac{1}{2}\, \|y_n(\tau)\|^2_{0,P} \Big|_s^t + \varepsilon \int_s^t \|y_n\|_1^2\, d\tau \leq \int_s^t (f_n, y_n)_{-1,P;1}\, d\tau \,. \tag{56}$$

From (56) and Lemma 1.1, Ch. 2 in [11] it follows that

$$y_n \in BS^2(\mathbb{R}\,; Y_1) \cap C_b(\mathbb{R}\,; Y_{0,P})\,,$$

and the sequence $\{y_n\}$ is bounded in this space, i.e. there is a constant $k_{12} > 0$ such that for $n = 1, 2, \ldots$.

$$\|y_n\|_{C_b(\mathbb{R}\,; Y_{0,P})} \leq k_{12} \tag{57}$$

and

$$\|y_n\|_{S^2} \le k_{12}\ . \tag{58}$$

In order to prove the existence of a solution it is by Lemma 2 sufficient to establish the existence of the strong limit $\lim_{n\to\infty} y_n = y_*$ in $C(\mathbb{R}\,; Y_{0,P})$.

From Lemma 1 we have for all $m, n \in \mathbb{N}$ the inequality

$$\frac{1}{2}\|y_n - y_m\|_{0,P}^2\,\Big|_s^t + \varepsilon \int_s^t \|y_n - y_m\|_1^2 d\tau \le \int_s^t (f_n - f_m, y_n - y_m)_{-1,P;1}\, d\tau\ . \tag{59}$$

Since $f_n(\tau) = f_m(\tau) = f(\tau)$ for $\tau \ge -\min\{m,n\}$ by using Lemma 1.3, Ch. 2, of [11] we obtain from (59) that $\{y_n\}$ is a Cauchy sequence in $C(\mathbb{R}\,; Y_{0,P})$.

By (57) and (58) its limit $y_* = \lim_{n\to\infty} y_n$ lies in $BS^2(\mathbb{R}\,; Y_1) \cap C_b(\mathbb{R}\,; Y_{0,P})$ and so the existence of a solution is proved.

Let us show the uniqueness. Suppose that there are two solutions, y_1 and y_2, on $\mathbb{R}$, such that $y_1(t_0) \neq y_2(t_0)$. Then from (48) it follows that $\lim_{t\to-\infty} \|y_1(t) - y_2(t)\|_{0,P} = +\infty$. But this contradicts the boundedness of y_1 and y_2. The exponential stability in the whole characterized by the estimate (55) follows from the forward estimate (48) stated in Lemma (1). ■

The following lemma is a slight modification of Theorem 3.1, Ch. 3 and Lemma 1.11, Ch. 4 from [11]. The proof is omitted.

Lemma 3 *Under the assumptions* **(H1)** – **(H6)** *the operator*

$$F_0 : BS^2(\mathbb{R}\,; Y_{-1}) \to C\,(\mathbb{R}\,; Y_{0,P})$$

is continuous and the operator

$$F_g : BS^2(\mathbb{R}\,; Y_{-1}) \to C_b(\mathbb{R}\,; Y_{0,P})$$

with $g \in BS^2(\mathbb{R}\,; Y_{-1})$ is locally Hölderian with exponent 1.

(H7) *For any bounded set $\mathcal{S} \subset \mathbb{R}$ the family of functions $\{\varphi(\cdot, w)\,|\, w \in \mathcal{S}\}$ is uniformly almost periodic.*

Theorem 5 *Suppose that the assumptions* **(H1)** – **(H7)** *are satisfied. If $f \in S^2(\mathbb{R}\,; Y_{-1})$ then the unique bounded solution y_* of (44) belongs to $CAP(\mathbb{R}\,; Y_{0,P}) \cap S^2(\mathbb{R}\,; Y_1)$.*

Proof. The operator-valued function $t \mapsto \mathcal{A}_P(t)$ with $\mathcal{A}_P(t)y = -A_P y - B_P\varphi(t, C_P y), y \in Y_1$, is extended to $\mathbb{R}_B$ as a function continuous in the $d_{Y_1,2}$-

metric. The function $f(\cdot+s)$ is Bohr a. p. in s and hence can also be extendend to a continuous function on $\mathbb{R}_B$ with values in $BS^2(\mathbb{R}\,;Y_{-1,P})$.

Now consider the family of equations depending on the parameter $q \in \mathbb{R}_B$

$$\frac{d}{dt}\,y_q(t) = A_P y_q(t) + B_P\varphi(t+q, C_P y_q(t)) + f(t+q). \tag{60}$$

Theorem 2 is applicable to any of these equations. Therefore, for any $q \in \mathbb{R}_B$ there is a unique solution $y_q \in BS^2(\mathbb{R}\,;Y_1)\cap C_b(\mathbb{R}\,;Y_{0,P})$. By uniqueness we have

$$y_q(\cdot) = y(\cdot+q)\,, \quad \forall\, q \in \mathbb{R} \subset \mathbb{R}_B\,. \tag{61}$$

Let F_q be the inverse of the operator corresponding to equation (60). Then

$$y_q = F_{q_0}[f(\cdot+q) - B_P\varphi(\cdot+q_0, C_P y_q) + B_P\varphi(\cdot+q, C_P y_q)]\,. \tag{62}$$

By Lemma 3 the set $\{y_q\}$ is bounded in the space $BS^2(\mathbb{R}\,;Y_1)$. Hence, by **(H7)**,

$$\lim[B_P(\cdot+q_0)y_q - B_P(\cdot+q)y_q] = 0 \tag{63}$$

in $B^2(\mathbb{R}\,;Y_{-1,P})$.

By Lemma 3 $F_{q_0} : BS^2(\mathbb{R}\,;Y_1) \to C_b(\mathbb{R}\,;Y_{0,P})$ is continuous. This, the representation (62) and (63) imply that y_s is continuous in $q \in \mathbb{R}_B$ as an element of $C(\mathbb{R}\,;Y_{0,P})$. If we put $y(q) := y_q(0), q \in \mathbb{R}_B$, we obtain the continuous extension of $y(t)$ to $\mathbb{R}_B$. Consequently, $y \in CAP(\mathbb{R}\,;Y_{0,P})$. In order to prove the second inclusion we use the second part of Lemma 3. Then $y_q \in BS^2(\mathbb{R}\,;Y_1)$ depends continuously on $q \in \mathbb{R}_B$. This together with (61) gives $y \in S^2(\mathbb{R}\,;Y_1)$. ■

3 Control systems in Lur'e form with a Duffing type nonlinearity

Let $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ be a Gelfand rigging of the real Hilbert space $\mathcal{V}_0$, i.e. a chain of Hilbert spaces with dense and continuous inclusions. Denote by $(\cdot,\cdot)_{\mathcal{V}_j}$ and $\|\cdot\|_{\mathcal{V}_j}, j = 1, 0, -1$, the scalar product resp. norm in $\mathcal{V}_j (j = 1, 0, -1)$ and by $(\cdot,\cdot)_{\mathcal{V}_{-1},\mathcal{V}_1}$ the pairing between $\mathcal{V}_{-1}$ and $\mathcal{V}_1$. Let $A_0 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$ be a linear operator, $b_0 \in \mathcal{V}_{-1}$ a generalized vector, $c_0 \in \mathcal{V}_0$ a vector and $d_0 < 0$ a number. According to the vectors c_0 and b_0 we introduce the linear operators $C_0 \in \mathcal{L}(\mathcal{V}_0, \mathbb{R})$ and $B_0 \in \mathcal{L}(\mathbb{R}, \mathcal{V}_{-1})$ by $C_0\nu = (c_0, \nu)_{\mathcal{V}_0}$, $\forall\, \nu \in \mathcal{V}_0$, and
$B_0\xi := \xi b_0$, $\forall \xi \in \mathbb{R}$.

Assume that $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two scalar-valued functions. Our aim is to study a system of indirect control, which is formally given as

$$\begin{aligned} \dot{\nu} &= A_0\nu + b_0[\phi(t,w) + g(t)]\,, \\ \dot{w} &= (c_0,\nu)_{\mathcal{V}_0} + d_0[\phi(t,w) + g(t)]\,. \end{aligned} \tag{64}$$

Let us demonstrate how (64) can be written as a standard control system. Consider for this the Gelfand rigging $Z_1 \subset Z_0 \subset Z_{-1}$, in which $Z_j := \mathcal{V}_j \times \mathbb{R}$, $j = 1, 0, -1$. The scalar product $(\cdot,\cdot)_{Z_j}$ in Z_j is introduced as $\big((\nu_1,w_1),(\nu_2,w_2)\big)_{Z_j} := (\nu_1,\nu_2)_{\mathcal{V}_j} + w_1w_2$, where $(\nu_1,w_1),(\nu_2,w_2) \in Z_j$ are arbitrary. The pairing between Z_{-1} and Z_1 is defined for $(h,\xi)\times\mathcal{V}_{-1}\times\mathbb{R} = Z_{-1}$ and $(\nu,\varsigma) \in \mathcal{V}_1 \times \mathbb{R} = Z_1$ through

$$((h,\xi),(\nu,\varsigma))_{Z_{-1},Z_1} := (h,\nu)_{\mathcal{V}_{-1},\mathcal{V}_1} + \xi\,\varsigma\,.$$

Let $\hat{b} := \begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \in Z_{-1}$ and $\hat{c} := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in Z_0$. Suppose further that the operators $\hat{C} \in \mathcal{L}(Z_0,\mathbb{R})$ and $\hat{B} \in \mathcal{L}(\mathbb{R}, Z_{-1})$ are given as

$$\hat{C}z = (\hat{c},z)_{Z_0}\,, \quad \forall\, z \in Z_0\,, \qquad \hat{B}\xi = \xi\hat{b}\,, \quad \forall\, \xi \in \mathbb{R}\,,$$

and the operator $\hat{A} \in \mathcal{L}(Z_1, Z_{-1})$ is defined as

$$\hat{A} := \begin{bmatrix} A_0 & 0 \\ C_0 & 0 \end{bmatrix}.$$

Consider now the system

$$\dot{z} = \hat{A}z + \hat{B}\,[\phi(t,w) + g(t)]\,,\ w = \hat{C}z\,, \tag{65}$$

which is equivalent to (64) through $z = (\nu,w)$. If $-\infty \le T_1 < T_2 \le +\infty$ are arbitrary, we define the norm for Bochner measurable functions in $L^2(T_1,T_2;Z_j)$, $j = 1, 0, -1$, by

$$\|z\|_{2,j} := \left(\int_{T_1}^{T_2} \|z(t)\|^2_{Z_j} dt\right)^{1/2}. \tag{66}$$

Let $\mathcal{W}(T_1,T_2;Z_1,Z_{-1})$ be the space of functions z such that $z \in L^2(T_1,T_2;Z_1)$ and $\dot{z} \in L^2(T_1,T_2;Z_{-1})$, equipped with the norm

$$\|z\|_{\mathcal{W}(T_1,T_2;Z_1,Z_{-1})} := \left(\|z\|^2_{2,-1} + \|\dot{z}\|^2_{2,-1}\right)^{1/2}. \tag{67}$$

Let us introduce the following assumptions **(A1)** – **(A6)** about the operator $A_0 \in \mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$, the vectors $b_0 \in \mathcal{V}_{-1}$ and $c_0 \in \mathcal{V}_0$, and the functions ϕ and g.

Note that **(A1)** – **(A6)** for (64) are related to the assumptions **(H1)** – **(H6)** for (65).

(A1) For any $T > 0$ and any $f = (f_1, f_2) \in L^2(0, T; \mathcal{V}_{-1} \times \mathbb{R})$ the problem

$$\dot{\nu} = A_0\nu + f_1(t)\,, \qquad (68)$$
$$\dot{w} = (c_0, \nu)_{\mathcal{V}_0} + f_2(t)\ , \quad (\nu(0), w(0)) = (\nu_0, w_0)$$

is well-posed, i.e. for arbitrary $(\nu_0, w_0) \in Z_0, (f_1, f_2) \in L^2(0, T; \mathcal{V}_{-1} \times \mathbb{R})$ there exists a unique solution $(\nu, w) \in \mathcal{W}(0, T; Z_1, Z_{-1})$ satisfying (68) in a variational sense and depending continuously on the initial data, i.e.

$$\|(\nu, w)\|^2_{\mathcal{W}(0,T;Z_1,Z_{-1})} \le k_{13}\|(\nu_0, w_0)\|^2_{\mathcal{V}_0\times\mathbb{R}} + k_{14}\|(f_1, f_2)\|^2_{2,-1}\ , \qquad (69)$$

where $k_{13} > 0$ and $k_{14} > 0$ are some constants .

(A2) There is a $\lambda > 0$ such that $A_0 + \lambda I$ is a Hurwitz operator .

(A3) For any $T > 0, (\nu_0, w_0) \in Z_1 \times \mathbb{R}, (\tilde{\nu}_0, \tilde{w}_0) \in Z_1 \times \mathbb{R}$ and $(f_1, f_2) \in L^2(0, T; \mathcal{V}_1 \times \mathbb{R})$ the solution of the direct problem (68) and the solution of the adjoint problem

$$\dot{\tilde{\nu}} = -(A_0^+ + \lambda\, I)\tilde{\nu} + f_1(t),$$
$$\dot{\tilde{w}} = -C_0^+\tilde{w} - \lambda\,\tilde{w} + f_2(t), \qquad (70)$$

are strongly continuous in t in the norm of $\mathcal{V}_1 \times \mathbb{R}$.

(A4) The pair (A_0, b_0) is L^2-controllable, i.e. for arbitrary $\nu_0 \in \mathcal{V}_0$ there exists a control $\alpha\,(\cdot) \in L^2(0, \infty; \mathbb{R})$ such that the problem

$$\dot{\nu} = A_0\nu + b_0\alpha\ , \quad \nu(0) = \nu_0$$

is well-posed in the variational sense on $(0, \infty)$.

Introduce by

$$\chi(p) = \left(c_0^c, (A_0^c - pI^c)^{-1}\, b_0^c\right)_{Z_0}\ , \quad p \in \rho(A_0^c)$$

the transfer function of the triple (A_0^c, b_0^c, c_0^c) .

(A5) Suppose $\lambda > 0$ and $\kappa_1 > 0$ are parameters, where λ is from **(A2)**. Then:

a) $\quad \lambda d_0 + \mathrm{Re}\,(-i\omega - \lambda)\chi(i\omega - \lambda) + \kappa_1\,|\,\chi(i\omega - \lambda) - d_0\,|^2 \le 0, \quad \forall\,\omega \ge 0\,. \quad (71)$

(A6) The function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\phi(t,0) = 0,\ \forall\, t \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ belongs to $L^2_{\rm loc}(\mathbb{R}\,;\mathbb{R})$. There are numbers $\kappa_1 > 0$ (from **(A5)**, $0 \le \kappa_2 < \kappa_3 < +\infty, q_1 < q_2$ and $\zeta_2 < \zeta_1$ such that:

$$a) \qquad q_1 < g(t) < q_2\,, \tag{72}$$

for a.a. t from an arbitrary compact time interval;

$$b) \qquad (\phi(t,w) + q_i)(w - \zeta_i) \le \kappa_1 (w - \zeta_i)^2, \qquad i = 1,2 \tag{73}$$
$$\forall\, t \in \mathbb{R},\ \forall\, w \in [\zeta_2, \zeta_1]\,;$$

$$c) \qquad \kappa_2 (w_1 - w_2)^2 \le (\phi(t,w_1) - \phi(t,w_2))(w_1 - w_2) \le \kappa_3 (w_1 - w_2)^2\,, \tag{74}$$
$$\forall\, t \in \mathbb{R},\ \forall\, w_1, w_2 \in [\zeta_2, \zeta_1]\,.$$

We assume in the next theorem that the solutions of (2) are for every $T > 0$ elements of the space $\mathcal{W}(0,T; Z_1, Z_{-1})$. Then we show the existence of solutions with initial states from a certain set.

Theorem 6 *Assume that for system (64) the hypotheses* **(A1)** – **(A7)** *are satisfied. Then there exists a closed, positively invariant and convex set* $\mathcal{G}$ *such that*

$$\{(\nu, w) \in \mathcal{V}_1 \times \mathbb{R} \,|\, \nu = 0, w \in [\zeta_2, \zeta_1]\} \subset \mathcal{G} \subset \{(\nu, w) \in \mathcal{V}_1 \times \mathbb{R} \,|\, w \in [\zeta_2, \zeta_1]\}\,. \tag{75}$$

In order to prove this theorem we need some auxilary results.

Suppose that $Y_1 \subset Y_0 \subset Y_{-1}$ is a Gelfand rigging of $Y_0, \|\cdot\|_j, (\cdot,\cdot)_j$ are the corresponding norms and scalar products, respectively, and $(\cdot,\cdot)_{-1,1}$ is the pairing between Y_{-1} and Y_1. Consider the linear system

$$\dot y = Ay\,, \quad w = (c, y)_0\,, \tag{76}$$

where $A \in \mathcal{L}(Y_1, Y_{-1})$ and $c \in Y_0$.

Assume that for each $y_0 \in Y_0$ there exists a unique solution $y(\cdot, y_0)$ of (76) in $\mathcal{W}(0,\infty)$ satisfying $y(0,y_0) = y_0$. In the sequel we need the following assumption.

(A7) The space Y_0 can be decomposed as $Y_0 = Y_0^+ \oplus Y_0^-$ such that the following holds:

a) For each $y_0 \in Y_0^+$ we have $\lim\limits_{t\to\infty} y(t, y_0) = 0$. For each $y_0 \in Y_0^-$ there exists a unique solution $y_-(t) = y(t, y_0)$ of (76), defined on $(-\infty, 0)$, such that $\lim\limits_{t\to-\infty} y_-(t) = 0$ and $(c, y(t, y_0))_0 = 0, \ \forall\, t \geq 0$, if and only if $y_0 = 0$.

b) For each $y_0 \in Y_0^+$ the equality $(c, y(t, y_0))_0 = 0, \ \forall\, t \leq 0$, holds if and only if $y_0 = 0$. For each $y_0 \in Y_0^-$ the equality $(c, y(t, y_0))_0 = 0, \ \forall\, t \leq 0$, holds if and only if $y_0 = 0$.

Remark 1 Assumption **(A7)** a) means that we assume for the linear system (76) the decomposition of Y_0 in $y = 0$ into a stable subspace $E^s \equiv Y_0^+$ and an unstable subspace $E^u \equiv Y_0^-$. Assumption **(A7)** b) characterizes the identifiability in the sense of Kalman of the pair (A, c) on Y_0^+ and Y_0^-, respectively.

In the following $L \geq 0$ for a linear operator $L \in \mathcal{L}(Z), Z$ a Hilbert space, means that L is *positive*, i.e. $(z, Lz)_Z > 0, \ \forall\, z \in Z\backslash\{0\}; \quad L \leq 0$ means that $-L$ is positive.

Lemma 4 *Suppose that system (76) satisfies* **(A7)** *and there exists a linear continuous operator* $P : Y_0 \to Y_0, P^* = P$, *such that for any* $s \leq t$ *and any solution* $y(\cdot, y_0)$ *of (76) we have with* $V(y) := (y, Py)_0, y \in Y_0$,

$$V(y(t, y_0)) - V(y(s, y_0)) \leq -\int_s^t (c, y(\tau, y_0))_0^2 \, d\tau \,. \tag{77}$$

$$\textit{Then} \quad P_{|Y_0^+} \geq 0 \,, \ \textit{i.e.}, \ (y, Py)_0 > 0 \quad \textit{for all} \quad y \in Y_0^+\backslash\{0\} \tag{78}$$

$$\textit{and} \quad P_{|Y_0^-} \leq 0 \,, \ \textit{i.e.}, \ (y, Py)_0 < 0 \quad \textit{for all} \quad y \in Y_0^-\backslash\{0\} \,. \tag{79}$$

Proof. Let $y_0 \in Y_0^+\backslash\{0\}$. Then by **(A7)** a) we have $\lim_{t\to\infty} y(t, y_0) = 0$ and, due to the boundedness of P, $\lim_{t\to\infty} V(y(t, y_0)) = 0$. It follows from (77) for $s = 0$ and $t \to \infty$ that

$$-V(y_0) \leq -\int_0^\infty (c, y(\tau, y_0))_0^2 \, d\tau \,. \tag{80}$$

Using again **(A7)** a), we conclude from (80) that

$$V(y_0) \geq \int_0^\infty (c, y(\tau, y_0))_0^2 \, d\tau > 0 \,.$$

Thus (78) is shown.

Let now $y_0 \in Y_0^- \backslash \{0\}$. Then by **(A7)** b) we have $\lim_{t\to-\infty} y(t, y_0) = 0$ and, consequently, $\lim_{t\to-\infty} V(y(t, y_0)) = 0$. If we take in (77) $s \to -\infty$ and $t \to 0$, we receive

$$V(y_0) \leq -\int_{-\infty}^{0} (c, y(\tau, y_0))_0^2 \, d\tau \,. \tag{81}$$

Assumption **(A1)** b) implies that $\int_{-\infty}^{0} (c, y(\tau, y_0))_0^2 \, d\tau > 0$. Thus we conclude from (81) that $V(y_0) < 0$. This proves (79). ■

The next lemma is concerned with the separation of quadratic cones by special functionals. Let us recall some definitions. Assume that H is a Hilbert space with scalar product $(\cdot, \cdot)$. A *cone* in H is a set $\mathcal{C} \subset H, \mathcal{C} \neq \varnothing$, such that $u \in \mathcal{C}, \zeta \in \mathbb{R}_+$ imply that $\zeta u \in \mathcal{C}$. It is easy to see that a cone $\mathcal{C}$ in H is convex if and only if $u, v \in \mathcal{C}$ imply that $u + v \in \mathcal{C}$.

Suppose that $P \in \mathcal{L}(H), P = P^*$. Then the set $\mathcal{C} := \{u \in H \,|\, (u, Pu) \leq 0\}$ is a cone which is called by us *quadratic.*

Assume that there is a decomposition $H = H^+ \oplus H^-$ such that $P_{|H^+} \geq 0$ and $P_{|H^-} \leq 0$. Then the quadratic cone $\{u \in H \,|\, (u, Pu) \leq 0\}$ is called by us *quadratic cone of dimension* $\dim H^-$.

Lemma 5 *Suppose that:*

1) $Y_1 \subset Y_0 \subset Y_{-1}$ *is a Gelfand rigging of the Hilbert space* Y_0 *with scalar products* $(\cdot, \cdot)_i$, *corresponding norms* $\|\cdot\|_i, i = 1, 0, -1,$ *and pairing* $(\cdot, \cdot)_{-1}$, *between* Y_{-1} *and* Y_1*;*
2) *There is an operator* $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, *self-adjoint and positive in* Y_0 *such that*

$$\mathcal{C} := \{y \in Y_0 \,|\, (y, Py)_0 \leq 0\} \quad \textit{is an } 1\textit{-dimensional quadratic cone;}$$

3) *There are vectors* $h \in Y_{-1}$ *and* $r \in Y_0$ *such that*

$$2\,(h, Py)_{-1,1} = (r, y)_0 \,, \ \forall\, y \in Y_1 \tag{82}$$

and
$$(h, r)_{-1,1} < 0 \,. \tag{83}$$

Then we have

$$\operatorname{int} \mathcal{C} \cap \{y \in Y_1 \,|\, (y, r)_0 = 0\} = \varnothing \,. \tag{84}$$

Proof. Suppose that (83) is not true, i.e., assume that there is a $y_0 \in Y_1$, $y_0 \neq 0$, such that

$$(y_0, Py_0)_0 < 0 \quad \text{and} \quad (y_0, r)_0 = 0\,. \tag{85}$$

Since $\mathcal{C}$ is a cone, we have $\xi y_0 \in \mathcal{C}$, $\forall\, \xi \in \mathbb{R}$, and

$$\operatorname{span}\{y_0\}\backslash\{0\} \subset \operatorname{int}\mathcal{C}\,. \tag{86}$$

Since the inclusions $Y_1 \subset Y_0 \subset Y_{-1}$ are dense, there exists a sequence $\{h_n\}_{n=1}^{\infty}$, $h_n \in Y_1\,(n = 1, 2, \ldots)$ such that $h_n \to h$ for $n \to \infty$ in the norm of Y_{-1}.

Because of (82) we have

$$2\,(h_n, Ph_n)_0 \to (r, h_n)_0 \quad \text{for} \quad n \to \infty\,. \tag{87}$$

Since $(\cdot,\cdot)_{-1,1}$ is the unique extension by continuity of the scalar product $(\cdot,\cdot)_0$ defined on $Y_0 \times Y_1$, it follows from (83) that there are numbers $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$(r, h_n)_0 \leq -\varepsilon_0 < 0\,, \quad \forall\, n \geq n_0\,. \tag{88}$$

Thus for each $\varepsilon_1 \in (0, \varepsilon_0)$ there is an $n_1 \in \mathbb{N}$ such that

$$4\,(h_n, Ph_n)_0 \leq -\varepsilon_1'\,, \quad \forall\, n \geq n_1\,, \tag{89}$$

where $\varepsilon_1' := \nu - \varepsilon_1$.

From (82) we conclude that $2\,(h_n, Py_0)_0 \to (r, y_0)_0 = 0$ for $n \to \infty$. Thus we have for each $\varepsilon_2 > 0$ a number $n_2 \in \mathbb{N}$ such that

$$2\,|\,(h_n, Py_0)_0\,| < \varepsilon_2\,, \quad \forall\, n \geq n_2\,. \tag{90}$$

Take now $\bar{n} := \max\{n_0, n_1, n_2\}$. Then the properties (87) – (90) are satisfied for $n \geq \bar{n}$. By (85) and the inequality $(\varepsilon_0 - \varepsilon_1) > 0$, we can choose the number ε_2 in (90) so small that

$$-(y_0, Py_0)_0\,(\varepsilon_0 - \varepsilon_1) - \varepsilon_2^2 > 0\,. \tag{91}$$

Let us show now that the plane $\Pi := \{\xi y_0 + \varsigma\, 2\, h_{\bar{n}} \,|\, \xi, \varsigma \in \mathbb{R}\}$, with exception of the point 0, is contained in $\operatorname{int}\mathcal{C}$. This will be a contradiction to assumption 2) of the theorem if we show that $\dim \Pi = 2$. Suppose that this is not the case. This means that there is a $\xi_0 \neq 0$ such that

$$\xi_0 y_0 = h_{\bar{n}}\,. \tag{92}$$

It follows from (88) and (92) that $(r, h_{\bar{n}})_0 < 0$, and from (85) and (93) that $(r, h_{\bar{n}})_0 = 0$. This contradiction shows that $\dim \Pi = 2$. It remains to demonstrate that $\Pi \setminus \{0\} \subset \operatorname{int} \mathcal{C}$. Consider for arbitrary $\xi, \varsigma \in \mathbb{R}$ with $\xi^2 + \varsigma^2 > 0$ the expression

$$\begin{aligned} &\big(\xi y_0 + \varsigma\, 2\, h_{\bar{n}},\ P\,(\xi y_0 + \varsigma\, 2\, h_{\bar{n}})\big)_0 \\ &= \xi^2 (y_0, P y_0)_0 + 4\, \xi \varsigma\, (h_{\bar{n}}, P y_0)_0 + \varsigma^2 4 (h_{\bar{n}}, P h_{\bar{n}})_0 \,. \end{aligned} \tag{93}$$

Under our conditions the quadratic form (93) is negative definite. Really, from (85) we have $(y_0, Py_0)_0 < 0$ and from (89) $4\,(h_{\bar{n}}, Ph_{\bar{n}})_0 < 0$. Thus by the Routh criterion the negative definiteness of the form is shown if the determinant D, associated to this form, is positive. The straight forward computation of D and the use of (89) – (91) gives the estimates

$$\begin{aligned} D &= (y_0, P\, y_0)_0\, 4\, (h_{\bar{n}}, P h_{\bar{n}})_0 - (4\, h_{\bar{n}}, P y_0)_0^2 \\ &\geq -(y_0, P y_0)_0 (\varepsilon_0 - \varepsilon_1) - \varepsilon_2^2 > 0 \,. \end{aligned}$$

■

Remark 2 Lemma 5 can be considered as generalized lemma about the separation of cones ([2, 5, 7, 13]). Really, in the finite-dimensional case we have $Y_1 = Y_0 = Y_{-1} = \mathbb{R}^n$, $(\cdot,\cdot)_{-1,1} = (\cdot,\cdot)_0 = (\cdot,\cdot)$ the Euclidean inner product and $P = P^*$, $\det P \neq 0$, a regular symmetric $n \times n$ matrix. Assumption (82) in Lemma 5 states that there are vectors $h, r \in \mathbb{R}^n$ such that

$$2\,(h, Py) = (r, y)\,, \quad \forall\, y \in \mathbb{R}^n\,. \tag{94}$$

It follows from (94) that

$$2\,h = P^{-1} r\,. \tag{95}$$

Equation (95) shows that assumption (83) of Lemma 5 takes the form

$$(r, P^{-1} r) < 0\,. \tag{96}$$

If (96) is satisfied, it follows from Lemma 5 for the 1-dimensional quadratic cone $\mathcal{C} = \{y \in \mathbb{R}^n | (y, Py) \leq 0\}$ that

$$\operatorname{int} \mathcal{C} \cap \{y \in \mathbb{R}^n \,|\, (y, r) = 0\} = \varnothing\,. \tag{97}$$

But this is exactly the sufficient part of the statement in [5].

The following lemma from [2] will be used in the proof of Theorem 6.

Lemma 6 *Suppose that* $t_0 \geq 0,\ k(\cdot),\ R(\cdot),\ V_i(\cdot),\ U_i(\cdot) : [t_0, \infty) \to \mathbb{R}$, $i = 1, 2$, *are continuous functions and* $\varkappa_1 > \varkappa_2$ *are numbers such that the following conditions are satisfied:*

1) *In some neighborhood of the set*

$$\mathbb{T}_1 := \{t \in (t_0, \infty) \,|\, R(t) = \varkappa_1\ ,\ V_i(t) \leq 0\ ,\ i = 1, 2\ ,\ U_1(t) \leq 0\}$$

the function R *is non-increasing, and in some neighborhood of the set*

$$\mathbb{T}_2 := \{t \in (t_0, \infty) \,|\, R(t) = \varkappa_2\ ,\ V_i(t) \leq 0\ ,\ i = 1, 2\ ,\ U_2(t) \geq 0\}$$

the function R *is non-decreasing.*

2) *In some neighborhood of the set*

$$\mathbb{T}_3 := \{t \in (t_0, \infty) \,|\, \varkappa_2 \leq R(t) \leq \varkappa_1\ ,\ V_i(t) \leq 0\ ,\ i = 1, 2\ ,\ U_1(t) = 0\}$$

the function U_1 *is non-increasing, and in some neighborhood of the set*

$$\mathbb{T}_4 := \{t \in (t_0, \infty) \,|\, \varkappa_2 \leq R(t) \leq \varkappa_1\ ,\ V_i(t) \leq 0\ ,\ i = 1, 2\ ,\ U_2(t) = 0\}$$

the function U_2 *is non-decreasing.*

3) *On the set* $\{t \in (t_0, \infty) \,|\, \varkappa_2 \leq R(t) \leq \varkappa_1\}$ *the function* $k(\cdot)$ *is non-negative and the functions* $t \mapsto V_i(t) + \int_0^t k(\tau) V_i(\tau) d\tau, i = 1, 2$, *are non-increasing.*

4) $R(t_0) \in [\varkappa_2, \varkappa_1],\ V_i(t_0) \leq 0, \quad i = 1, 2,\ U_1(t_0) \leq 0\,,\ U_2(t_0) \geq 0.$

Then for all $t \geq t_0$ *it holds* $R(t) \in [\varkappa_2, \varkappa_1],\ V_i(t) \leq 0,\ U_1(t) \leq 0,\ U_2(t) \geq 0.$

Proof.[Proof of Theorem 6] Let us consider system (64) in the form (2). The hypotheses **(A1)**, **(A3)**, **(A4)**, **(A6)** ensure ([8]) that there exists a linear continuous operator $\hat{P} \in \mathcal{L}(Z_{-1}, Z_0) \cap \mathcal{L}(Z_0, Z_1)$, selfadjoint in Z_0, such that the quadratic form in $Z_1 \times \mathbb{R}$

$$W(z, \xi) := 2\,((\hat{A} + \lambda I)z + \hat{b}\,\xi, \hat{P}z)_{Z_{-1}, Z_1} + (\kappa_1 (\hat{c}, z)_{Z_0} - \xi)\,(\hat{c}, z)_{Z_0}$$

satisfies the inequality

$$W(z, \xi) \leq 0\,, \quad \forall\, z \in Z_1, \quad \forall\, \xi \in \mathbb{R}\,. \tag{98}$$

Putting $\xi = 0$ in (98), we deduce

$$2\,((\hat{A} + \lambda I)z, \hat{P}z)_{Z_{-1}, Z_1} \leq -\kappa_1 (\hat{c}, z)^2_{Z_0}\,, \quad \forall\, z \in Z_1\,. \tag{99}$$

By **(A2)**, there exists a splitting $Z_0 = Z_0^+ \oplus Z_0^-$ with $\dim Z_0^- = 1$ such that **(A7)** is satisfied for $Y_j = Z_j, j = 1, 0, -1,\ A = \hat{A} + \lambda I$ and $c = \hat{c}$. From (99) it follows that for any $z_0 \in Z_0$ the solution $z(\cdot)$ of

$$\dot{z} = (\hat{A} + \lambda I)z\,, \quad z(0) = z_0 \tag{100}$$

satisfies inequality (77) with $V(z) = (z, \hat{P}z)_{Z_0}$ and $c = \hat{c}$. By Lemma 4 we conclude that

$$\hat{P}_{|Z_0^+} \geq 0 \qquad \text{and} \qquad \hat{P}_{|Z_0^-} \leq 0\,. \tag{101}$$

Thus the set $\hat{\mathcal{K}} := \{z \in Z_0 \,|\, (z, \hat{P}z)_{Z_0} \leq 0\}$ is a 1-dimensional quadratic cone. It follows also from (98) that

$$2\,(\hat{b}, \hat{P}z)_{Z_{-1},Z_1} = (\hat{c}, z)_{Z_0}\,, \quad \forall\, z \in Z_1\,. \tag{102}$$

Clearly, that in the pairing $(\cdot,\cdot)_{Z_{-1},Z_1}$ we have

$$(\hat{b}, \hat{c})_{Z_{-1},Z_1} = \kappa_1 < 0\,. \tag{103}$$

By (101) – (103) all hypotheses of Lemma 5 are satisfied with respect to the rigging $Z_1 \subset Z_0 \subset Z_{-1}$, the vector $r = \hat{c}$ and the generalized vector $h = \hat{b}$. Thus we have from this lemma the relation

$$\operatorname{int} \hat{\mathcal{K}} \cap \{z \in Z_1 \,|\, (\hat{c}, z)_{Z_0} = 0\} = \varnothing\,. \tag{104}$$

Take now the points $z_1, z_2 \in \mathcal{V}_1 \times \mathbb{R}$ as $z_1 = (0, \zeta_1)$ and $z_2 := (0, \zeta_2)$. It is clear that

$$(\hat{c}, z_1)_{Z_0} = \zeta_1\,,\ \hat{A}z_1 = 0\,,\ (\hat{c}, z_2)_{Z_0} = \zeta_2\,,\ \hat{A}z_2 = 0\,. \tag{105}$$

Define along an arbitrary solution $z(\cdot)$ of (2) the functions

$$\hat{V}_i(t) := (z(t) - z_i\,,\ \hat{P}(z(t) - z_i))_{Z_0}\,,$$
$$\hat{U}_i(t) := (\hat{c}, z(t) - z_i)_{Z_0}\,,\ i = 1, 2\,,$$

and introduce the set

$$\mathcal{G} := \{z \in Z_1 \,|\, (z - z_i, \hat{P}(z - z_i))_{Z_0} \leq 0\,,\ i = 1, 2\,,\ (\hat{c}, z)_{Z_0} \in [\zeta_2, \zeta_1]\}\,. \tag{106}$$

It follows from $\hat{P} \geq 0$, (101) and (104) that the set $\mathcal{G}$ is convex and bounded. Let us show that $\mathcal{G}$ is positively invariant for the solutions of (2). For this we applicate Lemma 4 for a given time interval $[t_0, \infty)$, the functions $k(t) \equiv$

$2\lambda, V_i(t) = \hat{V}_i(t), U_i(t) = \hat{U}_i(t)$ and the numbers $\varkappa_1 = w_1, \varkappa_2 = w_2$. From (98) it follows that for $i = 1, 2,\ t_0 \le s \le t$, along the solution $z(t)$ and $w(t) = (\hat{c}, z(t))_{Z_0}$

$$\begin{aligned} &\hat{V}_i(\tau)\,|_s^t + 2\,\lambda \int_s^t \hat{V}_i(\tau)d\tau \\ &\le - \int_s^t [\kappa_1(w(\tau) - \zeta_i) - \ (\phi(\tau, w(\tau)) + q_i)]\,(w(\tau) - \zeta_i)d\tau \\ &+ \int_s^t (g(\tau) - q_i)(w(\tau) - \zeta_i)d\tau\ . \end{aligned} \tag{107}$$

From **(A6)** we conclude that for $i = 1, 2$ and all $t \ge s \ge t_0$ such that

$$\begin{aligned} &w(\tau) \in [\zeta_2, \zeta_1], \tau \in [s, t], \\ &\int_s^t [\kappa_1(w(\tau) - \zeta_i) - (\phi(\tau, w(\tau)) + q_i)](w(\tau) - \zeta_i)\,d\tau \ge 0 \\ \text{and} \quad &\int_s^t (g(\tau) - q_i)\,(w(\tau) - \zeta_i)\,d\tau \le 0\ . \end{aligned} \tag{108}$$

Thus (107) and (108) imply that for $i = 1, 2$ and such $t \ge s \ge t_0$ we have

$$\hat{V}_i(\tau)\,|_s^t + 2\lambda \int_s^t \hat{V}_i(\tau)d\tau \le 0\ ,$$

i.e., the functions $t \mapsto \hat{V}_i(t) + 2\lambda \int_0^t \hat{V}_i(\tau)\,d\tau$ are non-increasing. That is, condition 3) of Lemma 4 is satisfied. Since $z(t_0) \in \mathcal{G}$, condition 4) of this lemma is also satisfied.

In the following $\mathbb{T}_i, i = 1, 2, 3, 4$, are the sets which are defined in Lemma 6. It follows from (104) that if $t \in \mathbb{T}_1$ then $z(t) = z_1$. Thus we have by (64) and (3.9a) that

$$\dot{w}(t) = d_0[\phi(t, w(t)) + g(t)] < 0\ . \tag{109}$$

In the same way one shows that $w(t)$ is non-decreasing in a neighborhood of $\mathbb{T}_2$.

From (104) and the inequality $d_0 = (\hat{b}, \hat{c})_{Z_{-1}, Z_1} < 0$ it follows that for $t \in \mathbb{T}_3$ we have $z(t) = z_1$ and this by (106) and **(A6)**

$$\begin{aligned} \dot{\hat{U}}_1(t) =& (\dot{z}(t), \hat{c})_{Z_{-1}, Z_1} = (\hat{A}z(t) + \hat{b}\,[\phi(t, w(t)) + g(t)], \hat{c})_{Z_{-1}, Z_1} \\ =& (\hat{b}, \hat{c})_{Z_{-1}, Z_1}[\phi(t, w_1) + g(t)] < 0\ . \end{aligned}$$

Similarly one can show that $\hat{U}_2(t)$ is non-decreasing near $\mathbb{T}_4$.

Thus we have verified all hypotheses of Lemma 6. By this lemma it follows that $\mathcal{G}$ is positively invariant. It remains to show the inclusion (75). Let $z = (0, w) \in \mathcal{V}_1 \times \mathbb{R}$ with $w \in [w_2, w_1]$. Since $(\hat{c}, z)_{Z_0} = w$, the inclusion (75) is shown if

$$(z - z_i\,,\ \hat{P}(z - z_i))_{Z_0} \leq 0\ , \quad i = 1, 2\,. \tag{110}$$

From (104) and (105) it follows that for (110) it is sufficient that $\hat{A}z = 0$ implies that $(z, \hat{P}z)_{Z_0} \leq 0$. But the last inequality results from (99) since

$$2\,\lambda(z, \hat{P}z)_{Z_0} \leq -\kappa_1(\hat{c}, z)^2_{Z_0} \leq 0\ .$$

∎

Now we prove for (64) the existence of solutions in $\mathcal{W}(0, T; \mathcal{V}_1 \times \mathbb{R}, \mathcal{V}_{-1} \times \mathbb{R})$ and the existence of at least one solution in $C_b(\mathbb{R}; \mathcal{V}_0 \times \mathbb{R}) \cap BS^2(\mathbb{R}\,; \mathcal{V}_1 \times \mathbb{R})$. We need for this the a priori inclusion given by Theorem 6 and two additional assumptions.

(A8) The imbedding $\mathcal{V}_1 \subset \mathcal{V}_0$ is compact.

(A9) The family of operators $\{\hat{\mathcal{A}}(t)\}_{t\in\mathbb{R}}, \hat{\mathcal{A}}(t) : Z_1 \to Z_{-1}$, given by $\hat{\mathcal{A}}(t)z := -\hat{A}z - \hat{B}\phi(t, \hat{C}z), \forall t \in \mathbb{R}, \forall z \in Z_1$, is monotone on the segment $\{z \in Z_1 \,|\, \hat{C}z \in [\zeta_2, \zeta_1]\}$, i.e. for any $t \in \mathbb{R}$ we have

$$(\hat{\mathcal{A}}(t)\eta - \hat{\mathcal{A}}(t)\vartheta, \eta - \vartheta)_{Z_1, Z_{-1}} \geq 0\,, \quad \forall\, \eta, \vartheta \in Z_1\,,$$
$$\text{such that} \quad \hat{C}\eta, \hat{C}\vartheta \in [\zeta_2, \zeta_1]\,. \tag{111}$$

There exists a continuous function $\tilde{\phi} : \mathbb{R} \times \mathbb{R}$ such that $\tilde{\phi}_{|\mathbb{R}\times[\zeta_2,\zeta_1]} = \phi$ and (111) with $\tilde{\phi}$ instead of ϕ is satisfied for all $\eta, \vartheta \in Z_1$.

Remark 3 If ϕ has the form $\phi(t, w) = \phi_1(t)\phi_2(w)$ with ϕ_1 and ϕ_2 continuous, it is clear that such a monotone extension exists.

Theorem 7 *Assume that for system (64) the assumptions* **(A1)** – **(A9)** *are satisfied. Then it holds:*

a) For any $g \in BS^2(\mathbb{R}; \mathbb{R})$ and any $(\nu_0, w_0) \in \mathcal{G}$, where $\mathcal{G}$ is the associated positively invariant set, there exists a solution $(\nu, w) \in \mathcal{W}(0, \infty; \mathcal{V}_1 \times \mathbb{R}, \mathcal{V}_{-1} \times \mathbb{R})$ of (64) such that $(\nu(0), w(0)) = (\nu_0, w_0)$.

b) For any $g \in BS^2(\mathbb{R}; \mathbb{R})$ there exists for (64) a solution

$$(\nu_*, w_*) \in C_b(\mathbb{R}\,; \mathcal{V}_0 \times \mathbb{R}) \cap BS^2(\mathbb{R}; \mathcal{V}_1 \times \mathbb{R})\,. \tag{112}$$

Proof. Consider system (64) written in the form (65). Introduce the new nonlinearity $\tilde{\phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by assumption **(A9)**. Thus we have the auxiliary system

$$\dot{z} = \hat{A}z + \hat{B}[\tilde{\phi}(t, w) + g(t)], \quad w = \hat{C}z \tag{113}$$

on the Gelfand triple $\mathcal{V}_1 \times \mathbb{R} \subset \mathcal{V}_0 \times \mathbb{R} \subset \mathcal{V}_{-1} \times \mathbb{R}$. It follows from **(A8)** that the embedding $\mathcal{V}_1 \times \mathbb{R} \subset \mathcal{V}_0 \times \mathbb{R}$ is compact. Under these conditions it was shown in [11] (Theorem 4.3, Ch. 3) that the above statements a) and b) are true for system (112) for all $(\nu_0, w_0) \in \mathcal{V}_0 \times \mathbb{R}$. Thus the positive invariance of $\mathcal{G}$ for (64) implies that solutions of (112) with initial states from $\mathcal{G}$ are also solutions of (64). The forward solutions of (64) in $\mathcal{G}$ can be used to construct for any $g \in BS^2(\mathbb{R}; \mathbb{R})$ a bounded solution of (64) which satisfies (112). ∎

(A10) Any continuous function ϕ which satisfies (3.9a) and (3.9b) has a continuous extension to a function $\tilde{\tilde{\phi}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies (3.9a) and (3.9b) for all $(t, w) \in \mathbb{R} \times \mathbb{R}$.

Theorem 8 *Assume that for system (64) the assumptions* **(A1**) – **(A9)** *are satisfied and in addition to this the following holds:*

(i) The operator $\begin{bmatrix} A_0 & \kappa_2 B_0 \\ C_0 & \kappa_2 d_0 \end{bmatrix}$ *from* $\mathcal{L}(Z_1, Z_{-1})$ *is Hurwitz ;*

(ii)

$$\frac{1}{\kappa_3 - \kappa_2} + \mathrm{Re}\, \frac{\chi(i\omega) - d_0}{i\omega + \kappa_2(\chi(i\omega) - d_0)} > 0, \ \forall\, \omega \in \mathbb{R}\,. \tag{114}$$

Then we have:

a) For any $g \in BS^2(\mathbb{R}; \mathbb{R})$ *system (64) has a unique solution* (ν_*, w_*) *inside* $\mathcal{G}$ *which satisfies (112) and this solution is exponentially stable inside* $\mathcal{G}$*.*

b) Let the families of functions $\{\phi(\cdot, w) \,|\, w \in [\zeta_2, \zeta_1]\}$ *and* $\{\tilde{\phi}(\cdot, w) \,|\, w \in \mathcal{S}\}$, *where* $\tilde{\phi}$ *is from* **(A9)** *and* $\mathcal{S} \subset \mathbb{R}$ *is an arbitrary bounded interval, be uniformly Bohr a.p.. Then for any* S^2*-a.p. forcing function* g *the unique in* $\mathcal{G}$ *bounded and exponentially stable solution* (ν_*, w_*) *is Bohr a.p..*

Proof. Consider in the Gelfand rigging structure $Z_1 \subset Z_0 \subset Z_{-1}$ the system

$$\dot{z} = \hat{A}z + \hat{B}\,[\tilde{\tilde{\phi}}(t, w) + g(t)]\,, \quad w = \hat{C}z\,, \tag{115}$$

where $\tilde{\tilde{\phi}}$ is the monotone extension of ϕ given by assumption **(A10)**. Introduce the new nonlinearity

$$\phi(t,w) := \tilde{\tilde{\phi}}(t,w) - \kappa_2 w\,, \quad t \in \mathbb{R},\, w \in \mathbb{R}\,. \tag{116}$$

Then system (115) can be written with $w = \hat{C}z$ and $\hat{f}(t) := \hat{B}g(t)$ as

$$\dot{z} = (\hat{A} + \kappa_2 \hat{C})z + \hat{B}\phi(t,w) + \hat{f}(t)\,. \tag{117}$$

Note that ϕ satisfies the inequality

$$0 \le (\phi(t,w_1) - \phi(t,w_2))(w_1 - w_2) \le (\kappa_3 - \kappa_2)(w_1 - w_2)^2\,, \quad \forall\, t, w_1, w_2 \in \mathbb{R}\,. \tag{118}$$

For system (117) with a nonlinearity of the type (118) we can apply Theorems 4 and 5. According to Theorem 4 there exists a Gelfand rigged chain

$$Z_1 \subset Z_{0,P} \subset Z_{-1,P} \tag{119}$$

with the property: For any $\hat{f} \in BS^2(\mathbb{R}; Z_{-1})$ there is for (117) an exponentially stable in the whole solution $z_{**} \in C_b(\mathbb{R}; Z_{0,P}) \cap BS^2(\mathbb{R}, Z_1)$. Theorem 5 says that for Bohr a.p. $\hat{f}$ and uniformly S^2-a.p. functions ϕ this solution is Bohr a.p. .

From Theorem 7 it follows that equation (117) has a bounded solution $z_* = (\nu_*, w_*)$ w.r.t. the rigging $Z_1 \subset Z_0 \subset Z_{-1}$. But z_* is also a solution w.r.t. the rigging $Z_1 \subset Z_{0,P} \subset Z_{-1,P}$. By uniqueness we have $z_* = z_{**}$. Thus

$$\hat{C}z_{**}(t) \in [\zeta_2, \zeta_1]\,,\ \forall\, t \in \mathbb{R}\,. \tag{120}$$

Inclusion (120) implies that z_{**} is also a solution of (64) and this solution is exponentially stable inside $\mathcal{G}$. ■

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