DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 1, 2019
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru

Functional differential equations

On Stability and Boundedness of Solutions of a Certain non-Autonomous Third-Order Functional Differential Equation with Multiple Deviating Arguments

Mohamed A. Abdel-Razek [1], Ayman M. Mahmoud [2], Doaa A. M. Bakhit [2]
[1] Department of Mathematics, Faculty of Science, Assiut University,
Assiut 71516, Egypt
e-mail: abdel_razek555@yahoo.com
[2] Department of Mathematics, Faculty of Science, New Valley University,
El-Khargah 72511, Egypt
e-mail: math_ayman27@yahoo.com & doaa_math90@yahoo.com

Abstract

This paper investigates the explicit criteria of the stability of the zero solution and the boundedness of all solutions for a certain non-autonomous third-order functional differential equation with multiple deviating arguments. To study the stability of the zero solution we construct the Lyapunov functional. The Gronwall-Reid-Bellman inequality is employed to establish the boundedness of all solutions of the addressing model. This study includes and improves some related results existing in the relevant literature. For illustration, two examples are given.

Keywords: Functional differential equations, stability, boundedness, third-order differential equation, Lyapunov functional, multiple deviating arguments.

1 Introduction

It is well-known that functional differential equations (FDEs), in particular, delay differential equations (DDEs) are very important in many areas of engineering and science. These equations are frequently encountered as mathematical models of most dynamical processes in mechanics, control theory, physics, chemistry, etc. (see [6, 12, 13, 14, 31, 39] and the references therein). One of the most attractive areas of the qualitative theory of FDEs is the stability and the boundedness of solutions. In the study of stability and boundedness problems for FDEs, it is well-known that Lyapunov's second method is very important and effective. This technique is also called the direct method because it can be applied to a differential equation directly, without any knowledge of solutions. Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time-lag, power system analysis, time varying non-linear feedback system and so on. It worth mentioning that there are numerous books studied the stability and the boundedness by Lyapunov's direct method (see for example [8, 10, 11, 13, 17, 18, 40]), etc. In the last few decades, the theory of FDEs has attracted much attention and numerous of papers have been published, we can mention the works in [1-3, 5, 9, 19-25, 27-30, 32-35, 37, 41], and the references therein. In the particular, many results on the stability and the boundedness of solutions of non-autonomous third-order FDEs have been studied. It can briefly be summarized as the following:

Sadek [30] investigated the asymptotic stability of the zero solution of the delay differential equation

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)f(x(t-r)) = 0,$$

where $a(t), b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$; where r is a positive constant; $f(x)$ is a continuous function and $f(0) = 0$.

Omeike [20] studied the stability and the boundedness of solutions of the third-order non-autonomous nonlinear differential equation with delays of the form

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = p(t),$$

where r is a positive constant, $a(t)$, $b(t)$, $c(t)$, $g(\dot{x})$ and $h(x)$ are real-valued functions continuous in their respective arguments; $g(0) = h(0) = 0$.

Mahmoud [19] established sufficient conditions for the asymptotic stability of the zero solution for a certain nonlinear non-autonomous third-order delay differential equation

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}(t - r(t))) + c(t)h(x(t - r(t))) = 0,$$

where $0 \le r(t) \le \gamma$, $\gamma > 0$, $a(t)$, $b(t)$, $c(t)$, $g(\dot{x})$ and $h(x)$ are real-valued functions continuous in their respective arguments; $g(0) = h(0) = 0$.

Besides it is worth-mentioning that according to our observation, there are few papers studied the behaviour of solutions of certain differential equations of third and fourth-order with multiple delays, (see, [4, 15, 16, 26, 36, 38]).

Tunç and Gözen [38] investigated the stability and the boundedness of solutions of the third-order FDE with multidelay of the form

$$\dddot{x}(t) + a(t)\ddot{x}(t) + nb(t)g(\dot{x}(t)) + c(t)\sum_{i=1}^{n} h_i(x(t - r_i)) = p(t),$$

where r_i are certain positive constants, $a(t)$, $b(t)$, $c(t)$, $g(\dot{x})$, $h(x)$ and $p(t)$ are real-valued and continuous functions in their respective arguments with $g(0) = h(0) = 0$.

Tunç [36] discussed the asymptotic stability of the zero solution and the boundedness of all solutions of the third-order nonlinear differential equation with multiple deviating arguments as

$$\begin{aligned}\dddot{x}(t) + &F(\dot{x}(t))\ddot{x}(t) + H(\dot{x}(t))\dot{x}(t) + \sum_{i=1}^{n} G_i(\dot{x}(t - g_i(t))) + \Psi(x(t)) \\ &= P(t, x(t), ..., x(t - g_n(t)), \dot{x}(t), ..., \dot{x}(t - g_n(t)), \ddot{x}(t)),\end{aligned}$$

where F, H, G_i, Ψ and P are continuous functions in their respective arguments, with $G_i(0) = \Psi(0) = 0$.

Ademola et al. [4] established the stability and the boundedness to a certain third-order delay differential equation with multiple deviating arguments as the following

$$\begin{aligned}\dddot{x} + \sum_{i=1}^{n} f_i(t, x, x(t - \tau_i(t)), \dot{x}, \dot{x}(t - \tau_i(t)), \ddot{x}, \ddot{x}(t - \tau_i(t))) + \sum_{i=1}^{n} g_i(\dot{x}(t - \tau_i(t))) \\ + \sum_{i=1}^{n} h_i(x(t - \tau_i(t))) = \sum_{i=1}^{n} p_i(t, x, x(t - \tau_i(t)), \dot{x}, \dot{x}(t - \tau_i(t)), \ddot{x}, \ddot{x}(t - \tau_i(t))),\end{aligned}$$

where f_i, g_i, h_i and p_i are continuous functions in their respective arguments.

Motivated by the above discussion, the present paper investigates the stability of the zero solution and the boundedness of all solutions of non-autonomous third-order FDE with multiple deviating arguments as follows

$$\dddot{x}+a(t)f(\dot{x})\ddot{x}+b(t)\sum_{i=1}^{n}g_i(x(t-r_i(t)),\dot{x}(t-r_i(t)))+c(t)\sum_{i=1}^{n}h_i(x(t-r_i(t)))$$
$$=p(t,x,\dot{x},\ddot{x},x(t-r(t))), \tag{1.1}$$

where $a(t),b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$; f, g_i and p_i are continuous functions for all values of respective arguments, with $h_i(0)=g_i(x,0)=0$.
All of the functions which appear and the solutions considered are supposed to be real. The dots indicate differentiation with respect to the independent variable t. Also the derivatives $\frac{\partial}{\partial x}g_i(x,\dot{x})$, $\frac{\partial}{\partial y}g_i(x,\dot{x})$, $h_i'(x)$, $a'(t)$, $b'(t)$, $c'(t)$ and $r'(t)$ exist and are continuous moreover, the existence and uniqueness of the solutions of (1.1) will be assumed.

However, to the best of our knowledge, there is no previous literature on stability and boundedness of solutions to non-autonomous third-order FDE with multiple deviating arguments (1.1).

2 Preliminary Results

We consider the following general non-autonomous finite delay differential system:

$$\dot{x}=f(t,x_t),\ x_t=x(t+\theta),\ \ -r\leq\theta\leq 0, \tag{2.1}$$

where $f:[0,\infty)\times C_H\to\mathbb{R}^n$ is a continuous mapping, $f(t,0)=0$, and we suppose that f takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C,\|.\|)$ is the Banach space of continuous function $\varphi:[-r,0]\to\mathbb{R}^n$ with the supremum norm, C_H is the open $H-$ball in C, $r>0$ for $H>0, C_H=\{\phi\in C:\|\phi\|<H\}$, $C=C([-r,0],\mathbb{R}^n)$. We will give some important definitions (see Burton [9]).

Definition 2.1 *A continuous function $V:[0,\infty)\times C_H\to[0,\infty)$, which is locally Lipschitz in ϕ and the derivative of this function is defined as*

$$\dot{V}(t,x_t)=\limsup_{h\to 0}\frac{V(t+h,x_{t+h}(t_0,\phi))-V(t,x_t(t_0,\phi))}{h},$$

is called a Lyapunov functional for (2.1), *if there is a wedge* W *satisfies the following conditions*

(i) $W(|\phi(0)|) \leq V(t,\phi), \quad V(t,0) = 0 \quad$ *and*

(ii) $\dot{V}_{(2.1)}(t,x_t) \leq 0.$

Definition 2.2 *The zero solution of* (2.1) *is said to be stable at* $t \geq t_0$, *if for each* $\varepsilon > 0$ *and* $t_0 \in \mathbb{R}$ *there exists a positive constant* $\delta = \delta(\varepsilon, t_0)$ *such that, if* $\|\phi\| < \delta$ *then* $|x(t,t_0,\phi)| < \varepsilon$ *for* $t \geq t_0$.

Definition 2.3 *The zero solution of* (2.1) *is said to be uniformly stable, if it is stable for* $t \geq t_0$ *and the positive constant* δ *is independent of* t_0.

Theorem 2.1 [7, 41] *Let* $V(\phi) : C_H \to \mathbb{R}$ *be a continuous functional satisfying a local Lipschitz condition,* $V(0) = 0$ *and functions* $W_i(r)$, $(i = 1,2)$ *are wedges such that*

(i) $W_1(|\phi(0)|) \leq V(\phi) \leq W_2(\|\phi\|)$ *and*

(ii) $\dot{V}_{(2.1)}(\phi) \leq 0, \quad for\ \phi \in C_H.$

Then the zero solution of (2.1) *is uniformly stable.*

Assumptions:
In addition to the basic assumptions on f, g_i and h_i of equation (1.1), suppose that there are positive constants a_0, a_1, a_2, a_3, a_4, a_5, a_i, b_0, b_1, b_i, l_i, with $a_1a_0b_0a_i - c_0l_i > 0$, L_i, N_i; for all $i\,(i = 1,2,3,...,n)$, γ and β, which satisfy the following assumptions:

(i) $\frac{g_i(x,y)}{y} \geq a_i$, $\frac{h_i(x)}{x} \geq b_i$, for all $x \neq 0$ and $y \neq 0$.

(ii) $a_0 \leq f(y) \leq a_3$, $\sup\{h_i'(x)\} = l_i$, $\ y\frac{\partial g_i(x,y)}{\partial x} \leq 0$, for all x and y.

(iii) $a_1 \leq a(t) \leq a_2$, $c_1 \leq b_0 \leq b(t) \leq b_1$, $c_1 \leq c(t) \leq c_0$ such that $c_0 > 2c_1$, for $t \geq 0$.

(iv) $\frac{l_i}{a_i} < \mu$, $\ b'(t) \leq c'(t) \leq 0$, $a_5 \leq a'(t) \leq a_4$, for $t \geq 0$, $a_i \neq 0$.

(v) $r_i(t) \leq \gamma$, $r_i'(t) \leq \beta$, $0 < \beta < 1$, for $t \geq 0$.

(vi) $|h_i'(x)| \leq L_i$, $|\frac{\partial g_i(x,y)}{\partial y}| \leq N_i$, for all x and y.

3 Main Results

Theorem 3.1 *By assuming that the assumptions (i) – (vi) hold true with $h_i(0) = g_i(x, 0) = 0$, suppose that the positive constant γ is also satisfied*

$$\gamma < min\Bigg[\frac{(1-\beta)(a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n l_i - \mu a_3a_4)}{2\Big\{\mu(1-\beta)(b_1\sum_{i=1}^n N_i + c_0\sum_{i=1}^n L_i) + c_0(\mu+1)\sum_{i=1}^n L_i\Big\}},$$
$$\frac{(1-\beta)(a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n l_i)}{2b_0\sum_{i=1}^n a_i\Big\{(1-\beta)c_0\sum_{i=1}^n L_i + (\mu-\beta+1+b_1)\sum_{i=1}^n N_i\Big\}}\Bigg],$$

where

$$\mu = \frac{a_0a_1b_0\sum_{i=1}^n a_i + c_0\sum_{i=1}^n l_i}{2b_0\sum_{i=1}^n a_i}.$$

Then the zero solution of (1.1) with $p = 0$ is uniformly stable.

Proof.
By considering $p = 0$, equation (1.1) is equivalent to the following system

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -a(t)f(y)z - b(t)\sum_{i=1}^n g_i(x,y) + b(t)\sum_{i=1}^n\int_{t-r_i(t)}^t \frac{\partial g_i(x(s),y(s))}{\partial x}y(s)ds \\
&\quad + b(t)\sum_{i=1}^n\int_{t-r_i(t)}^t \frac{\partial g_i(x(s),y(s))}{\partial y}z(s)ds \\
&\quad - c(t)\sum_{i=1}^n h_i(x) + c(t)\sum_{i=1}^n\int_{t-r_i(t)}^t h_i'(x(s))y(s)ds.
\end{aligned} \tag{3.1}$$

The proof of the theorem depends entirely on some fundamental properties of a certain differentiable Lyapunov functional $V = V(x_t, y_t, z_t)$ of the system (3.1) defined by

$$\begin{aligned}
V(x_t,y_t,z_t) =& \mu c(t)\sum_{i=1}^n\int_0^x h_i(\xi)d\xi + c(t)y\sum_{i=1}^n h_i(x) \\
&+ \mu a(t)\int_0^y f(\eta)\eta d\eta + b(t)\sum_{i=1}^n\int_0^y g_i(x,\eta)d\eta + \mu yz + \frac{1}{2}z^2 \\
&+ \sum_{i=1}^n\lambda_i\int_{-r_i(t)}^0\int_{t+s}^t y^2(\theta)d\theta + \sum_{i=1}^n\delta_i\int_{-r_i(t)}^0\int_{t+s}^t z^2(\theta)d\theta.
\end{aligned} \tag{3.2}$$

By differentiating (3.2) in terms of t, we get

$$
\begin{aligned}
\frac{dV}{dt} =&\mu c'(t)\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c'(t)y\sum_{i=1}^{n}h_i(x) + b'(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta \\
&+ \mu a'(t)\int_0^y f(\eta)\eta d\eta + c(t)y^2\sum_{i=1}^{n}h_i'(x) + b(t)\sum_{i=1}^{n}\int_0^y y\frac{\partial g_i(x,\eta)}{\partial x}d\eta \\
&- a(t)f(y)z^2 + \mu z^2 + b(t)(\mu y + z)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\frac{\partial g_i(x(s),y(s))}{\partial x}y(s)ds \\
&- \mu y b(t)\sum_{i=1}^{n}g_i(x,y) + b(t)(\mu y + z)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\frac{\partial g_i(x(s),y(s))}{\partial y}z(s)ds \\
&+ c(t)(\mu y + z)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}h_i'(x(s))y(s)ds + y^2\sum_{i=1}^{n}\lambda_i r_i(t) + z^2\sum_{i=1}^{n}\delta_i r_i(t) \\
&- \sum_{i=1}^{n}\lambda_i(1-r_i'(t))\int_{t-r_i(t)}^{t}y^2(\theta)d\theta ds - \sum_{i=1}^{n}\delta_i(1-r_i'(t))\int_{t-r_i(t)}^{t}z^2(\theta)d\theta ds.
\end{aligned}
$$

From the assumptions (ii), (vi) and by using the inequality $xy \le \frac{1}{2}(x^2+y^2)$, we have

$$
\begin{aligned}
\frac{dV}{dt} \le&\mu c'(t)\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c'(t)y\sum_{i=1}^{n}h_i(x) + \mu a'(t)\int_0^y f(\eta)\eta d\eta \\
&+ b'(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta + c(t)y^2\sum_{i=1}^{n}l_i - a(t)f(y)z^2 + \mu z^2 \\
&- \mu y b(t)\sum_{i=1}^{n}g_i(x,y) + \frac{\mu}{2}b(t)y^2\sum_{i=1}^{n}N_i r_i(t) + \frac{b(t)}{2}z^2\sum_{i=1}^{n}N_i r_i(t) \\
&+ \frac{\mu}{2}c(t)y^2\sum_{i=1}^{n}L_i r_i(t) + \frac{c(t)}{2}z^2\sum_{i=1}^{n}L_i r_i(t) + y^2\sum_{i=1}^{n}\lambda_i r_i(t) + z^2\sum_{i=1}^{n}\delta_i r_i(t) \\
&+ \left\{\frac{\mu}{2}c(t)\sum_{i=1}^{n}L_i + \frac{c(t)}{2}\sum_{i=1}^{n}L_i - \sum_{i=1}^{n}\lambda_i(1-r_i'(t))\right\}\int_{t-r_i(t)}^{t}y^2(\theta)d\theta ds \\
&+ \left\{\frac{\mu}{2}b(t)\sum_{i=1}^{n}N_i + \frac{b(t)}{2}\sum_{i=1}^{n}N_i - \sum_{i=1}^{n}\delta_i(1-r_i'(t))\right\}\int_{t-r_i(t)}^{t}z^2(\theta)d\theta ds.
\end{aligned}
$$

Therefore by using the assumptions $(i)-(v)$, we obtain

$$\begin{aligned}\frac{dV}{dt} \leq & \mu c'(t)\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c'(t)y\sum_{i=1}^{n}h_i(x) + \mu a'(t)\int_0^y f(\eta)\eta d\eta \\ & + \left\{c_0\sum_{i=1}^{n}l_i - \mu b_0\sum_{i=1}^{n}a_i + \frac{\mu}{2}b_1\gamma\sum_{i=1}^{n}N_i + \frac{\mu}{2}c_0\gamma\sum_{i=1}^{n}L_i + \gamma\sum_{i=1}^{n}\lambda_i\right\}y^2 \\ & + b'(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta + \left\{\mu - a_0a_1 + \frac{b_1}{2}\gamma\sum_{i=1}^{n}N_i + \frac{c_0}{2}\gamma\sum_{i=1}^{n}L_i + \gamma\sum_{i=1}^{n}\delta_i\right\}z^2 \\ & + \left\{\frac{c_0(\mu+1)\sum_{i=1}^{n}L_i}{2} - (1-\beta)\sum_{i=1}^{n}\lambda_i\right\}\int_{t-r_i(t)}^{t}y^2(s)ds \\ & + \left\{\frac{b_1(\mu+1)\sum_{i=1}^{n}N_i}{2} - (1-\beta)\sum_{i=1}^{n}\delta_i\right\}\int_{t-r_i(t)}^{t}z^2(s)ds.\end{aligned}$$

If we take

$$\sum_{i=1}^{n}\lambda_i = \frac{(\mu+1)c_0\sum_{i=1}^{n}L_i}{2(1-\beta)} \ \ and \ \ \sum_{i=1}^{n}\delta_i = \frac{b_1(\mu+1)\sum_{i=1}^{n}N_i}{2(1-\beta)},$$

and from assumptions (ii) and (iv), we can obtain

$$\mu a'(t)\int_0^y f(\eta)\eta d\eta \leq \mu\frac{a_3a_4}{2}y^2.$$

Then we find

$$\begin{aligned}\frac{dV}{dt} \leq & G - \left\{\mu b_0\sum_{i=1}^{n}a_i - c_0\sum_{i=1}^{n}l_i - \frac{\mu}{2}a_3a_4 - \frac{\mu}{2}b_1\gamma\sum_{i=1}^{n}N_i - \frac{\mu}{2}c_0\gamma\sum_{i=1}^{n}L_i\right. \\ & \left. - \gamma\sum_{i=1}^{n}\lambda_i\right\}y^2 - \left\{a_0a_1 - \mu - \frac{b_1}{2}\gamma\sum_{i=1}^{n}N_i - \frac{c_0}{2}\gamma\sum_{i=1}^{n}L_i - \gamma\sum_{i=1}^{n}\delta_i\right\}z^2,\end{aligned} \tag{3.3}$$

where

$$G = \mu c'(t)\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c'(t)y\sum_{i=1}^{n}h_i(x) + b'(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta.$$

Since $b'(t) \leq c'(t) \leq 0$ and $\int_0^y g_i(x,\eta)d\eta \geq 0$, we find the following two cases:
Case (1): if $c'(t) = 0$, it follows that

$$G = b'(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta \leq 0.$$

Case (2): if $c'(t) \neq 0$ and since $\frac{b'(t)}{c'(t)} < 1$, then we have

$$G \leq c'(t)\Big(\mu \sum_{i=1}^{n} \int_0^x h_i(\xi)d\xi + y\sum_{i=1}^{n} h_i(x) + \sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta\Big).$$

Since $\sup\{h_i'(x)\} = \sum_{i=1}^n l_i$; by (ii) and $\sum_{i=1}^n \frac{g_i(x,y)}{y} \geq \sum_{i=1}^n a_i$; by (i), it follows that

$$G \leq c'(t)\Big\{\frac{1}{2\sum_{i=1}^n a_i}\Big(y\sum_{i=1}^n a_i + \sum_{i=1}^n h_i(x)\Big)^2 + \int_0^x \Big(\mu - \frac{\sum_{i=1}^n h_i'(\xi)}{\sum_{i=1}^n a_i}\Big)\sum_{i=1}^n h_i(\xi)d\xi\Big\}.$$
$$\leq c'(t)\int_0^x \Big(\mu - \frac{\sum_{i=1}^n l_i}{\sum_{i=1}^n a_i}\Big)\sum_{i=1}^n h_i(\xi)d\xi.$$

Then we obtain

$$G \leq c'(t)\delta_1 \int_0^x \sum_{i=1}^n h_i(\xi)d\xi \leq 0.$$

By using the assumption (iv), and considering $\delta_1 = \mu - \frac{\sum_{i=1}^n l_i}{\sum_{i=1}^n a_i} > 0$, so that we can write (3.3) as the following

$$\frac{dV}{dt} \leq -\Big\{\frac{a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n l_i}{2} - \frac{\mu}{2}a_3a_4 - \frac{\mu(1-\beta)(b_1\sum_{i=1}^n N_i + c_0\sum_{i=1}^n L_i) + c_0(\mu+1)\sum_{i=1}^n L_i}{2(1-\beta)}\gamma\Big\}y^2$$
$$-\Big\{\frac{a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n L_i}{2b_0\sum_{i=1}^n a_i} - \frac{c_0(1-\beta)\sum_{i=1}^n L_i + (\mu-\beta+1+b_1)\sum_{i=1}^n N_i}{2(1-\beta)}\gamma\Big\}z^2.$$

Therefore, if

$$\gamma < min\Big[\frac{(1-\beta)(a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n l_i - \mu a_3a_4)}{2\Big\{\mu(1-\beta)(b_1\sum_{i=1}^n N_i + c_0\sum_{i=1}^n L_i) + c_0(\mu+1)\sum_{i=1}^n L_i\Big\}},$$
$$\frac{(1-\beta)(a_0a_1b_0\sum_{i=1}^n a_i - c_0\sum_{i=1}^n l_i)}{2b_0\sum_{i=1}^n a_i\Big\{(1-\beta)c_0\sum_{i=1}^n L_i + (\mu-\beta+1+b_1)\sum_{i=1}^n N_i\Big\}}\Big].$$

Thus for the positive constant D_1, we obtain

$$\frac{dV}{dt} \leq -D_1(y^2+z^2). \tag{3.4}$$

Because of $\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative, then from equation (3.2), we have

$$V(x_t,y_t,z_t) \geq \mu c(t)\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c(t)y\sum_{i=1}^{n}h_i(x) + \mu a(t)\int_0^y f(\eta)\eta d\eta$$
$$+ b(t)\sum_{i=1}^{n}\int_0^y g_i(x,\eta)d\eta + \mu yz + \frac{1}{2}z^2.$$

From the assumptions $(i)-(v)$, we get

$$V(x_t,y_t,z_t) \geq \mu c_1\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi + c_1 y\sum_{i=1}^{n}h_i(x) + \frac{\mu a_1 a_0}{2}y^2 + \frac{b_0}{2}\sum_{i=1}^{n}a_i y^2$$
$$+ \mu yz + \frac{1}{2}z^2.$$

Now we can write the previous inequality as the following form

$$V(x_t,y_t,z_t) \geq \frac{1}{2b_0\sum_{i=1}^{n}a_i}\left(b_0 y\sum_{i=1}^{n}a_i + c_1\sum_{i=1}^{n}h_i(x)\right)^2 + \frac{1}{2}(\mu y + z)^2$$
$$+ \int_0^x\left\{c_1\mu - \frac{c_1^2\sum_{i=1}^{n}h_i'(\xi)}{b_0\sum_{i=1}^{n}a_i}\right\}\sum_{i=1}^{n}h_i(\xi)d\xi + \frac{\mu}{2}(a_1a_0 - \mu)y^2.$$

By using the assumptions $(i)-(iii)$ and since $a_0a_1 - \mu = \frac{a_0a_1b_0\sum_{i=1}^{n}a_i - c_0\sum_{i=1}^{n}l_i}{2b_0\sum_{i=1}^{n}a_i} > 0$, then we obtain

$$V(x_t,y_t,z_t) \geq \int_0^x\left\{c_1\mu - \frac{c_1^2\sum_{i=1}^{n}l_i}{b_0\sum_{i=1}^{n}a_i}\right\}\sum_{i=1}^{n}h_i(\xi)d\xi + \frac{\delta_2}{2}(y^2+z^2).$$

Suppose that $\delta_3 = c_1\mu - \frac{c_1^2\sum_{i=1}^{n}l_i}{b_0\sum_{i=1}^{n}a_i} > 0$, by the assumptions $b_0 > c_1$ and $c_0 > 2c_1$, with the assumption (i), we have

$$V(x_t,y_t,z_t) \geq \frac{\delta_3}{2}x^2\sum_{i=1}^{n}b_i + \frac{\delta_2}{2}(y^2+z^2).$$

So we can write Lyapunov functional $V(x_t,y_t,z_t)$ as

$$V(x_t,y_t,z_t) \geq D_2(x^2+y^2+z^2), \tag{3.5}$$

where $D_2 = \min\{\frac{\delta_3}{2}\sum_{i=1}^{n}b_i, \frac{\delta_2}{2}\}$, D_2 is a positive constant.
Since $|\sum_{i=1}^{n}h_i'(x)| \leq \sum_{i=1}^{n}L_i$, $|\sum_{i=1}^{n}\frac{\partial g_i(x,y)}{\partial y}| \leq \sum_{i=1}^{n}N_i$ and $h_i(0) = g_i(x,0) =$

0, then by using the mean-value theorem we can write equation (3.2) as the following

$$V(x_t,y_t,z_t) \leq \mu c(t)\sum_{i=1}^{n}\int_0^x L_i\xi d\xi + c(t)\sum_{i=1}^{n} L_i xy + \mu yz + \mu a(t)\int_0^y f(\eta)\eta d\eta + b(t)\sum_{i=1}^{n} N_i\int_0^y \eta d\eta + \frac{1}{2}z^2 + \sum_{i=1}^{n}\lambda_i\int_{t-r_i(t)}^{t}\Big(\theta - t + r_i(t)\Big)y^2(\theta)d\theta + \sum_{i=1}^{n}\delta_i\int_{t-r_i(t)}^{t}\Big(\theta - t + r_i(t)\Big)z^2(\theta)d\theta.$$

From the assumptions $(i)-(v)$ and by using the inequality $uv \leq \frac{1}{2}(u^2+v^2)$, we have

$$V(x_t,y_t,z_t) \leq \frac{\mu c_0}{2}x^2\sum_{i=1}^{n} L_i + \frac{c_0}{2}x^2\sum_{i=1}^{n} L_i + \frac{c_0}{2}y^2\sum_{i=1}^{n} L_i + \frac{\mu a_2 a_3}{2}y^2 + \frac{b_1}{2}y^2\sum_{i=1}^{n} N_i + \frac{\mu}{2}y^2 + \frac{\mu}{2}z^2 + \frac{1}{2}z^2 + \frac{\gamma^2}{2}\|y\|^2\sum_{i=1}^{n}\lambda_i + \frac{\gamma^2}{2}\|z\|^2\sum_{i=1}^{n}\delta_i.$$

So that the above inequality becomes as

$$V(x_t,y_t,z_t) \leq \frac{1}{2}\Big\{c_0(\mu+1)\sum_{i=1}^{n} L_i\Big\}\|x\|^2 + \frac{1}{2}\Big\{c_0\sum_{i=1}^{n} L_i + b_1\sum_{i=1}^{n} N_i + \mu(a_2a_3+1) + \gamma^2\sum_{i=1}^{n}\lambda_i\Big\}\|y\|^2 + \frac{1}{2}\Big\{(\mu+1) + \gamma^2\sum_{i=1}^{n}\delta_i\Big\}\|z\|^2.$$

Then we obtain

$$V(x_t,y_t,z_t) \leq D_3(x^2+y^2+z^2),\ D_3 > 0. \tag{3.6}$$

Then, from inequalities (3.4), (3.5) and (3.6), we conclude that all the assumptions of Theorem 2.1 are satisfied, so that the zero solution of equation (1.1) is uniformly stable. Hence, the proof of Theorem 3.1 is now complete.

Remark 3.1 *If we consider equation* (1.1) *is a differential equation with a deviating arguments, and if we let* $f(\dot{x}) = 1$ *and* $g(x(t-r(t)), \dot{x}(t-r(t))) = g(x(t-r(t)))$, *we find that the results of the equation discussed by Mahmoud* [19].

Example 3.1. Consider the non-autonomous third-order nonlinear functional differential equation with multiple deviating arguments as:

$$\dddot{x} + (4+\sin t)(\dot{x})^2\ddot{x} + \left(\frac{1}{4} + \frac{1}{t^3+2}\right)\sum_{i=1}^{n}\left[3x(t-r_i(t)) + \frac{x(t-r_i(t))}{2+\sin t}\right]$$
$$+\left(2+\frac{1}{t^3+1}\right)\sum_{i=1}^{n}\left[4\dot{x}(t-r_i(t)) + \frac{\dot{x}(t-r_i(t))}{2+|x(t-r_i(t))|+|\dot{x}(t-r_i(t))|}\right] = 0. \tag{3.7}$$

The previous equation is equivalent to the system

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= -(4+\sin t)y^2 z - \left(2+\frac{1}{t^3+1}\right)\left(4y + \frac{y}{2+|x|+|y|}\right)\\
&\quad + \left(2+\frac{1}{t^3+1}\right)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left[\frac{-y}{(2+|x|+|y|)^2}\right]y(s)ds\\
&\quad + \left(2+\frac{1}{t^3+1}\right)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left[4+\frac{2+x}{(2+|x|+|y|)^2}\right]z(s)ds\\
&\quad - \left(\frac{1}{4}+\frac{1}{t^3+2}\right)\left(3x + \frac{x}{2+\sin t}\right)\\
&\quad + \left(\frac{1}{4}+\frac{1}{t^3+2}\right)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left[3+\frac{1}{2+\sin s}\right]y(s)ds.
\end{aligned}$$

Then, we can write the function

$$a(t) = 4+\sin t,\quad a_1 = 3 \le a(t) \le 4 = a_2, a'(t) = \cos t,\ -1 \le a'(t) \le 1.$$

Also the function

$$f(y) = y^2, a_0 = 2 \le y^2 \le 4 = a_3,$$

and

$$b(t) = (2+\frac{1}{t^3+1}),\ b_0 = 2 \le b(t) \le 2.5 = b_1,$$

therefore we have

$$b'(t) = \frac{-3t^2}{(t^3+1)^2} \le 0.$$

Also

$$c(t) = (\frac{1}{4}+\frac{1}{t^3+2}),\ c_1 = \frac{1}{4} \le\ c(t) \le \frac{3}{4} = c_0,$$

therefore we have

$$c'(t) = \frac{-3t^2}{(t^3+2)^2} \leq 0,$$

it is clear that

$$b'(t) \leq c'(t) \leq 0.$$

Therefore from the assumption (ii), we have $b_0 = 2 > c_1 = \frac{1}{4}$ and $c_0 = \frac{3}{4} > 2c_1 = \frac{1}{2}$.
Now, let the function

$$\sum_{i=1}^{n} g_i(x,y) = \sum_{i=1}^{n} \left[4y + \frac{y}{(2+|x|+|y|)}\right],$$

then, we have

$$\sum_{i=1}^{n} \frac{g_i(x,y)}{y} = \sum_{i=1}^{n} \left[4 + \frac{1}{(2+|x|+|y|)}\right] \geq 4 = \sum_{i=1}^{n} a_i.$$

It follows that the derivative of this function in terms of x is

$$\sum_{i=1}^{n} \frac{\partial g_i(x,y)}{\partial x} = \sum_{i=1}^{n} \left[\frac{-y}{(2+|x|+|y|)^2}\right] \leq 0,$$

and in terms of y is

$$\sum_{i=1}^{n} \frac{\partial g_i(x,y)}{\partial y} = \sum_{i=1}^{n} \left[4 + \frac{2+x}{(2+|x|+|y|)^2}\right] \leq 4.5 = \sum_{i=1}^{n} N_i.$$

Finally, the function

$$\sum_{i=1}^{n} h_i(x) = \sum_{i=1}^{n} \left[3x + \frac{x}{2+\sin t}\right],$$

and

$$\sum_{i=1}^{n} \frac{h_i(x)}{x} = \sum_{i=1}^{n} \left[3 + \frac{1}{2+\sin t}\right] \geq 3 = \sum_{i=1}^{n} b_i.$$

Therefore the derivative in terms of x becomes

$$\sum_{i=1}^{n} \frac{\partial h_i(x)}{\partial x} = \sum_{i=1}^{n} \left[3 + \frac{1}{2+\sin t}\right] \leq 4 = \sum_{i=1}^{n} L_i.$$

Then $\sup\{h_i'(x)\} = 4 = \sum_{i=1}^{n} l_i$.
Then we have

$$a_0a_1b_0\sum_{i=1}^{n} a_i - c_0\sum_{i=1}^{n} l_i = 48 - 3 = 45 > 0.$$

Thus all the assumptions $(i) - (vi)$ of Theorem 3.1 are satisfied.
The following is the second main result of boundedness of solutions for (1.1) for $p \neq 0$.

Theorem 3.2 *In addition to the assumptions imposed on the functions that appeared in equation* (1.1), *we have the assumption*

$$|p(t, x, \dot{x}, \ddot{x}, x(t - r(t)))| \leq q(t),$$

where $\max\{q(t)\} < \infty$ *and* $q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is space of integrable Lebesgue functions. Then, there exists a finite positive constant k such that the solution $x(t)$ of equation* (1.1) *defined by initial functions*

$$x(t) = \phi(t),\ \ \dot{x}(t) = \dot{\phi}(t),\ \ddot{x}(t) = \ddot{\phi}(t)$$

satisfies the inequalities

$$|x(t)| \leq k,\ \ |\dot{x}(t)| \leq k,\ |\ddot{x}(t)| \leq k,$$

for all $t \geq t_0$, where $\phi \in C^2([t_0 - r, t_0], \mathbb{R})$.

Proof.
Now, if $p \neq 0$, then the equation (1.1) is equivalent to the following system

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= -a(t)f(y)z - b(t)\sum_{i=1}^{n} g_i(x,y) + b(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\frac{\partial g_i(x(s),y(s))}{\partial x}y(s)ds\\
&\quad + b(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\frac{\partial g_i(x(s),y(s))}{\partial y}z(s)ds - c(t)\sum_{i=1}^{n} h_i(x)\\
&\quad + c(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} h_i'(x(s))y(s)ds + p(t,x,y,z,x(t-r(t))).
\end{aligned} \tag{3.8}$$

From the assumptions $(i) - (vi)$ of Theorem 3.1 and the equation (3.2), we get

$$\begin{aligned}
\frac{dV(x_t,y_t,z_t)}{dt} &\leq -D_1(y^2+z^2) + (\mu y + z)p(t,x,y,z,x(t-r(t)))\\
&\leq (\mu|y| + |z|)|p|\\
&\leq D_3(|y| + |z|)q(t),
\end{aligned}$$

where $D_3 = \min(\mu, 1)$.
Therefore, we obtain

$$\frac{dV(x_t, y_t, z_t)}{dt} \le D_3(|y| + |z|)q(t).$$

Since by the inequalities $|y| < 1 + y^2$, $|z| < 1 + z^2$, then we get

$$\frac{dV(x_t, y_t, z_t)}{dt} \le D_3(2 + y^2 + z^2)q(t). \tag{3.9}$$

From the inequality (3.5), we have $y^2 + z^2 \le D_2^{-1}V$.
Then the inequality (3.9) becomes

$$\frac{dV(x_t, y_t, z_t)}{dt} \le D_3(2 + D_2^{-1}V)q(t),$$

and by integrating the last inequality from 0 to t, therefore we get

$$V(x_t, y_t, z_t) \le V(x_0, y_0, z_0) + 2D_3 \int_0^t q(s)ds + D_3 D_2^{-1} \int_0^t Vq(s)ds,$$

since $q(t) \in L^1(0, \infty)$ and by using the Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned} V(x_t, y_t, z_t) &\le \left[V(x_0, y_0, z_0) + 2D_3 \int_0^\infty q(s)ds\right] \exp\left(D_3 D_2^{-1} \int_0^\infty q(s)ds\right) \\ &= k_1 < \infty, \end{aligned}$$

for $k_1 > 0$.

Again, since $V(x_t, y_t, z_t) \ge D_2(x^2 + y^2 + z^2)$; by (3.5), then we have

$$x^2 + y^2 + z^2 \le D_2^{-1}V \le D_2^{-1}k_1 = K.$$

Thus we conclude

$$|x(t)| \le K, \quad |\dot{x}(t)| \le K, \quad |\ddot{x}(t)| \le K, \text{ for all } t \ge t_0.$$

Therefore the proof of Theorem 3.2 is now complete.
Example 3.2. Consider the third-order nonlinear functional differential equation as

$$\begin{aligned} &\dddot{x} + (4 + \sin t)(\dot{x})^2\ddot{x} + \left(\frac{1}{4} + \frac{1}{t^3 + 2}\right) \sum_{i=1}^{n} \left[3x(t - r_i(t)) + \frac{x(t - r_i(t))}{2 + \sin t}\right] \\ &+ \left(2 + \frac{1}{t^3 + 1}\right) \sum_{i=1}^{n} \left[4\dot{x}(t - r_i(t)) + \frac{\dot{x}(t - r_i(t))}{2 + |x(t - r_i(t))| + |\dot{x}(t - r_i(t))|}\right] \\ &= \frac{1}{4 + t^2 + x^2(t) + y^2(t) + z^2(t) + x^2(t - r(t))}. \end{aligned} \tag{3.10}$$

Then, the function

$$p = \frac{1}{4 + t^2 + x^2(t) + y^2(t) + z^2(t) + x^2(t - r(t))} \leq \frac{1}{4 + t^2} = q(t),$$

for all $t \in \mathbb{R}^+$.
It follows that

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{4 + s^2} ds = \frac{\pi}{4} < \infty,$$

then $q(t) \in L^1(0, \infty)$.
Since

$$\mu = \frac{a_0 a_1 b_0 \sum_{i=1}^n a_i + c_0 \sum_{i=1}^n l_i}{2b_0 \sum_{i=1}^n a_i} = \frac{51}{16},$$

then, we obtain

$$\frac{dV(x_t, y_t, z_t)}{dt} \leq D_3(|y| + |z|)\frac{1}{4 + t^2},$$

where $D_3 = \min(\frac{51}{16}, 1) = 1$.
Therefore, we get

$$\frac{dV(x_t, y_t, z_t)}{dt} \leq \frac{2 + y^2 + z^2}{4 + t^2} \leq \frac{2}{4 + t^2} + \frac{D_2^{-1} V}{4 + t^2}.$$

By integrating the previous inequality from 0 to t, using the fact that $\frac{1}{4+t^2} \in L^1(0, \infty)$, we have

$$V(x_t, y_t, z_t) \leq (V_0 + \frac{\pi}{2}) \exp{(D_2^{-1} \frac{\pi}{4})} < \infty.$$

So, we can conclude the boundrdness of all solutions of the equation (3.10).

4 Conclusion

We know that the differential equations of third-order play extremely important and useful roles in many scientific areas such as atomic energy, biology, chemistry, control theory, economy, engineering, information theory, mechanics, medicine, physics, etc. Sufficient conditions for stability and boundedness of solutions of third-order functional differential equation with multiple delays were established. The appropriate Lyapunov functional is used to obtain the results. The results of this paper improve and complement existing results in the literature.

Acknowledgments

The author is grateful to the anonymous referee(s) and the handling editor for their valuable comments and useful suggestions that have improved the quality of this work.

References

[1] Abou-El-Ela, A. M. A., Sadek, A. I. and Mahmoud, A. M., Stability and boundedness of solutions of certain third-order nonlinear delay differential equation, *ICGST International Journal on Automatic Control and System Engineering*, 9(1), 9-15, 2009.

[2] Ademola, A. T. and Arawomo, P. O., Uniform stability and boundedness of solutions of nonlinear delay differential equations of third-order, *Math. J. Okayama Univ.*, 55, 157-166, 2013.

[3] Ademola, A. T., Arawomo, P. O., Ogunlaran, O. M. and Oyekan, E. A., Uniform stability, boundedness and asymptotic behaviour of solutions of some third-order nonlinear delay differential equations, *Differential Equations and Control Processes*, 4, 43-66, 2013.

[4] Ademola, A. T., Ogundare, B. S. and Adesina, O. A., Stability, boundedness, and existence of periodic solutions to certain third-order delay differential equations with multiple deviating arguments, *International Journal of Differential Equations*, 2015,1-12.

[5] Afuwape, A. U. and Omeike, M. O., On the stability and boundedness of solutions of a kind of third-order delay differential equations, *Appl. Math. Comput.*, 200(1), 444-451, 2008.

[6] Arino, O., Hbid, M. L. and Dads, E. A., *Delay Differential Equations and Applications*, Springer, P.O. Box 17, 3300 AA Dordrecht, The Netherlands, 2006.

[7] Burton, T. A., *Voltera Integral and Differential Equations*, Academic Press, 1983.

[8] Burton, T. A., *Stabitity and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, 1985.

[9] Burton, T. A., Asymptotic stability for functional differential equations, *Acta Math. Hungar*, 65, 243-251, 1994.

[10] Burton, T. A. and Hering, R. H., Lyapunov theory for functional differential equations, Rocky Mountain J. Math., 24, 3-17, 1994.

[11] Deo, S. G. and Raghavendra, V., *Ordinary Differential Equations and Stability Theory*, Tata Mc Graw-Hill Publishing Company Limited New Delhi, 1980.

[12] Green, K. and Wagenknechtn, T., Spectra and pseudospectra of neutral delay differential equations with application to real-time substructuring, Meccanica, *Springer Science and Business Media B.V.*, 45, 249-263, 2010.

[13] Hale, J. K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[14] Iannelli, M., *Mathematics of Biology*, Springer-Verlag Berlin Heidelberg, 2010.

[15] Korkmaz, E. and Tunç, C., Stability and boundedness to certain differential equations of fourth-order with multiple delays, *Faculty of Science and Mathematics of Nis, Serbia,* 5(2014), 1049-1058, 2014.

[16] Korkmaz, E and Tunç, C., On some qualitative behaviour of certain differential equations of fourth-order with multiple retardations, *Journal of Applied Analysis and Computation*, 6(2), 336-349, 2016.

[17] Krasovskii, N. N., *Stability of Motion*, Stanford University Press, 1963.

[18] Lyapunov, A. M., *Stability of Motion*, Academic Press, New York, NY, USA, 1966.

[19] Mahmoud, A. M., On the asymptotic stability of solutions for a certain non-autonomous third-order delay differential equation, *British Journal of Mathematics and Computer Science,* 16(3), 1-12, 2016.

[20] Omeike, M. O., Stability and boundedness of solutions of some non-autonomous delay differential equations of the third-order, *An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S)*, 55(1), 49-58, 2009.

[21] Omeike, M. O., New results on the stability of solutions of some non-autonomous delay differential equations of the third-order, *Differential Equations and Control Processes*, 1, 18-29, 2010.

[22] Omeike, M. O., Stability and boundedness of solutions of a certain system of third-order nonlinear delay differential equations, *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, 54(1), 109-119, 2015.

[23] Remili, M. and Beldjerd, D., On the asymptotic behavior of the solutions of third-order delay differential equations, *Rend. Circ. Mat. Palermo*, 63(3), 447-455, 2014.

[24] Remili, M. and Oudjedi, L. D., Stability and boundedness of the solutions of non autonomous third-order differential equations with delay, *Agta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, 53(2), 139-147, 2014.

[25] Remili, M. and Oudjedi, L. D., Uniform ultimate boundedness and asymptotic behaviour of third-order nonlinear delay differential equation, *African Mathematical*, Springer-Verlag, Berlin, Heidelberg, 27, 1227-1237, 2016.

[26] Remili, M. and Oudjedi, L. D., Boundedness and stability in third-order nonlinear differential equations with multiple deviating arguments, *Archivum Mathematicum (BRNO)*, 52(2), 79-90, 2016.

[27] Remili, M., Oudjedi, L. D. and Beldjerd, D., Stability and ultimate boundedness of solutions of some third-order differential equation with delay, *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 23, 90-95, 2017.

[28] Sadek, A. I., Stability and boundedness of kind of third-order delay differential equations, *Appl. Math. Lett.*, 16(5), 657-662, 2003.

[29] Sadek, A. I., On the stability of solutions of certain fourth-order delay differential equations, *Appl. Math. comput.*, 148, 587-597, 2004.

[30] Sadek, A. I., On the stability of solutions of some non-autonomous delay differential equations of the third-order, *Asymptotic Anal.*, 43, 1-7, 2005.

[31] Smith, H., *An Introduction to Delay Differential Equations with Sciences Applications to the Life Sciences*, Springer Science and Business Media, LLC., 2011.

[32] Tunç, C., New results about stability and boundedness of solutions of certain nonlinear third-order delay differential equations, *Arab. J. Sci. Eng.*, 31(2), 185-196, 2006.

[33] Tunç, C., Stability and boundedness of solutions of nonlinear differential equations of third-order with delay, *Differential Equations and Control Processes*, 3, 1-13, 2007.

[34] Tunç, C., On the stability and boundedness of solutions to third-order nonlinear differential equations with retarded argument, *J. Nonlinear Dyn.*, 57, 97-106, 2009.

[35] Tunç, C., Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments, *Elect. J. Qual. Theory Diff. Eqs.*, 1, 1-12, 2010.

[36] Tunç, C., Stability and boundedness of the nonlinear differential equations with multiple deviating arguments, *African Mathematical,* **24**, 381-390, 2013.

[37] Tunç, C., Stability and boundedness in differential systems of third-order with variable delay, *Proyecciones Journal of Mathematics*, 35(3), 317-338, 2016.

[38] Tunç, C. and Gözen, M., Stability and uniform boundedness in multidelay functional differential equations of third-order, *Abstr. Appl. Anal.*, 2013, 1-7.

[39] Yang, K., *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, 1993.

[40] Yoshizawa, T., *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, 1966.

[41] Zhu, Y., On stability, boundedness and existence of periodic solutions of a kind of third-order nonlinear delay differential system, *Ann. of Diff. Eqs.*, 8(2), 249-259, 1992.