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MATHEMATICAL MODELS OF DISCONTINUOUS DYNAMICAL SYSTEMS AND THEIR GEOMETRIC INVARIANTS.

Abstract

An autonomous discontinuous system, defined by a set of vector fields on a compact manifold is studied. A multigraph, describing possible transitions of trajectories from one cell to another, is constructed. It is shown that there exists a canonical algorithm allowing to reduce this graph to its canonical form which is the same for all topologically conjugated systems of vector fields and persists under perturbations of general systems of vector fields. Bifurcations, leading to changing of the normal form of the corresponding graph, are studied.

Keywords: Partitions, cell dynamics, discontinuous systems.

An autonomous discontinuous system, defined by a set of vector fields on a compact manifold is studied. A multigraph, describing possible transitions of trajectories from one cell to another, is constructed. It is shown that there exists a canonical algorithm allowing to reduce this graph to its canonical form which is the same for all topologically conjugated systems of vector fields and persists under perturbations of general systems of vector fields. Bifurcations, leading to changing of the normal form of the corresponding graph, are studied.

1. Introduction. In many engineering applications, characteristics of the system can be either discontinuous or non-smooth. As well-known examples, one may point an oscillator with clearance analyzed in [17], piecewise linear oscillators [7,11,19,24], Jeffcott rotor with bearing clearances [4,8,14], systems with Coulomb friction [3,20] and metal cutting processes [6,21,23].

The main trouble of dealing with such a kind of system is that even in the simplest cases the basic properties of solutions, such as existence, uniqueness and continuous dependence on initial data and parameters may be violated [10,12,18]. This is one of the reasons why the results, obtained by numerical methods (like Euler or Runge-Kutta method) may contradict to theoretical and experimental ones. There are three simplest reasons of non-uniqueness (of course, they are much more multiple): sliding, chattering and Coulomb friction. In all these cases the low velocity motions are stopped in a finite segment of time and the information of the motion is lost or, in other words, the dimension of the dynamics declines. This is the most principle difference between the smooth and the piecewise continuous (piecewise smooth) dynamics.

There are several possible approaches to describe the non-smooth dynamical systems.

The first one can be found in [22]. It includes modeling of non-smooth systems by discontinuous functions and modeling of discontinuities by smooth functions. In the latter case extra care is required as smoothing discontinuities can produce a ghost solution.

The main aim of the paper is to outline a general methodology for solving of non-smooth dynamical systems and to apply it to practical problems. A new type of models of the dynamical systems on the chain complexes is offered. A system of smooth vector fields on cells of the highest dimension is extended to all the complex. A graph, corresponding to this system of vector fields is constructed. Actually, this graph demonstrates all possible ways of transitions between distinct cells. It is shown that this graph can be reduced to the so-called normal form and this normal form does is in general structurally stable and is the same for topologically equivalent systems of vector fields.

These invariants are calculated for some examples. This approach allows to reduce the systems with discontinuous right hand sides to several o.d.e. system of different dimensions, see [16].

The bifurcations of dynamical system, leading to changes of the normal form of the graph (the so-called spatial bifurcations), are studied.

Periodic responses of a vibro-impact system with a drift are investigated through a semi-analytical method, developed by Pavlovskaia and Wiercigroch [15], which allows to determine the favorable operating conditions. The model accounts for visco-elastic impacts and is capable to mimic drift - dynamics of a bounded progressive motion.

Consider a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T \in \Omega$ is the state space vector,

$$\mathbf{p} = [p_1, p_2, \ldots, p_m]^T$$

is a vector of the system parameters, and $\mathbf{f} = [f^1, f^2, \ldots, f^N]^T$ is the vector function, which is dependent upon the process being modeled. Then we assume that the dynamical system (1) is continuous on every one of k subsets $\mathbf{X}_i$ of the global hyperspace Ω (Fig. 1), therefore, the right hand side of (1) will be piecewise smooth. For all $i = 1, \ldots, k$ suppose $f(\mathbf{x}, \mathbf{p}) = f_i(\mathbf{x}, \mathbf{p})$ if $\mathbf{x} \in \mathbf{X}_i$. Here and later all vector values are denoted by bold letters.

Adiabatic invariants of the impact systems were studied in [5].

There are several ways to reduce the non-smooth dynamical system to mappings on the Euclidean space. One can consider a transversal section of a periodic closed trajectory, provided the corresponding periodic solution is unique at least forward and this trajectory is a graph of a smooth function and consider the corresponding Poincaré mapping. Alternatively, if the solution $\mathbf{x}(t, \mathbf{x}_0, \mathbf{p})$ of system (1) with the initial data $\mathbf{x}(0+) = \mathbf{x}_0$ is unique, we may fix a positive value T and consider the mapping $\mathbf{P}_T(\mathbf{x}_0) = \mathbf{x}(T + 0, \mathbf{x}_0, \mathbf{p})$. Similarly, we may consider instead of system (1) a T – periodic system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}). \tag{2}$$

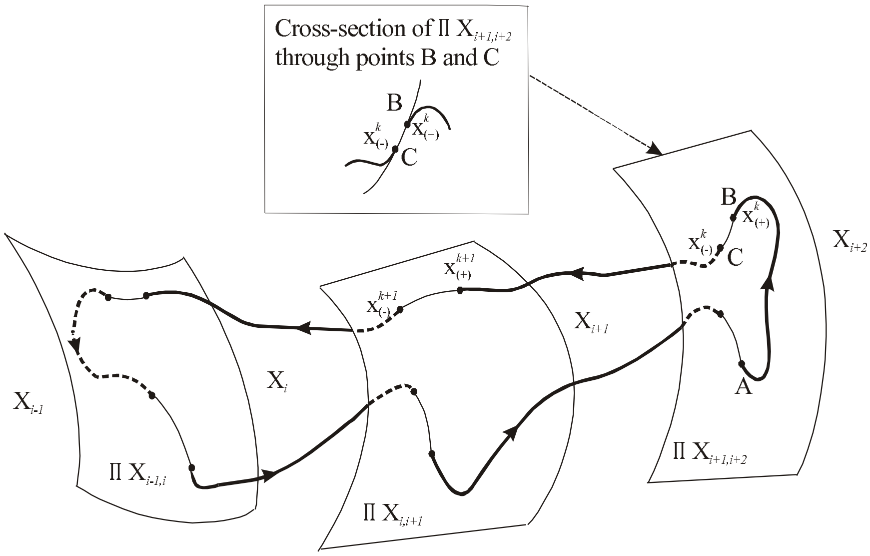


Figure 1: *A possible structure of the hyperspace Ω (adopted from [16]).*

However, there is another way, typical for the systems with a discontinuous right hand side. For any point x_0 where the right hand side of the considered system is discontinuous (denote the set of all such points by Λ) let L be the trajectory of the point $\mathbf{x}_0$. Suppose, the next forward-in-time intersection of L and Λ is well-defined. Then we may consider the so-called *stroboscopic mapping* $\mathbf{S}$, supposing $\mathbf{S}(\mathbf{x}_0) = \mathbf{x}_1$.

If Λ is a smooth manifold in neighborhoods of $\mathbf{x}_0$ and $\mathbf{x}_1$ and the intersections with the trajectory L are transversal at the points $\mathbf{x}_0$ and $\mathbf{x}_1$, the mapping $\mathbf{S}$ is a local diffeomorphism of a small vicinity of $\mathbf{x}_0$. The nonauthonomous periodic systems may be treated by this way as well, by adding an equation $\dot{t} = 1$, $t \in S^1$. Let T be the period of the right hand side of system (2). For the simplicity we assume that the discontinuity set Λ does not depend on t (the general situation is not much more difficult). Suppose there is a solution $\mathbf{x}(t)$ has two successive transversal intersections with the surface Λ, corresponding to the time instants t_0 and t_1 and the points $\mathbf{x}_0$ and $\mathbf{x}_1 \in \Lambda$. Let $\tau_i = t_i \mod T$, $i = 0, 1$. Then we denote

$$\mathbf{S}(\tau_0, \mathbf{x}_0) = (\tau_1, \mathbf{x}_1).$$

The troubles start then the trajectory L is not transversal to Λ either at the point $\mathbf{x}_0$ or at the point $\mathbf{x}_1$. Then the mapping $\mathbf{S}$ near $\mathbf{x}_0$ or its inverse $\mathbf{S}^{-1}$ near $\mathbf{x}_1$ may be not defined or at least not smooth. If the corresponding solution $\mathbf{x}(t)$ is periodic, this situation is called *grazing* [13]. The Lyapunov exponents of the periodic solution tend to infinity as the parameter value approaches to one, corresponding to the grazing. At the critical value of parameters the Poincaré mapping is still continuous but non-smooth. There are several cases then the normal forms for this type of bifurcation may be written down [1] (see also the references therein). The Poincaré mapping may be presented as a composition of one, which has a finite limit while the parameter tends to the critical value and the so-called discontinuity mapping [2], which has a square root type singularity in the case of impulse type conditions and the 2/3 type singularity for piecewise-smooth systems. The stroboscopic mapping is, generally speaking, discontinuous.

2. General model of piecewise continuous systems. We consider a piecewise continuous dynamical system on a smooth manifold, described by a system of vector fields. Provided some non-degeneracy conditions are satisfied, we can classify the discontinuity points. The points of different type form unions of smooth submanifolds (we call them cells) of the initial manifolds. We construct the multigraph, which shows how the solutions of the discontinuous systems may pass from one cell to others. Some of the points of discontinuity are reducible in the sense of topological equivalence of dynamical systems, some of them are irreducible. If we delete all the nodes and arrows, corresponding to the reducible points from the constructed multigraph, we obtain the so-called normal form of the multigraph of the dynamical system, which is a topological invariant of the system and describes the structure of the phase space. New bifurcations, corresponding to the changes of the structure of this multigraph, are described. The introduced model is applied to study one of important cases of discontinuous dynamics – the vibro-impact systems. The experimental data, justifying the offered approach, are provided.

Let Ω be a smooth manifold of the dimension N. Later on, we assume for

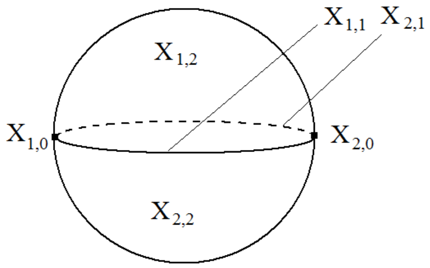


Figure 2: *A cell division of the unit sphere.*

the simplicity that either Ω is a C^∞ smooth compact manifold M or a Euclidean space $\mathbb{R}^N$ or a product $M \times \mathbb{R}^q$ of these two sets. Then it can be artificially compactified by adding the set $M \times \{\infty\}$ and becomes diffeomorphic to $M \times S^q$, where S^q is the unit sphere in $\mathbb{R}^{q+1}$. Suppose, that Ω consists of a finite number of non-intersecting cells

$$\Omega = \bigcup_{n=0}^{N} \bigcup_{k=1}^{K_n} \mathbf{X}_{k,n},$$

$K_n \in \mathbb{Z}^+$, such that every $\mathbf{X}_{k,n}$ is an n – dimensional submanifold of Ω. Let

$$\Omega_n = \bigcup_{m=0}^{n} \bigcup_{k=1}^{K_m} \mathbf{X}_{k,m}.$$

Denote $I = \{(k,n) : n \in \{0,\ldots,N\}, k \in \{1,\ldots,K_n\}\}$. We say that the set Ω, endowed with this structure is a *complex*. Unlike the classical case of the simplicial complex we define a cell $\mathbf{X}_{k,n}$ as an n-dimensional manifold (not necessarily a ball) with a piecewise smooth boundary. Also, we suppose by definition that for any two cells the intersection of their closures is either the empty set or a union of some other cells.

In Figure 2 the partition of the 2D sphere is given. Here we have 2 cells of the dimension 2 ($\mathbf{X}_{1,2}$ and $\mathbf{X}_{2,2}$), 2 cells of the dimension 1 ($\mathbf{X}_{1,1}$ and $\mathbf{X}_{2,1}$) and 2 cells of the dimension 0 ($\mathbf{X}_{1,0}$ and $\mathbf{X}_{2,0}$). For this case $\Omega = \Omega_2$, Ω_1 is the equator circle and Ω_0 is the union of the "west pole" $\mathbf{X}_{1,0}$ and the "east pole" $\mathbf{X}_{2,0}$.

Denote by $\partial_t \mathbf{X}$ the topological boundary of $\mathbf{X}$. For any pair $(k,n) \in I$

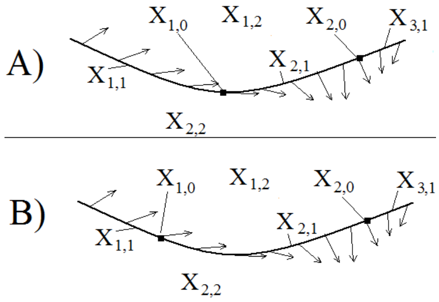


Figure 3: *Structure of vector fields on boundaries of cells.*

define

$$I_{k,n} = \{(l,m) : m \in \{0,\dots,n-1\}, l \in \{1,\dots,K_m\} : \mathbf{X}_{l,m} \subset \partial_t \mathbf{X}_{k,n}\}.$$

Due to the definition of the chain complex, $\partial_t \mathbf{X}_{k,n} = \bigcup_{(l,m)\in I_{k,n}} \mathbf{X}_{l,m}$ and $\partial_t \mathbf{X}_{k,n} \bigcap \mathbf{X}_{l,m} = \emptyset$ if $(l,m) \notin I_{k,n}$.

Definition 1. We say that the set of vector fields $\mathbf{f}_n$ on the complex Ω is *regular* if the cells $\mathbf{X}_{k,n}$ are chosen in such a way that for any $\mathbf{x} \in \Omega_{N-1}$, except ∞, and any k such that $(k,N) \in I_{\mathbf{x}}$,

$$\mathbf{f}_k(\mathbf{x}) \notin T\Omega_m \quad \text{if} \quad \mathbf{x} \notin \Omega_{m-1}, \quad m = 1,\dots,N-1. \tag{3}$$

These condition implies that the cell structure and its relations with vector fields are not sensitive with respect to small perturbations of the system parameters.

Later on we suppose that either the set Ω_{N-1} is compact or we artificially compactify it, adding ∞. Then, grace to the Sard theorem, the existence of the finite cell structure, providing regularity, can be obtained by a C^1 small perturbation of arbitrary set of smooth functions $\mathbf{f}_k$.

In the part A) of Figure 3 we show how a regular set of vector fields may look like in a neighborhood of the border between two 2D cells. The part B) shows how Condition (3) may be violated.

3. Constructing the discontinuous system. Suppose that the considered system is defined by equations

$$\dot{\mathbf{x}} = \mathbf{f}_k(\mathbf{x}) \tag{4}$$

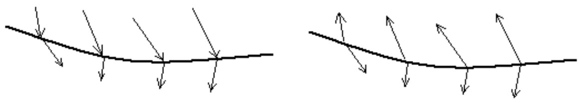


Figure 4: *Non-sliding and sliding cells.*

provided $\mathbf{x} \in \mathbf{X}_{k,N}$, $k = 1, \dots, K_N$. Denote by $\mathbf{x}_k(t, \mathbf{x}_0)$ the solutions of systems (4) for fixed values of k with the initial data $\mathbf{x}_k(0) = \mathbf{x}_0$ provided they are correctly defined.

Definition 2. We call the point $\mathbf{x}_0 \in \Omega_{N-1}$ *non-sliding* or there exist two values $k_-, k_+ \in \{1, \dots, K_N\}$, such that

1. $\mathbf{x}_0 \in \partial_{k_-,k_+} = \partial_t \mathbf{X}_{k_-,N} \bigcap \partial_t \mathbf{X}_{k_+,N}$,
2. the set ∂_{k_+,k_-} is a local smooth manifold in a neighborhood of the point $\mathbf{x}_0$,
3. For any k such that $(k, N) \in I_{\mathbf{x}}$ there is $\Delta > 0$ such that
$$\mathbf{x}_k(t, \mathbf{x}_0) \in \mathbf{X}_{k_+,N}$$
for all $t \in (0, \Delta)$ and $\mathbf{x}_k(t, \mathbf{x}_0) \in \mathbf{X}_{k_-,N}$ for all $t \in (-\Delta, 0)$.

Though the trajectory of the non-sliding point $\mathbf{x}_0$ may be non-smooth in this point, the general properties of solutions of a smooth system, such as existence, uniqueness, continuity etc. held true in a neighborhood of $\mathbf{x}_0$. The behavior of solutions in neighborhoods of non-sliding points is quite similar to one of the smooth systems.

We say the point $\mathbf{x}_0 \in \Omega_{N-1}$ is *sliding* if it is not non-sliding. The left and the right side of Figure 4 demonstrate the difference between the non-sliding (A.) and the sliding (B.) type of boundary points.

Consider the orthogonal projections $\mathbf{f}_k^m(\mathbf{x})$ of vector fields $\mathbf{f}_k(\mathbf{x})$ to $T\Omega_m$ (this tangent space is defined on the m-dimensional cells only).

Definition 3. We call the point $\mathbf{x}_0 \in \Omega$ *first order sliding* if it is sliding with respect to the initial set of vector fields and non-sliding with respect to set of vector fields $\mathbf{f}_k^{N-1}(\mathbf{x})$ on the complex Ω_{N-1}.

We say the point is is of the m – th order sliding ($m = 1, \ldots, N-1$) if is not sliding of any order $l \in \{0, \ldots, m-1\}$, but is non-sliding with respect to the system of vector fields $\mathbf{f}_k^{N-m}(\mathbf{x})$ on the complex Ω_{N-m}.

4. Filippov's method. We construct the set of vector fields $\mathbf{f}_{k,n}$ on the cells $\mathbf{X}_{k,n} \subset \Omega_{N-1}$ by the following procedure. Any cell $\mathbf{X}_{k,N-1}$ appertains to borders of exactly two N – dimensional cells $\partial_t \mathbf{X}_{k_\pm,N}$. If this cell is non-sliding, we define $\mathbf{f}_{k,N-1}(\mathbf{x}) = \mathbf{f}_{k_0}(\mathbf{x})$, where k_0 is the minimum of k_- and k_+. If the cell is sliding, we define

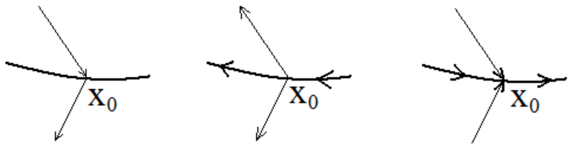
$$\mathbf{f}_{k,N-1}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{f}_{k_+}(\mathbf{x}) + (1-\alpha(\mathbf{x}))\mathbf{f}_{k_-}(\mathbf{x})$$

so that $\alpha(x) \in (0,1)$ and $\mathbf{f}_{k,N-1}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}_{k,N-1}$. These functions α are uniquely defined and smooth due to the regularity of the considered system of vector fields. We have constructed the piecewise smooth system of vector fields on $N-1$ – dimensional cells. Then we regularize it, if possible (an additional division of $N-1$ dimensional cells may be needed) and apply the similar procedure to define the dynamics on the cells of the complex Ω_{N-2}. The only difference is that considering Ω_{N-2}, we consider the first order sliding cells as non-sliding (they *are* non-sliding with respect to the system of vector field on Ω_{N-1}). Then we similarly treat Ω_{N-3} and so on. Finally, we get a collection of vector fields $\mathbf{f}_{k,n}$ for all the cells of the complex Ω.

Select a point $\mathbf{x}_0 \in \mathbf{X}_{k,n}$ ($n < N$). We say that the point $\mathbf{x}_0$ is of

a) the 0 type if it is non-sliding or its sliding order is less than $N-n$ (see the part A. of Fig. 5);

b) the + type if there is a local solution $\mathbf{x}(t, \mathbf{x}_0)$ of the considered system with the initial data $(0, \mathbf{x}_0)$ and $\Delta > 0$ such that $\mathbf{x}(t, \mathbf{x}_0) \in \mathbf{X}_{k,N}$ for all $t \in (-\Delta, 0)$ and $\varphi_k(t, \mathbf{x}_0) \notin \Omega_n$ for all $t \in (0, \Delta)$ (see the part B. of Fig. 5);



A. B. C.

Figure 5: *Dynamics on non-sliding and sliding cells.*

c) the $-$ type if there is a local solution $\mathbf{x}(t, \mathbf{x}_0)$ of the considered system with the initial data $(0, \mathbf{x}_0)$ and $\Delta > 0$ such that $\mathbf{x}(t, \mathbf{x}_0) \in \mathbf{X}_{k,N}$ for all $t \in (0, \Delta)$ and $\varphi_k(t, \mathbf{x}_0) \notin \Omega_n$ for all $t \in (-\Delta, 0)$ (see the part C. of Fig. 5).

All the points of the complex Ω_{N-1} are of one of these three types due to Condition (3). Two last types correspond to the points of non-uniqueness. We divide the cells in such a way that for any cell all the points of any cell are of the same type. This can be done due to the regularity of the considered system of vector fields.

We divide the cells in such a way that for any cell al the points are of the same type. This partition can be done because of the regularity of the considered system of vector fields.

5. Definition of the multigraph. Construct the oriented multigraph G, joining the cells $\mathbf{X}_{k,m} \subset \Omega_N$ by arrows (oriented lines) according to the following principle.

Let for a fixed cell $\mathbf{X}^0$ there exist a point $\mathbf{x}_0 \in \mathbf{X}^0$, a positive value ε, a cell $\mathbf{X}^- \neq \mathbf{X}^0$ and a solution $\mathbf{x}(t, \mathbf{x}_0)$ such that $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, $\mathbf{x}(t, \mathbf{x}_0) \in \mathbf{X}^-$ for all $t \in (-\varepsilon, 0)$. Then we draw the arrow from $\mathbf{X}^-$ to $\mathbf{X}^0$. If there exist a point $\mathbf{x}_0 \in \mathbf{X}^0$, a positive value ε, cell $\mathbf{X}^+ \neq \mathbf{X}^0$ and a solution $\mathbf{x}(t, \mathbf{x}_0)$ such that $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, $\mathbf{x}(t, \mathbf{x}_0) \in \mathbf{X}^+$ for all $t \in (0, \varepsilon)$ we draw the arrow from $\mathbf{X}^0$ to $\mathbf{X}^+$. Note that neither two non-sliding nor two cells of the same order of sliding can be linked directly.

An example of the dynamical system on the 2D sphere with the cell struc-

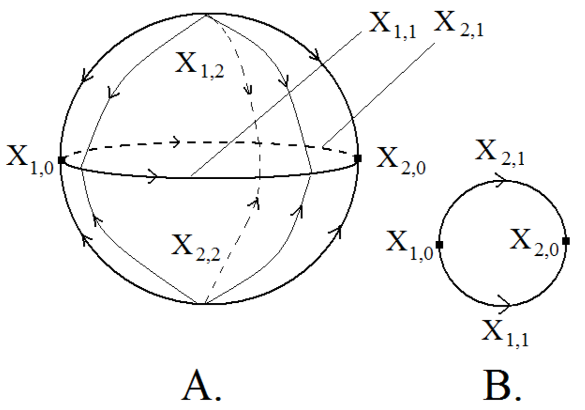


Figure 6: *A discontinuous system on the unit sphere.*

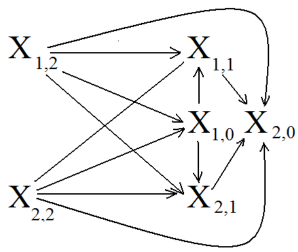


Figure 7: *The corresponding multigraph.*

ture from Fig. 2 is given at Figure 6. Here the north and the south pole of the sphere Ω are the sources, both the 1D cells are of the $-$ type and the corresponding dynamics is given at the part B of the figure. The points $\mathbf{X}_{1,0}$ and $\mathbf{X}_{2,0}$ are equilibria. This happens due to the sliding phenomena despite all the vector fields $\mathbf{f}_k$ are non-zero at these points.

The multigraph, corresponding to the considered system, is given at Figure 7.

Any periodic (or non-wandering) orbit of the considered system corresponds to a cycle in this oriented multigraph. If this cycle consists of the cells, whose dimension is not greater than n, the considered system may have a periodic solution, corresponding to the dynamics of the dimension, which is not less than n.

We prove that the oriented multigraph G does not change if we C^1 slightly change the regular set of vector fields $\mathbf{f}_k$. As we show at the section 3, the dynamical system may be smoothed in neighborhoods of non-sliding points. This means that some of the 0 type cells may be erased and some new ones may be added. So, the multigraph G does depend on the choice of the mathematical model, we shall find an invariant, which does not.

In the theory of dynamical systems, one usually say that two systems are equivalent if there exists a homeomorphism, which transfers the trajectories of one system to ones of another system [9]. Homeomorphisms may transfer the smooth trajectories of ordinary dynamical systems to non-smooth trajectories of the discontinuous systems. In the other words, sometimes, the topologically equivalent discontinuous systems may have a different cell structure. We prove that in this case there are some invariants for these systems which are the same, anyway.

Note that the cell structure may be not unique for a fixed dynamical system. For example, we may split an interval $(a, b) \subset \mathbb{R}$ with the dynamical system $\dot{x} = 1$ in the following way

$$(a, b) = (a, (a + b)/2) \bigcup \{(a + b)/2\} \bigcup ((a + b)/2, b).$$

We start with an obvious statement.

For any cell $\mathbf{X}_{k,N}$ we consider a neighborhood U_k, homeomorphic to a domain in $\mathbb{R}^N$ and so small that for any $k, l = 1, \dots K_N$ the corresponding neighborhoods intersect if and only if the closures of the cells do. For any point $\mathbf{x} \in \Omega$, we define the set $I_{\mathbf{x}} = \{(k, n) \in I : \mathbf{x} \in U_k\}$.

Let $\mathbf{f}_k(\mathbf{x}) : \Omega \to T\Omega$ $(k = 1, \dots, K_N)$ be a system of smooth vector fields on neighborhoods $U_{k,N}$. The following result shows the stability of the constructed structure.

Theorem 1. *Let the complex Ω_{N-1} be a compact set (without the point ∞) and let the set of functions $\mathbf{f}_k$ on Ω be regular. Then there exists such $\varepsilon > 0$ such that for any set $\mathbf{g}_k$ of vector fields, regular with respect to the same complex Ω such that*

$$\|\mathbf{f}_k - \mathbf{g}_k\|_{C^1(U_k)} < \varepsilon$$

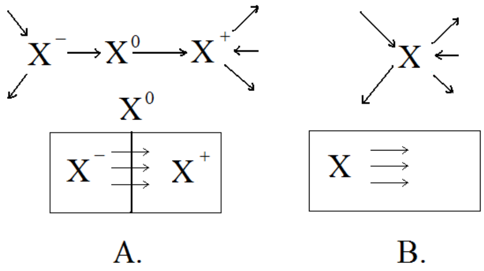


Figure 8: *Erasing the non-sliding boundary cells. Case 1.*

the multigraph G, corresponding to the complex Ω and vector fields $\mathbf{f}_{k,n}$ coincide with one, corresponding to the complex Ω and vector fields $\mathbf{g}_{k,n}$.

Corollary. *If the complex Ω is a compact set and the functions $\mathbf{f}_{k,n}(\mathbf{x},\mathbf{p})$ are C^1 smooth on the set $\{(\mathbf{x},\mathbf{p}) : \mathbf{x} \in \Omega, \mathbf{p} \in D\}$, where D is a domain in $\mathbb{R}^m$, then for any fixed oriented multigraph G the set of parameters $\mathbf{p}$, corresponding to G, is open.*

6. Normal forms for multigraphs of discontinuous systems.

Definition 4. We say that two dynamical systems

$$\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x}) \tag{5}$$

and

$$\dot{\mathbf{y}} = \mathbf{F}_2(\mathbf{y}) \tag{6}$$

defined on two complexes Ω^1 and Ω^2, are equivalent if there is a homeomorphism $\mathbf{h} : \Omega^1 \to \Omega^2$ such that the image of any trajectory of system (5) is a trajectory of (6).

The homeomorphism $\mathbf{h}$ is not necessarily an isomorphism of complexes, so in general it does **not** transfer cells to cells.

Fix a complex Ω with a regular system of vector fields $\mathbf{f}$. Let G be the corresponding multigraph. We say that the multigraph G_0 is the normal form of G if it is obtained by the following procedure (Fig. 8 or Fig. 9).

Step 1. Let $\mathbf{X}^+$ and $\mathbf{X}^-$ be two cells of the dimension N, such that the intersection of their boundaries contains at least one non-sliding cell $\mathbf{X}^0$ of the

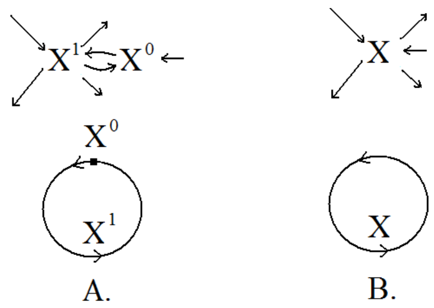


Figure 9: *Erasing the non-sliding boundary cells. Case 2.*

dimension $N-1$. Suppose that either $\mathbf{X}^+ \neq \mathbf{X}^-$ or $\mathbf{X}^0$ does not appertain to a boundary of any other cell except the cell $\mathbf{X}^+ = \mathbf{X}^- = \mathbf{X}^1$. Then we erase the lines of the graph G, joining $\mathbf{X}^0$ with $\mathbf{X}^+$ and $\mathbf{X}^-$, consider the new cell $\mathbf{X} = \mathbf{X}^+ \bigcup \mathbf{X}^- \bigcup \mathbf{X}^0$ instead of the initial three and redirect all the lines which were going to/from $\mathbf{X}^+$ and $\mathbf{X}^-$, except the raised ones, to/from the cell $\mathbf{X}$. We repeat this procedure while it is possible to find such a triple of cells $\mathbf{X}^+$, $\mathbf{X}^-$ and $\mathbf{X}^0$.

Steps 2,... Then we apply, while possible, the same procedure to the order 1 sliding $N-1$ dimensional cells of the complex Ω_{N-1}, separated by a order 1 sliding $N-2$ dimensional cell. Then we similarly treat with the complex Ω_{N-3} and so on.

Let G_n $(n = 0, \dots, N-1)$ be subgraphs of G, corresponding to the dynamical system on the sliding cells of the order $\geqslant N-n$ for complexes Ω_n, endowed with the vector fields $\mathbf{f}_{k,n}$.

Similarly, we define subgraphs $G_{n,0}$ of the normal form G_0 of the multigraph G.

The graph at Figure 7 coincide with its normal form. To have a non-trivial example of the normal form, see Figure 12 below.

Remark 1. If G_0 is the normal form for G then for all $n = 0, \dots N-1$ the multigraph $G_{n,0}$ is the normal form for G_n, concerning the system of vector fields $\mathbf{f}^n$ on complexes Ω^n.

Theorem 2. *Let the vector fields of systems (5) and (6) be regular and*

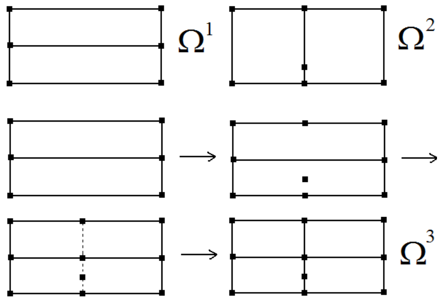


Figure 10: *Merging of two cell structures.*

equivalent and the normal forms of the corresponding multigraphs G^i be G_0^i, $i = 1, 2$. Then $G_0^1 = G_0^2$.

Proof. Suppose that every cell of Ω^1 intersects transversally every cell of Ω^2 and the intersection consists of a finite number of connected components. We may obtain this by a C^1 small diffeomorphic perturbation of the complex Ω^1_{N-1} and the related perturbation of vector fields. The normal forms of multigraphs are obviously invariant with respect to the complex isomorphisms, providing equivalence of vector fields, so we may consider two different cell partitions Ω^1 and Ω^2 of the same manifold endowed with the same discontinuous dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}).$$

Let Ω^3 be the "merged" complex, consisting of all nonempty intersections of cells of the complexes Ω^1 and Ω^2. The illustration is given in Figure 10, where we show how we construct step by step the merged complex from initial ones.

We show that the normal forms, corresponding to complexes Ω^1 and Ω^3 with the same dynamical system coincide. Then by transitivity, ones, corresponding to the complexes Ω^1 and Ω^2, do.

The complex Ω^3 may be obtained from Ω^1 by a finite number q of cell divisions and cell inserts. In the first case we take a cell $\mathbf{X}_{k,n}$ of the dimension n and replace it with the union of two cells $\mathbf{X}^+_{k,n}$ and $\mathbf{X}^-_{k,n}$ of the dimension n and one cell $\mathbf{X}^0_{k,n-1}$ of the dimension $n-1$. In the second one we insert some low dimensional cells inside the interior of an initial one.

Suppose without loss of generality that $q = 1$. Since Condition (3) is satisfied for the complexes Ω^1 and Ω^2, it is true for the complex Ω^3 as well. Denote the multigraph, corresponding to the complexes Ω^3 by G^3.

Note that the new cell $\mathbf{X}^0_{k,n-1}$ is same order of sliding as the cells $\mathbf{X}^+_{k,n}$ and $\mathbf{X}^-_{k,n}$ and is of the 0 type with respect to the system of vector fields on the complex Ω^3_n. If it was of the + or of the − type then every point of this cell would be the non-uniqueness point for the system on the complex Ω^3_n and, consequently, one on the complex Ω^2_n. However, it is a uniqueness point for the same vector field on the complex Ω^1_n. Hence, the sets of non-uniqueness points for (5) and (6) are not be homeomorphic and these two systems may not be equivalent.

The types and orders of sliding for the cells $\mathbf{X}^+_{k,n}$ and $\mathbf{X}^-_{k,n}$ coincide and are the same with ones of the cell $\mathbf{X}_{k,n}$. If these new cells are non-sliding, we just add two more nodes and two more lines to the multigraph G^1 and either, replacing Figure 8B with 8A if $\mathbf{X}^+_{k,n} \neq \mathbf{X}^-_{k,n}$ or 9B with 9A if $\mathbf{X}^+_{k,n} = \mathbf{X}^-_{k,n}$.

So, what we are doing while passing from G^1 to G^3 is exactly inverse to what we are doing, obtaining the normal form. Then, if we find a normal form for the multigraph G_3, we may obtain G_1 in one of the intermediate steps. All the farther steps are the same and, hence the normal forms of multigraphs coincide. □

If we have a general type system (1), the normal forms of multigraphs may change in a neighborhood of a fixed value $\mathbf{p}_0$ of the parameter $\mathbf{p}$. We call this phenomena *spatial bifurcation* of Eq. (1), because not only the topological structure of the solutions, but even one of the phase space changes. One of the simplest examples of this bifurcation is the following.

Consider the equation

$$\dot{x} = \sqrt{x^2 + p}, \qquad x, p \in \mathbb{R}. \tag{7}$$

For positive values of the parameter p Eq. (7) is defined on all the real line. If p is negative, the phase space consists of two disjoint components

$(-\infty, -\sqrt{-p}]$ and $[\sqrt{-p}, \infty)$. So, $p = 0$ corresponds to the spatial bifurcation.

Remark 2. We can smooth the considered system on all the complex Ω. Consider the functions $\chi_k : \Omega \to [0, 1]$, $k = 1, \ldots, K_N$ with the following properties.

1. $\chi_k \in C^\infty$ for all $k = 1, \ldots, K_N$;
2. $\chi_k(\mathbf{x}) = 0$ if $x \notin U_k$;
3. for all $\mathbf{x} \in \Omega$
$$\sum_{k=1}^{K_N} \chi_k(x) = 1.$$

We consider the system

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) = \sum_{k=1}^{K_N} \chi_k(\mathbf{x})\mathbf{f}_k(\mathbf{x}). \tag{8}$$

The right hand side $\mathbf{g}$ is as smooth as the vector fields $\mathbf{f}_k$ in this case. This procedure can be done locally in neighborhoods of non-sliding points. However, system (8) is not equivalent to the initial discontinuous system, since all their solutions are unique.

Remark 3. Alternatively we may smooth not the vector fields but the solutions of the discontinuous system. For any point $\mathbf{x}_0 \in \Omega$ we find solutions $\varphi_k(t, \mathbf{x}_0)$ of systems (4) with the initial data $(0, \mathbf{x}_0)$ such that $\mathbf{x}_0 \in U_k$. Suppose all of them are well-defined for $t \in [-\Delta, \Delta]$. Then we define the solution of our system with initial data $0, \mathbf{x}_0$ by the formula

$$\mathbf{x}(t) = \sum_{k=1}^{K_N} \mathbf{x}_{k,N}(t)\varphi(\mathbf{x}_{k,N}(t)).$$

If the domains U_k and the constant Δ (uniform for all $\mathbf{x}_0$) are chosen small enough, we may not care of the additives, that correspond to $(k, N) \notin I_{\mathbf{x}_0}$. This model has two "contradictive" properties: the solution is uniquely defined and continuously depends on the initial data and parameters, but there may be troubles to extend it and the trajectories of different points may intersect since the group property of autonomous systems may be violated.

7. Example. Let us use this construction to describe a simple mathematical model of a linear s.d.f dynamical system with impacts.

We consider a linear differential equation of the form

$$\ddot{x} + p\dot{x} + qx = f(t) = a \sin t + b. \tag{9}$$

Assume that $p > 0$, $p^2 < 4q$.

Eq. (9) is defined for $x > 0$ and the following impact conditions take place:

1. if $x(t_0) = 0$ then
$$\dot{x}(t_0 + 0) = -\dot{x}(t_0 - 0); \tag{10}$$
2. if $x(t_0 - 0) = \dot{x}(t_0 - 0) = 0$ then $x(t) = \dot{x}(t) = 0$ for any t, such that $f|_{[t_0,t]} \leqslant 0$.

We take a small value e_0, corresponding to the Coulomb friction in the considered system of the considered system and suppose that $-a < b - e_0 < a$, the function $f(t) - e_0$ has exactly two zeros over the period: $\sin^{-1}(b - e_0)/a$ and $\pi - \sin^{-1}(b - e_0)/a$.

Reduce the introduced system to an autonomous piecewise continuous one in the 3–dimensional space. Let the complex $\Omega = \mathbb{R}^2 \times S^1 \bigcup \{\infty\}$ consist of the following cells (Fig. 11):

$$\mathbf{X}_3^+ = \{(x, y, t) : x > 0\}, \qquad \mathbf{X}_3^- = \{(x, y, t) : x < 0\},$$
$$\mathbf{X}_2^+ = \{(0, y, t) : y > 0\}, \qquad \mathbf{X}_2^- = \{(0, y, t) : y < 0\},$$
$$\mathbf{X}_1^+ = \{(0, 0, t) : f(t) > e_0\}, \qquad \mathbf{X}_1^- = \{(0, 0, t) : f(t) < e_0\},$$
$$\mathbf{X}_0^+ = \{(0, 0, t_+) : f(t_+) = e_0,\ f'(t_+) > 0\},$$
$$\mathbf{X}_0^- = \{(0, 0, t_-) : f(t_-) = e_0,\ f'(t_-) < 0\}.$$

We consider the systems

$$\dot{x} = y, \qquad \dot{y} = -qx - py + f(t), \qquad \dot{t} = 1$$

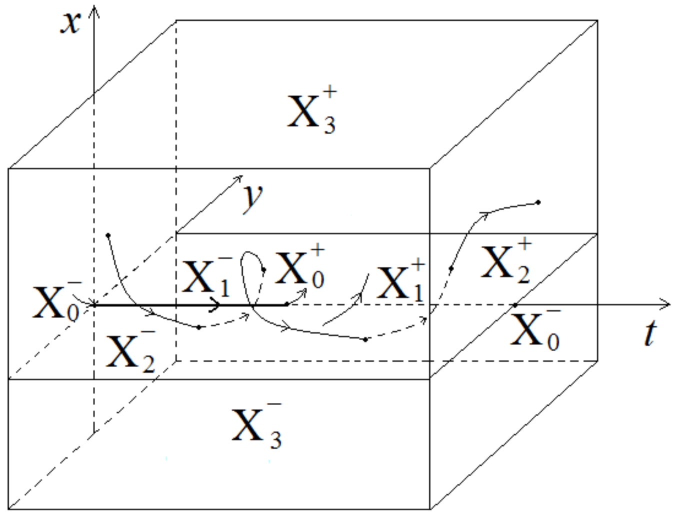


Figure 11: *An example of a discontinuous dynamical system.*

if $(x, y, t) \in \mathbf{X}_3^+$;

$$\dot{x} = y, \qquad \dot{y} = -x, \qquad \dot{t} = 0 \tag{11}$$

if $(x, y, t) \in \mathbf{X}_3^- \bigcup \mathbf{X}_2^- \bigcup \mathbf{X}_2^+$ (this is the replacement of Condition (10)); and

$$\dot{x} = 0, \qquad \dot{y} = f(t), \qquad \dot{t} = 1$$

if $x = y = 0$, $f(t) > e_0$,

$$\dot{x} = 0, \qquad \dot{y} = f(t), \qquad \dot{t} = 1$$

if $x = y = 0$, $f(t) \geqslant e_0$;

$$\dot{x} = 0, \qquad \dot{y} = 0, \qquad \dot{t} = 1$$

if $x = y = 0$, $f(t) < e_0$.

The equation $\dot{t} = 0$ in system (11) reflects the fact that the jump of the velocity during the impact is instantaneous.

We have 2 cells of every dimension: ones of the dimension 2 are of the 0 type, the cell $\mathbf{X}_1^-$ is of the i type, $\mathbf{X}_1^+$ and $\mathbf{X}_0^\pm$ are of the 0 type.

The corresponding multigraph G and its normal form are given at Fig. 12A and 12B respectively.

Here $\mathbf{X}_3 = \mathbf{X}_3^+ \bigcup \mathbf{X}_3^- \bigcup \mathbf{X}_2^+ \bigcup \mathbf{X}_2^- \bigcup \mathbf{X}_1^+$.

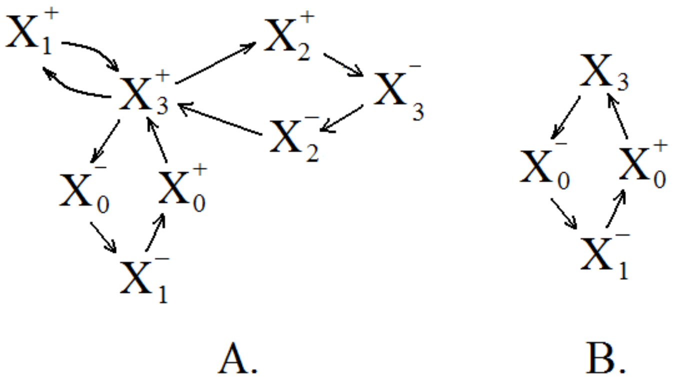


Figure 12: *The corresponding multigraph and its normal form. Case 1.*

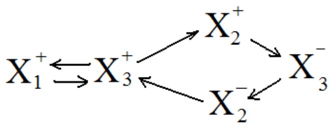


Figure 13: *The corresponding multigraph and its normal form. Case 2.*

Similarly, one may consider the case $b > a + e_0$ when $f(t) > e_0$ for all t. For this case, we have the cells $\mathbf{X}_3^{\pm}$, $\mathbf{X}_2^{\pm}$, and $\mathbf{X}_1^{+}$ only. Moreover, the cell $\mathbf{X}_1^{+}$ is of the 0 type and we do not have no 1 dimensional dynamics any more. The multigraph G (and its normal form) in this case have a more simple structure (Fig. 13.).

The normal form of this multigraph is the graph, which contains the single node Ω and no lines.

The case $b < e_0 - a$ is a bit different. There is no more cell $\mathbf{X}_1^{+}$ and no more 0 dimensional cells. The graph and its normal form are given at Fig. 14A and Fig. 14B respectively.

Here $\mathbf{X}_3 = \mathbf{X}_3^{+} \bigcup \mathbf{X}_3^{-} \bigcup \mathbf{X}_2^{+} \bigcup \mathbf{X}_2^{-}$.

Consequently, the values $b = \pm a + e_0$ correspond to spatial bifurcations.

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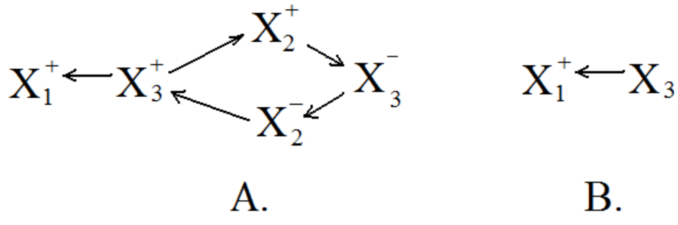

Figure 14: *The corresponding multigraph and its normal form. Case 3.*

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