DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 2, 2013
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru

On the Existence and Uniqueness of solution of Impulsive Quantum Stochastic Differential Equation

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Abstract

This paper is concerned with the existence and uniqueness of solutions of quantum stochastic differential equations (QSDE) subject to certain impulse effects. The QSDE in our work is within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus.

AMS Subject Classification: 81S25, 34A37.

Key Words and Phrases: Quantum stochastic processes,Impulse operators, fixed points.

1 Introduction

In the formulation of quantum stochastic calculus of Hudson-Parthasarathy[12], the quantum noises which are operator-valued quantum stochastic processes arose from quantum field operators; annihilation, creation and gauge operators. In their work the existence of unitary solution of the resulting quantum stochastic differential equation which was a non-commutative generalization of the famous Ito stochastic differential equation was established. The problem of the existence of solution for a discontinuous case was established via the differential inclusions in [8]. The multivalued stochastic processes considered in [8]

lie in certain locally convex spaces and the cases involving other locally convex spaces were established in [9]. Some properties of the solution sets of quantum stochastic differential inclusions were established in [2], [3] and [4]. Results on the strong solutions of quantum stochastic differential equations were established in [5] on a more general regularity condition.
The aim of this work is to consider the quantum stochastic differential equation which are subject to certain impulse effects which is a situation that arises more often in quantum physics. For instance sudden occurrences in non-deterministic quantum optics can be modeled as a problem of impulsive quantum stochastic differential equations. The recurrence of such problems is the motivation for this work and hence will be a further extension of the results on QSDE in the literatures. In the classical differential equation, the existence of solution of impulsive differential equations and inclusions of various kinds had been established by authors, see [6], [1], [11] and the references cited in them.
By using fixed point theorem, we shall establish the existence and uniqueness of a mild solution of quantum stochastic differential equation subject to impulse effects. The QSDE which shall be considered will follow the Hudson-Parthasarathy formulation of quantum stochastic calculus and domain of our quantum stochastic processes will follow the ones in [8].

2 Preliminaries

In this subsection we shall introduce the notations and definitions on Quantum stochastic differential equations as applicable in subsequent sections.

2.1 Notations and Definitions

Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; γ is a fixed Hilbert and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable γ-valued maps on $\mathbb{R}_+$.
The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\| \, . \, \|$ the norm induced by $\langle .,. \rangle$.
Let $\mathbb{E}$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$. We define the locally convex space $\mathcal{A}$ of noncommutative stochastic processes whose topology τ_w, is generated by the family of seminorms $\{\| \; x \; \|_{\eta\xi} = | \; \langle \eta, x\xi \rangle \; |, x \in \mathcal{A}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. The completion of $(\mathcal{A}, \tau_w)$ is denoted by $\widetilde{\mathcal{A}}$. The underlying elements of $\widetilde{\mathcal{A}}$ consist of linear maps from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D}\underline{\otimes}\mathbb{E}$. For a

fixed Hilbert space γ, the spaces $L^p_{loc}(\widetilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^p_{loc}(I\times\widetilde{\mathcal{A}})$ are adopted as in [8].
For a topological space $\mathcal{N}$, let $clos(\mathcal{N})$ be the collection of all nonempty closed subsets of $\mathcal{N}$; we shall employ the Hausdorff topology on $clos(\widetilde{\mathcal{A}})$ as defined in [8]. Moreover, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, we define the Hausdorff distance, $\rho(A, B)$ as :

$$\mathbf{d}(x,B) \equiv \inf_{y\in B} \mid x-y \mid, \qquad \delta(A,B) \equiv \sup_{x\in A} \mathbf{d}(x,B)$$
$$\text{and } \rho(A,B) \equiv \max(\delta(A,B), \delta(B,A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.
By a stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$, we mean a function on I with values in $clos(\widetilde{\mathcal{A}})$.
A stochastic process Φ will be called
(i) adapted if $\Phi(t) \subseteq \widetilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$; (iii) locally absolutely p-integrable if $t \mapsto \parallel \Phi(t) \parallel_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\widetilde{\mathcal{A}})$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$
The set of all absolutely p-integrable stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathcal{A}})$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \widetilde{\mathcal{A}})$ is the set of maps $\Phi : I \times \widetilde{\mathcal{A}} \to clos(\widetilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathcal{A}})$.
Consider stochastic processes $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})$ and $(0, x_0)$ be a fixed point in $[0, T] \times \widetilde{\mathcal{A}}$. Then, a relation of the form

$$\begin{aligned} X(t) = x_0 + \int_0^t &(E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) \\ &+ G(s,X(s))dA^+_g(s) + H(s,X(s))ds, \quad t\in[0,T]). \end{aligned}$$

will be called a stochastic integral equation driven by operator-valued stochastic processes annihilation, creation and gauge operators.
The stochastic differential equation corresponding to the integral equation above is;

$$\begin{aligned} dX(t) &= E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) \\ &\quad + G(t,X(t))dA^+_g(t) + H(t,X(t))dt \\ X(0) &= x_0 \ \text{ almost all } \ t\in[0,T] \end{aligned} \tag{2.1}$$

Let $\mathbb{P} : [0, T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ be sesquilinear form valued stochastic process defined in [8] in terms of E, F, G, H by using the matrix elements in Hudson and Parthasarathy quantum stochastic calculus [12], it was established that problem (2.1) is equivalent to

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &= \mathbb{P}(t, X(t))(\eta, \xi) \\ \langle \eta, X(0)\xi\rangle &= \langle \eta, x_0\xi\rangle \quad \text{for almost all } t \in [0, T] \end{aligned} \tag{2.2}$$

In what follows, if U is a topological space, we denote by clos(U), the collection of all non-empty closed subsets of U.
As explained in [8], the map $\mathbb{P}$ cannot in general be written in the form:

$$\mathbb{P}(t, x)(\eta, \xi) = \widetilde{\mathbb{P}}(t, \langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
The notion of solution of (2.1) or equivalently (2.2) is defined as follows:

Definition 2.1 *By a solution of (2.1) or equivalently* (2.2)*, we mean a stochastic process* $\varphi \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$ *such that*

$$\begin{aligned} d\varphi(t) &= E(t, \varphi(t))d\Lambda_\pi(t) + F(t, \varphi(t))dA_f(t) \\ &\quad + G(t, \varphi(t))dA_g^+(t) + H(t, \varphi(t))dt \quad \textit{almost all } t \in I \\ \varphi(t_0) &= \varphi_0 \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle &= \mathbb{P}(t, \varphi(t))(\eta, \xi) \\ \varphi(t_0) &= \varphi_0 \end{aligned}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *, almost all* $t \in I$.

Let $I_0 = [0, t_1]$, $I_1 = (t_1, t_2]$, ..., $I_m = (t_m, b]$

$$\{I_k = (t_k, t_{k+1}]; k = 1, 2, ..., m\}, \quad I' = I \setminus \{t_1, t_2, ..., t_m\}, \quad t_0 = 0, \quad t_{m+1} = b$$

$$\begin{aligned} PC(I, \widetilde{\mathcal{A}}) = &\{x : I \to \widetilde{\mathcal{A}} : x(t) \quad \text{is continuous everywhere except for some } \; t_k \\ &\text{at which } x(t_k^-) \text{ and } x(t_k^+), k = 1, ..., m \text{ exist and } x(t_k^-) = x(t_k)\}. \end{aligned}$$

and

$$\begin{aligned} PC^1(I, \widetilde{\mathcal{A}}) = &\{x : I \to \widetilde{\mathcal{A}} : x(t) \quad \text{is continuous differentiable everywhere except} \\ &\text{for some } t_k \; \text{ at which } x'(t_k^-) \text{ and } x'(t_k^+), k = 1, ..., m \\ &\text{exist and } x'(t_k^-) = x'(t_k)\} \end{aligned}$$

The sesquilinear equivalent forms $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ and $PC^1(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ are defined in a similar manner with $x : I \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$. $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ equip with the norm

$$\| x \|_{PC} = \sup\{| x(t)(\eta, \xi) | : t \in I\}$$

is a Banach space.
Let $I = [0, b]$ and $0 < t_1 < ... < t_{m+1} = b$. Let A be the infinitesimal generator of a family of semigroup $\{T(t) : t \geq 0\}$, we consider the existence of solution of the quantum stochastic evolution problem

$$\begin{aligned} dx(t) &= A(t)x(t) + (E(t, x(t))d\Lambda_\pi(t) + F(t, x(t))dA_f(t) \\ &\quad + G(t, x(t))dA_g^+(t) + H(t, x(t))dt), \\ &\quad \text{almost all } \; t \in I = [0, b], \; t \neq t_k, k = 1, ..., m \\ \triangle x \mid_{t=t_k} &= J_k(x(t_k^-)), \quad k = 1, ..., m \\ x(0) &= x_0. \end{aligned} \tag{2.3}$$

$J_k \in C(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}})$, $(k = 1, 2, ..., m)$ and $\triangle x \mid_{t=t_k} = x(t_k^+) - x(t_k^-)$

Remark 2.1 *This work would have applications in the theory of quantum continuous measurements. For instance in quantum optics, if the mean number of photons up to time t_i is momentary then we have impulses on the counting stochastic processes associated with the observables $X(t_i)$ concerned.*

3 Main Results

By a mild solution to (2.3) we mean an adapted stochastic process $x \in P(I, \widetilde{\mathcal{A}})$ which is a solution to the impulsive stochastic integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\Big(E(s, x(s))d\Lambda_\pi(s) + F(s, x(s))dA_f(s) + G(s, x(s))dA_g^+(s) + H(s, x(s))ds\Big) + \sum_{0<t_k<t} T(t - t_k)J_k(x(t_k))$$

In this section we establish the existence of mild solution of (2.3) via a fixed point method.

Theorem 3.1 *: Suppose that $\Phi \in \{E, F, G, H\} : [0, b] \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ are quantum stochastic processes such that $x(t) \mapsto \Phi(t, x(t))$ is continuous and $t \mapsto \Phi(t, x(t))$*

is L^1-measurable. Assume that
(i) there exist constants c_k such that $\| J_k(x) \| \leq c_k$, $k = 1, 2, 3, ..., m$ for each $x \in \widetilde{\mathcal{A}}$
(ii) there exists a constant M such that $\| T(t) \|_{\eta\xi} \leq M$ for each $t \geq 0$
(iii) there exists a continuous nondecreasing function

$\Psi : [0, \infty] \to (0, \infty)$ *and* $p^\Phi \in L^1(I, \mathbb{R}_+)$ *such that* $\| \Phi(t, x) \|_{\eta\xi} \leq p^\Phi(t)\Psi(\| x \|_{\eta\xi})$

for a.e. $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $x \in \widetilde{\mathcal{A}}$ with $\int_0^b m^\Phi(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}$ where

$$m^\Phi(s) = Mp^\Phi(s), \qquad c = M\left[\| x_0 \|_{\eta\xi} + \sum_{k=1}^m c_k \right] \quad I = [0, b]$$

for each $B \subseteq PC(I, \widetilde{\mathcal{A}})$ and $t \in I$ such that $\sup_{t\in I} \| x(t) \|_{\eta\xi} < \infty$, the set

$$\left\{ T(t)x_0 + \int_0^t T(t-s)\Phi(s, x(s))ds + \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k^-)) : x \in B \right\}$$

is relatively compact in $\widetilde{\mathcal{A}}$. Then the problem has at least one mild solution.

Proof: By following the transformation to the sesquilinear-form problem stated above; if $\Phi \in \{E, F, G, H\}$ are L^1-Caratheodory, then $\mathbb{P}$ is L^1-Caratheodory and for a continuous non-decreasing function ψ and $p \in L^1(I, \mathbb{R}_+)$ where $p = max\{p^\Phi\}$,

$$| \mathbb{P}(t, x)(\eta, \xi) | \leq p(t)\psi(\| x \|_{\eta\xi})$$

for a.e. $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $x \in \widetilde{\mathcal{A}}$ with $\int_0^b m(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}$ where $m(s) = Mp(s)$. We will transform the problem to a fixed point problem. Consider the problem $N : PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E})) \to PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ defined by

$$\begin{aligned} N(x)(t)(\eta, \xi) &= \langle \eta, T(t)x_0\xi \rangle \\ &\quad + \int_0^t T(t-s)\Phi(s, x(s))(\eta, \xi)ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k^-))(\eta, \xi). \end{aligned} \tag{3.1}$$

We shall show that N is continuous and maps compact subset to a relatively compact subset in $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ and it preserves convexity. Then we conclude by Schauder-Tychonov's theorem that N has a fixed point which is a mild solution of the problem. Let x_n be a sequence in $PC(I, \widetilde{\mathcal{A}})$ such that $x_n \to x$, for each $t \in I$

$$\begin{aligned} N(x_n)(t)(\eta,\xi) = \langle \eta, T(t)x_0\xi\rangle \\ &+ \int_0^t T(t-s)\Phi(s,x_n(s))(\eta,\xi)ds \\ &+ \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k^-))(\eta,\xi). \end{aligned}$$

Then we have

$$\begin{aligned} \mid N(x_n)(t)(\eta,\xi) - N(x)(t)(\eta,\xi) \mid &\leq \int_0^t \mid T(t-s) \mid\mid \mathbb{P}(s,x_n(s))(\eta,\xi) \\ &- \mathbb{P}(s,x(s))(\eta,\xi) \mid ds \\ &+ \sum_{0<t_k<t} \mid T(t-t_k) \mid\mid J_k(x_n(t_k^-))(\eta,\xi) \\ &- J_k(x(t_k^-))(\eta,\xi) \mid \\ &\leq M \int_0^t \mid \mathbb{P}(s,x_n(s))(\eta,\xi) \\ &- \mathbb{P}(s,x(s))(\eta,\xi) \mid ds \\ &+ M \mid J_k(x_n(t_k^-))(\eta,\xi) - J_k(x(t_k^-))(\eta,\xi) \mid . \end{aligned}$$

Since $J_k,\ k = 1, 2, ..., m$ are continuous and $\mathbb{P}$ is an L^1-Caratheodory stochastic process, we have by Lebesgue dominated convergence theorem that

$$\begin{aligned} \parallel N(x_n) - N(x) \parallel &\leq M \int_0^t \mid \mathbb{P}(s,x_n(s))(\eta,\xi) - \mathbb{P}(s,x(s))(\eta,\xi) \mid ds \\ &+ M \mid J_k(x_n(t_k^-))(\eta,\xi) - J_k(x(t_k^-))(\eta,\xi) \mid \end{aligned}$$

So, $\parallel N(x_n) - N(x) \parallel \to 0$ as $n \to \infty$. Therefore N is continuous.
Let B be a bounded subset of $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$, for any $x \in B$,

$$\begin{aligned} \mid N(x)(t)(\eta,\xi) \mid &= \mid \langle \eta, T(t)x_0\xi\rangle + \int_0^t T(t-s)\mathbb{P}(s,x(s))(\eta,\xi)ds \\ &+ \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k^-))(\eta,\xi) \mid \\ &\leq M \mid x_0 \mid + M \int_0^b \mid \mathbb{P}(s,x(s))(\eta,\xi) \mid ds + M\sum_{k=1}^m c_k \end{aligned}$$

Since $\mathbb{P}$ is L^1-Caratheodory, for each $t \in I$ the last inequality is less than

$$M \mid x_0 \mid +M \int_0^b \varphi_{\eta\xi} ds + M \sum_{k=1}^{m} c_k := l$$

for some real constants l. Therefore, N maps bounded sets into bounded sets in $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$

We want to show that N maps bounded sets into equicontinuous sets in $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$. Let $\tau_1, \tau_2 \in I'$, $\tau_1 < \tau_2$ and $B_r(0) = \{x \in PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E})) : \| x \|_{PC} \leq r\}$.

Case 1; $t \neq t_i$.

Let $x \in B_r(0)$ then for each $t \in I$, we have

$$\begin{aligned} \mid N(x)(\tau_2)(\eta,\xi) - N(x)(\tau_1)(\eta,\xi) \mid \leq & \mid \langle \eta, (T(\tau_2) - T(\tau_1))x_0\xi \rangle \mid \\ & + \int_0^{\tau_1} \mid T(\tau_2 - s) - T(\tau_1 - s) \mid \varphi_{r,\eta\xi}(s) ds \\ & + \int_{\tau_1}^{\tau_2} \mid T(\tau_2 - s) \mid \varphi_{r,\eta\xi}(s) ds \\ & + \sum_{\tau_1 < t < \tau_2} c_k \mid T(\tau_2 - t_k) - T(\tau_1 - t_k) \mid \end{aligned}$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$ This proves the equicontinuity for the case where $t \neq t_i$

Case 2; $t = t_i^-$.

Let $\delta_1 > 0$ be fixed such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$, we have that

$$\begin{aligned} \mid N(x)(t_i)(\eta,\xi) - N(x)(t_i - h)(\eta,\xi) \mid \leq & \mid \langle \eta, (T(t_i) - T(t_i - h))x_0\xi \rangle \mid \\ & + \int_0^{t_i - h} \mid [T(t_i - h - s) \\ & - T(t_i - s)]\varphi_{r,\eta\xi}(s) \mid ds \\ & + \int_{t_i - h}^{t_i} \mid T(t_i - h - s)\varphi_{r,\eta\xi}(s) \mid ds \\ & + \sum_{k=1}^{i-1} \mid [T(t_i - h - t_k) \\ & - T(t_i - t_k)]J(x(t_k^-)) \mid \end{aligned}$$

The right hand side tends to zero as $h \to 0$.

Case 3; $t = t_i^+$.

Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. For $0 < h < \delta_2$, we have that

$$\begin{aligned} | N(x)(t_i+h)(\eta,\xi) - N(x)(t_i)(\eta,\xi) | \leq & | \langle \eta, (T(t_i+h) - T(t_i))x_0\xi \rangle | \\ & + \int_0^{t_i} | [T(t_i+h-s) \\ & - T(t_i-s)]\varphi_{r,\eta\xi}(s) | ds \\ & + \int_{t_i}^{t_i+h} | T(t_i-h)\varphi_{r,\eta\xi}(s) | ds \\ & + \sum_{0<t_k<t_i} | [T(t_i-h-t_k) \\ & - T(t_i-t_k)]J_k(x(t_k^-)) | \\ & + \sum_{t_i<t_k<t_{i+h}} | T(t_i-h-t_k)J_k(x(t_k^-)) | . \end{aligned}$$

The right hand side tends to zero as $h \to 0$. This and hypothesis (iv) together with Arzela-Ascoli implies that N is completely continuous.
The last step is to show that there exists $r > 0$ such that the set

$$R(N) = \{x \in PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E})) : x = \lambda Nx, 0 < \lambda < 1\} \subseteq \overline{B}_r(0).$$

i.e. we'll show that the set $R(N)$ is bounded.
Let $x \in R(N)$, then $x = \lambda N(x)$ for some $0 < \lambda < 1$. Thus for each $t \in I$;

$$\begin{aligned} x(t)(\eta,\xi) = & \lambda[\langle \eta, T(t)x_0\xi \rangle + \int_0^t T(t-s)\mathbb{P}(s, x(s))(\eta,\xi)ds \\ & \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k^-))(\eta,\xi)] \end{aligned}$$

Then by the hypothesis (i)-(iii), it implies that for each $t \in I$, we have

$$\begin{aligned} | x(t)(\eta,\xi) | \leq & M | x_0 | + \int_0^t m(s)(| x(s)(\eta,\xi) | + \psi(\| x(s) \|_{\eta\xi})ds \\ & + M\sum_{k=1}^m c_k. \end{aligned}$$

Let

$$v(t)(\eta,\xi) = M | x_0 | + \int_0^t m(s)(| x(s)(\eta,\xi) | + \psi(\| x(s) \|_{\eta\xi})ds + M\sum_{k=1}^m c_k.$$

Therefore $v(0)(\eta,\xi) = M\bigg[\mid x_0 \mid + \sum_{k=1}^{m} c_k \bigg]$

$$v'(t)(\eta,\xi) = m(t)\big(\mid x(t)(\eta,\xi) \mid + \psi(\mid x(t)(\eta,\xi) \mid)\big) \quad \text{for a.e. } \ t \in I$$

by the increasing property of ψ, we have

$$\mid v'(t)(\eta,\xi) \mid \leq \mid m(t)(v(t)(\eta,\xi) + \psi(v(t)(\eta,\xi))) \mid .$$

Then for each $t \in I$, we have

$$\int_{v(0)}^{v(t)} \frac{du}{u+\psi(u)} \leq \int_0^b m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u+\psi(u)}$$

Consequently there exists a constant r such that $\mid v(t)(\eta,\xi) \mid \leq r$, $t \in I$, and hence $\parallel x \parallel_{PC} \leq r$. This shows that $R(N)$ is bounded. In a similar manner, the convexity property is preserved by the operator N. Hence by Schauder-Tychonov's theorem [14], we conclude that N has a fixed point which is a mild solution of the problem $\square$.
We now prove the uniqueness of the solution of the problem.

Theorem 3.2 *Assume that* $\Phi \in \{E,F,G,H\}$ *are as in theorem above and suppose condition (ii) holds. Assume the following conditions are also satisfied*
(i) There exists $L^{\Phi}_{\eta\xi} \in L^1_{loc}(\mathbb{R}_+)$ *such that*

$$\parallel \Phi(t,x) - \Phi(t,\overline{x}) \parallel_{\eta\xi} \leq L^{\Phi}_{\eta\xi}(t) \parallel x - \overline{x} \parallel_{\eta,\xi} \quad \textit{for each} \ \ t \in I, x, \overline{x} \in \widetilde{\mathcal{A}}$$

(ii) There exists constants c_k *and* $\mid L_{\eta\xi}(t) \mid < L$ *such that*

$$\mid J_k(x) - J_k(\overline{x}) \mid \leq c_k \parallel x - \overline{x} \parallel_{\eta,\xi} \quad \textit{for each} \ \ k = 1,2,...,m; \forall x, \overline{x} \in \widetilde{\mathcal{A}}$$

If $MbL + M\sum_{k=1}^{m} c_k < 1$, *then the problem has a unique mild solution.*

Proof: It was established in [8] that if $\Phi \in \{E,F,G,H\}$ are Lipschitzian, then there exists $L_{\eta\xi}(t) \in L^1_{loc}(I)$ such that $\mathbb{P}$ is $L_{\eta\xi}(t)$-Lipschitz continuous. Let the operator N be defined as above and for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $L > 0$ is such that $\mid L_{\eta\xi}(t) \mid < L$ for all $t \in I$. We will show that N is a contraction and hence has a unique fixed point.
Let $N : PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E})) \to PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$, consider $x, \overline{x} \in$

$PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$. Then we have for each $t \in I$

$$\begin{aligned} \mid N(x)(t)(\eta,\xi) - N(\overline{x})(t)(\eta,\xi) \mid &\leq \int_0^t M \mid \mathbb{P}(t,x)(\eta,\xi) - \mathbb{P}(t,\overline{x})(\eta,\xi) \mid \\ &+ M \sum_{k=1}^{m} \mid J_k(x(t_k^-)) - J_k(\overline{x}(t_k^-)) \mid \\ &\leq ML \int_0^t \| x(s) - \overline{x}(s) \|_{\eta\xi} \, ds \\ &+ M \sum_{k=1}^{m} c_k \| x(t_k^-) - \overline{x}(t_k^-) \|_{\eta\xi} \\ &\leq MbL \| x - \overline{x} \|_{PC} + M \sum_{k=1}^{m} c_k \| x - \overline{x} \|_{PC} \\ &= \left(MbL + M \sum_{k=1}^{m} c_k \right) \| x - \overline{x} \|_{PC} \\ &< \| x - \overline{x} \|_{PC} . \end{aligned}$$

This implies that N is a contraction and hence it has a unique fixed point which is a mild solution of the problem □.

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