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On the Existence of a limiting regime in the sense of Demidovic for a certain third order nonlinear differential equation

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Abstract

In this paper we prove the existence of a limiting regime in the sense of Demidovic for certain third order nonlinear differential equations. We also prove that such a solution is periodic (or almost periodic) whenever the forcing term is periodic (or almost periodic). The generalized Theorems of Demidovic are used in the proofs of our results. The results generalize earlier results in the literature.

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1 Introduction

In generalizing the results of Demidovic [5] on the existence of a limiting regime, Ezeilo [7] considered the system of equations of the form

$$\dot{X} = f(t, X) + g(t, X) \tag{1.1}$$

where $f(t, X)$ satisfies either

$$||f(t, 0)|| \leq m < \infty \quad for \quad all \quad t \in \mathrm{I\!R}$$

or

$$\int_{-\infty}^{\infty} ||f(t, 0)||^p dt < \infty, \quad 1 \leq p < 2,$$

while $g(t, X)$ satisfies Lipschitz condition, with $g(t, 0) \equiv 0$.

Precisely, the following theorem was proved:

Theorem 1 *[7], Suppose that*

i) there exists a positive definite $n \times n$ matrix A such that the eigenvalues of $\{D + D^T\}$, where $D = A\frac{\partial f}{\partial X}$, are all negative,

ii) $f(t, 0)$ satisfies either

$$||f(t, 0)|| \leq m < \infty \quad for \quad all \quad t,$$

or

$$\int_{-\infty}^{\infty} ||f(t, 0)||^p dt < \infty, \quad 1 \leq p < 2.$$

iii) $g(t, 0) \equiv 0$ and

$$||g(t, X) - g(t, Y)|| \leq \gamma(t)||X - Y||$$

for all X, Y, t with $\gamma(t)$ satisfying

$$\int_{-\infty}^{\infty} \gamma^q(t) dt < \infty, \quad 1 \leq q \leq 2.$$

Then, there exists a unique solution $X^*(t)$ of (1.1) such that

$$||X^*(t)|| \leq m, \qquad for \quad t \in IR, \tag{1.2}$$

and every other solution $X(t)$ of (1.1) converges to $X^*(t)$ as $t \to +\infty$.

Also, the following was proved [7]

Theorem 2 *Suppose conditions (i) and (iii) of Theorem 1 hold, and if in addition the following conditions hold*

I) if $f(t, X)$ and $g(t, X)$ are uniformly almost periodic in t for $||X|| \leq m$, then the unique solution $X^*(t)$ of (1.1) is uniformly almost periodic (u.a.p) in t;

II) if $f(t, X)$ and $g(t, X)$ are both periodic functions of t, for $||X|| \leq m$ and have the same period ω, then $X^*(t)$ is periodic in t, with a least period ω.

Definition 1 *We shall say that a solution $X^*(t)$ of (1.1) is a limiting regime in the sense of Demidovic, if there exists a constant m, $0 < m < \infty$ such that $||X^*(t)|| \leq m, \quad -\infty < t < \infty$.*

It is known that Ezeilo in [6, 7, 8, 10] applied the ideas of these theorems to second order and third order nonlinear equations. Also, Afuwape [2] applied these theorems to a fourth order non-linear differential equation. We also note that Afuwape [1], Afuwape and Omeike [3, 4] and Tejumola [13, 14] studied the convergence of some third order nonlinear differential equations. Other considerations pre 1974 are also recorded in [12]. The existence of limiting regime in the sense of Demidovic has only been addressed by few.

Our objective in this paper is to prove the existence of a limiting regime in the sense of Demidovic by applying theorems 1 and 2 to third order non-linear differential equations of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}). \tag{1.3}$$

Our method shall be in the use of a complete Lyapunov function. This equation is more general than that considered by Ezeilo in [10] and unlike in [14] , we use a complete Lyapunov function in our proofs. We also give an example to illustrate the results.

We shall assume that $p(t, x, \dot{x}, \ddot{x})$ is separable in the form $q(t) + r(t, x, \dot{x}, \ddot{x})$.

We shall write (1.3) in the equivalent system form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z + Q \\ \dot{z} &= -az - g(y) - h(x) + r(t, x, y, z + Q) - aQ \end{aligned} \tag{1.4}$$

with $Q(t) = \int_0^t q(s)ds$, and $g(y),\ h(x)$ continuous in their respective arguments. We shall also assume that the incrementary ratio $H_\xi(\eta) = \frac{h(\eta+\xi)-h(\xi)}{\eta}$, $(\eta \neq 0)$, of $h(x)$ lies in a closed sub-interval $[\Delta_0, kab]$ of the Routh-Hurwitz interval $(0, ab)$, for some constant $b > 0$, such that $g(y)/y > b$, for all $y \neq 0$, and $g(0) \equiv 0$, where $k < 1$, (this will be determined later).

2 The Main Results

Throughout this paper, we shall denote by D_j, $(j = 0, 1, 2, 3, \cdots)$ a positive constant.

We shall assume that the solutions of (1.3) are uniquely determined by their initial conditions. Since, we can always replace $q(t)$ by $q(t) + r(t, 0, 0, 0)$, we shall assume that $r(t, 0, 0, 0) \equiv 0$. Moreover, we shall assume that the function $r(t, x, y, z + Q)$ satisfies

$$\begin{aligned} |r(t, x_2, y_2, z_2 + Q) &- r(t, x_1, y_1, z_1 + Q)| \\ &\leq \phi(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|\}, \end{aligned} \tag{2.1}$$

for all $t \in \mathbb{R}$ and $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$, with $\phi(t)$ a continuous function satisfying for $1 \leq \nu \leq 2$

$$\int_{-\infty}^{\infty} \phi^\nu(t)dt < \infty. \tag{2.2}$$

The main result of this paper on the existence of a limiting regime in the sense of Demidovic is the following:

Theorem 3 *Suppose that*

(i) (a) $h(0) = 0$ *and that the incrementary ratio*

$$H_\xi(\eta) = \frac{h(\xi + \eta) - h(\xi)}{\eta} \in I_0 \equiv [\Delta_0, kab], \quad (\eta \neq 0) \tag{2.3}$$

a closed sub-interval of the Routh-Hurwitz interval $(0, ab)$, *with* $k < 1$,
(b) $g(0) = 0$ *and that*

$$G_\xi(\eta) = \frac{g(\eta + \xi) - g(\eta)}{\xi} \in [b, b_0], \quad (\xi \neq 0); \tag{2.4}$$

(ii) for some $D_0 > 0$,

$$|Q(t)| = |\int_0^t q(s)ds| \leq D_0; \tag{2.5}$$

(iii) $r(t, x, \dot{x}, \ddot{x})$ *satisfies (2.1) with* $\int_{-\infty}^{\infty} \phi^{\nu}(t)dt < \infty, \quad (1 \leq \nu \leq 2)$.

Then, there exists a unique solution $x^*(t)$ *of (1.3) satisfying*

$$[x^*(t)]^2 + [\dot{x}^*(t)]^2 + [\ddot{x}^*(t)]^2 \leq D_1,$$

for $t \in IR$. *Moreover, every other solution* $x(t)$ *of equation (1.3) converges to* $x^*(t)$ *as* $t \to \infty$.

Our other result on the almost periodicity or periodicity of the limiting regime $x^*(t)$ is as follows:-

Theorem 4 *Suppose that* $h(0) = 0$ *and that hypotheses (i) and (iii) of Theorem 3 hold. Suppose further that there exists a solution* $x(t)$ *of equation (1.3) such that*

$$[x(t)]^2 + [\dot{x}(t)]^2 + [\ddot{x}(t)]^2 \leq D_1.$$

Then

(I) if $Q(t)$ *is almost periodic and* $r(t, x, \dot{x}, \ddot{x})$ *is almost periodic in* t, *for* $[x(t)]^2 + [\dot{x}(t)]^2 + [\ddot{x}(t)]^2 \leq D_1$, *then* $x^*(t)$ *is almost periodic in* t.

(II) if $Q(t)$ *and* $r(t, x, \dot{x}, \ddot{x})$ *are periodic in* t, *with period* τ, *for* $[x(t)]^2 + [\dot{x}(t)]^2 + [\ddot{x}(t)]^2 \leq D_1$, *then* $x^*(t)$ *is periodic in* t, *with period* τ.

Remark 1 *If in equation (1.3)* $g(\dot{x}) = b\dot{x}$, *then the earlier results of Ezeilo in [10] will be obtained from Theorems 3 and 4.*

3 Some Preliminary results

Let (x, y, z) be any solution of system (1.4). Our main tool, in the use of Theorems 1 and 2, for the proofs of the results is the following function, defined by

$$2V(x, y, z) = \beta(1-\beta)b^2x^2 + b(\beta + \alpha a^{-1})y^2 + \alpha a^{-1}z^2 + (z + ay + b(1-\beta)x)^2, \tag{3.1}$$

where α, β are constant parameters such that $0 < \beta < 1$, and $0 < \alpha < a^{-1}$

Clearly $V(x, y, z)$ is positive definite and satisfies

$$D_2(x^2 + y^2 + z^2) \le V(x, y, z) \le D_3(x^2 + y^2 + z^2) \tag{3.2}$$

where

$$D_2 = \frac{1}{2}\min\{\beta(1-\beta)b^2; b(\beta + \alpha a^{-1}); \alpha a^{-1}\}$$

and

$$D_3 = \frac{1}{2}\min\{\beta(1-\beta)(1+a+b); b(\beta+\alpha a^{-1})+a[1+b(1-\beta)+a]; 1+\alpha a^{-1}+a+b(1-\beta)\}.$$

Furthermore, the derivative of $V(t) = V(x(t), y(t), z(t))$ with respect to t for all solutions $(x(t), y(t), z(t))$ of (1.4) gives

$$\begin{aligned} \frac{dV}{dt}\,|_{(1.4)} \;=\;& -b(1-\beta)xh(x) - [ayg(y) - ab(1-\beta)y^2] \\ & -\alpha z^2 + [b^2(1-\beta)xy - ayh(x) - b(1-\beta)xg(y)] \\ & +[b(1+\alpha a^{-1})yz - (1+\alpha a^{-1})zh(x) - (1+\alpha a^{-1})zg(y)] \\ & +[(1+\alpha a^{-1})z + ay + b(1-\beta)x]r(t, x, y, z+Q) \\ & +[b(\beta + \alpha a^{-1})y - \alpha z]Q(t) \end{aligned} \tag{3.3}$$

This we can rewrite as

$$\frac{dV}{dt}|_{(1.4)} \;=\; -U_1 + U_R + U_Q \tag{3.4}$$

where, with some re-arrangements

$$U_1 \;=\; U_{11} + U_{12} + U_{13} + U_{14} + U_{15} \tag{3.5}$$

with

$$\begin{aligned} U_{11} &= \gamma_1 b(1-\beta)H_0x^2 + \eta_1 a(G_0 - b(1-\beta))y^2 + \xi_1\alpha z^2 \\ U_{12} &= \gamma_2 b(1-\beta)H_0x^2 + (1+\alpha a^{-1})H_0xz + \xi_2\alpha z^2 \\ U_{13} &= \gamma_3 b(1-\beta)H_0x^2 + aH_0xy + \eta_1 a(G_0 - b(1-\beta))y^2 \\ U_{14} &= \gamma_1 b(1-\beta)H_0x^2 + b(1-\beta)(G_0 - b)xy + \eta_3 a(G_0 - b(1-\beta))y^2 \\ U_{15} &= \eta_4 a(G_0 - b(1-\beta))y^2 + (1+\alpha a^{-1})(G_0 - b)yz + \xi_3\alpha z^2 \\ U_Q &= [b(\beta + \alpha a^{-1})y - \alpha z]Q(t) \\ U_R &= [(1+\alpha a^{-1})z + ay + b(1-\beta)x]r(t, x, y, z+Q) \end{aligned}$$

for $\gamma_j > 0$; $\eta_j > 0; (j = 1, 2, 3, 4)$, $\xi_i > 0$; $(i = 1, 2, 3)$ and

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 1; \quad \eta_1 + \eta_2 + \eta_3 + \eta_4 = 1; \qquad \text{and} \quad \xi_1 + \xi_2 + \xi_3 = 1,$$

and $H_0 = \frac{h(x)}{x}$, for $x \neq 0$, $G_0 = \frac{g(y)}{y}$ for $y \neq 0$ and as $h(0) = 0$ and $g(0) = 0$ by hypotheses of the Theorem.

Lemma 3.1 *Subject to a conveniently chosen value of k in (2.3), we have for all $(x, y, z) \in \mathbb{R}$*

$$U_{1j} \geq 0; \quad (j = 2, 3, 4, 5).$$

Proof We note that for any two numbers u, v and for some $K > 0$, we always have

$$uv = (Ku + \frac{1}{2K}v)^2 - K^2u^2 - \frac{1}{4K^2}v^2.$$

On a re-arrangement of U_{12} we obtain

$$\begin{aligned} U_{12} &\geq \gamma_2 b(1-\beta)H_0x^2 - (1+\alpha a^{-1})H_0(K_1^2x^2 + \frac{1}{4K_1^2}z^2) + \xi_2\alpha z^2 \\ &= H_0\{\gamma_2 b(1-\beta) - K_1^2(1+\alpha a^{-1})\}x^2 + \{\xi_2\alpha - \frac{1}{4K_1^2}(1+\alpha a^{-1})H_0\}z^2 \\ &\geq 0. \end{aligned}$$

whenever

$$K_1^2 \leq \frac{\gamma_2 b(1-\beta)}{1+\alpha a^{-1}}; \quad \text{and} \quad H_0 \leq \frac{4\xi_2\alpha\gamma_2 b(1-\beta)}{(1+\alpha a^{-1})^2}. \tag{3.6}$$

Also, a re-arrangement of U_{13} gives

$$\begin{aligned} U_{13} &\geq \gamma_3 b(1-\beta)H_0x^2 - aH_0(K_2^2x^2 + \frac{1}{4K_2^2}y^2) + \eta_2 a(G_0 - b(1-\beta))y^2 \\ &= H_0\{\gamma_3 b(1-\beta) - aK_2^2\}x^2 + a\{\eta_2(G_0 - b(1-\beta)) - \frac{1}{4K_2^2}H_0\}y^2 \\ &\geq H_0\{\gamma_3 b(1-\beta) - aK_2^2\}x^2 + a\{\beta\eta_2 b - \frac{1}{4K_2^2}H_0\}y^2 \\ &\geq 0 \end{aligned}$$

whenever

$$K_2^2 \leq \frac{\gamma_3 b(1-\beta)}{a} \quad \text{and} \quad H_0 \leq \frac{4\eta_2\gamma_3 b^2\beta(1-\beta)}{a}. \tag{3.7}$$

Combining all the inequalities in (3.6) and (3.7), we have that for all (x, y, z) in IR, $U_{12} \geq 0$ and $U_{13} \geq 0$, if $H_0 \leq kab$ with

$$k = \left\{ \frac{4\xi_2 \alpha\gamma_2(1-\beta)}{a(1+\alpha a^{-1})^2}; \frac{4\eta_2\gamma_3 b\beta(1-\beta)}{a^2} \right\} < 1. \tag{3.8}$$

Similarly, a rearrangement of U_{14} gives

$$\begin{aligned} U_{14} &\geq \gamma_4 b(1-\beta)H_0 x^2 - b(1-\beta)(G_0-b)(K_3^2 x^2 + \frac{1}{4K_3^2}y^2) + \eta_3 a(G_0 - b(1-\beta))y^2 \\ &= b(1-\beta)\{\gamma_4 H_0 - K_3^2(G_0-b)\}x^2 + \{\eta_3 a(G_0-b(1-\beta)) - \frac{b}{4K_3^2}(1-\beta)(G_0-b)\}y^2 \\ &\geq b(1-\beta)\{\gamma_4\Delta_0 H_0 - K_3^2(b_0-b)\}x^2 + \{\eta_3 ab\beta - \frac{b}{4K_3^2}(1-\beta)(b_0-b)\}y^2 \\ &\geq 0 \end{aligned}$$

whenever

$$\frac{b(1-\beta)(b_0-b)}{4ab\eta_3\beta} \leq K_3^2 \leq \frac{\gamma_4\Delta_0}{b_0-b}. \tag{3.9}$$

Finally, a re-arrangement of U_{15} gives

$$\begin{aligned} U_{15} &\geq \eta_4 a(G_0 - b(1-\beta))y^2 - (1+\alpha a^{-1})(G_0-b)(K_4^2 y^2 + \frac{1}{4K_4^2}z^2) + \xi\alpha z^2 \\ &= \{\eta_4 a(G_0 - b(1-\beta)) - K_4^2(1+\alpha a^{-1})(G_0-b)\}y^2 \\ &\quad + \{\xi_3\alpha - \frac{1}{4K_4^2}(1+\alpha a^{-1})(G_0-b)\}z^2 \\ &\geq \{\eta_4 ab\beta - K_4^2(1+\alpha a^{-1})(b_0-b)\}y^2 + \{\xi_3\alpha - \frac{1}{4K_4^2}(1+\alpha a^{-1})(b_0-b)\}z^2 \\ &\geq 0 \end{aligned}$$

whenever

$$\frac{(1+\alpha a^{-1})(b_0-b)}{4\alpha\xi_3} \leq K_4^2 \leq \frac{\eta_4 ab\beta}{(1+\alpha a^{-1})(b_0-b)}$$

Lemma 3.2 *For all values of $(x, y, z) \in \mathbb{R}$ there exists a $D_4 > 0$ such that*

$$\begin{aligned} U_{11} &= \gamma_1 b(1-\beta)H_0 x^2 + \eta_1 a(G_0 - b(1-\beta))y^2 + \xi_1\alpha z^2 \\ &\geq D_4(x^2+y^2+z^2) \end{aligned}$$

whenever $H_0 \leq kab$, *with* k *satisfying (3.8).*

Proof: Choose $D_4 = \min\{\gamma_1 b(1-\beta)\Delta_0, \eta_1 ab\beta, \xi_1\alpha\}$.

Lemma 3.3 *For all values of* $(x, y, z) \in \mathbb{R}$

$$U_Q \leq D_5(|x| + |y| + |z|); \quad \text{and} \quad U_R \leq D_6(|x| + |y| + |z|)|r(t, x, y, z + Q)|$$

Proof: Choose $D_5 = \max\{b(\beta + \alpha a^{-1}), \alpha a^{-1} z\} D_0$
and
$D_6 = \max\{(1 + \alpha a^{-1}), a, b(1-\beta)\}$.

Lemma 3.4

$$\frac{dV}{dt}|_{(1.4)} \;\leq\; -(D_7 - D_8\phi(t))V(t) + D_9 V^{1/2}(t). \tag{3.10}$$

Proof: This follows by combining Lemmas 3.1, 3.2 and 3.3, and using condition (2.1) on equation (3.3), where $D_7 = D_4/D_3$, $D_8 = D_6/D_2$ and $D_9 = D_5/D_2$.

In line with Ezeilo [7], it suffices to prove the following lemma:

Lemma 3.5 *Assume that the conditions (i), (ii), (iii) and (iv) of Theorem 1 hold. Then, for arbitrary* t_0*, there exist positive constants* k_1, k_2 *depending on* a, g, h *and* q, r *such that for* $t \geq t_0$,

$$V(x(t), y(t), z(t)) \leq k_1 V(x(t_0), y(t_0), z(t_0)) + k_2 \tag{3.11}$$

Moreover there are finite constants τ_0 *and* k_0*, also dependent only on* a, g, h *and* q, r*, such that if*
$V(x(t_0), y(t_0), z(t_0)) \leq k_0$*, then*

$$V(x(t_0 + \tau), y(t_0 + \tau), z(t_0 + \tau)) \leq k_0 \tag{3.12}$$

for every τ *with* $\tau_0 \leq \tau < \infty$.

Proof: If we set $u(t) = V(t)^{1/2} = V(x(t), y(t), z(t))^{1/2}$, we have from (3.10) that

$$\frac{d}{dt}\{u(t)\exp\{L(t)\}\} \;\leq\; \frac{1}{2} D_9 \exp\{L(t)\} \tag{3.13}$$

where $L(t) = \frac{1}{2}[D_7 t - D_8 \int_0^t \phi(t)dt]$.

On integration from t_0 to t_0+s, $s \geq 0$, we have

$$u(t_0+s)\exp\{L(t_0+s)\} \leq u(t_0)\exp\{L(t_0)\} + \frac{1}{2}D_9\int_{t_0}^{t_0+s}\exp\{L(t)\}dt. \quad (3.14)$$

Now, if we define

$$D_{10} = \exp\{\frac{1}{2}D_8\int_{-\infty}^{\infty}\phi(t)dt\} < \infty$$

we can have on some arrangements of terms that

$$u(t_0+s) \leq D_{10}u(t_0)\exp\{-\tfrac{1}{2}D_7 s\} + D_{11}, \quad (3.15)$$

where D_{11} is a positive number depending on D_7, D_9 and D_{10}.

Now, if $D_{10}u(t_0) \leq D_{11}$, we have that

$$u(t_0+s) \leq 2D_{11}, \quad \text{for} \quad s \geq 0. \quad (3.16)$$

That is

$$V(t_0+s) \leq \{2D_{11}\}^2, \quad \text{provided that} \quad s \geq 0.$$

Also, if $D_{10}u(t_0) > D_{11}$, we have from (3.15) that

$$u(t_0+s) < 2D_{10}u(t_0), \quad \text{for} \quad s \geq 0.$$

That is

$$V(t_0+s) < \{2D_{10}\}^2 V(t_0), \quad \text{provided that} \quad s \geq 0.$$

Hence in all situations, we have

$$V(t_0+s) \leq \{2D_{10}\}^2 V(t_0) + \{2D_{11}\}^2, \quad \text{provided that} \quad s \geq 0.$$

which is equivalent to (3.11) with $k_1 = \{2D_{10}\}^2$ and $k_2 = \{2D_{11}\}^2$.

To complete the proof of the lemma, we have to show that for some number τ_0 (to be chosen later),

$$V(t_0+\tau) \leq k_0$$

for every $\tau_0 \leq \tau < \infty$ and k_0 such that $V(t_0) \leq k_0$.

Define $D_{12} = k_2 = (2D_{11})^2$.

First, if $V(t_0) \geq D_{12}$, we have that $D_{11} < \frac{1}{2}u(t_0)$.

Thus from (3.15), we have

$$\begin{aligned} u(t_0+s) &< u(t_0)D_{10}\exp\{-\frac{1}{2}D_7 s\} + \frac{1}{2}u(t_0) \\ &\leq u(t_0) \quad \text{provided} \quad s \geq (\log 2D_{10})/D_7. \end{aligned} \quad (3.17)$$

That is

$$V(t_0 + s) \leq V(t_0)$$

whenever $V(t_0) \geq D_{12}$. Now, if $V(t_0) < D_{12}$, we have that $u(t_0) \leq D_{12}^2$. Thus from (3.15) we have

$$\begin{aligned} u(t_0+s) &< D_{10}\exp\{-\frac{1}{2}D_7 s\}D_{12}^{\frac{1}{2}} + D_{12}^{\frac{1}{2}} \\ &\leq 2D_{12}^{\frac{1}{2}} \ \text{ provided } \ s \geq (\log\frac{2}{3}D_{10})/D_7. \end{aligned} \tag{3.18}$$

That is

$$V(t_0 + s) < 2D_{12}, \ \text{ provided } \ s \geq (\log 2D_{10})/D_7.$$

Thus on choosing $k_0 = 2D_{12}$ and $\tau_0 \geq (\log 2D_{10})/D_7$ we complete the proof of the lemma.

Combining all these, Lemma 3.5 is proved.

To complete the proof of Theorem 3 we need to prove that any two solutions of (1.4) converges.

That is

Lemma 3.6 *Suppose that conditions (i) and (iii) of Theorem 3 hold. Suppose further that that there exist positive constants k_5 and k_6 whose magnitudes depend on a, g, q and r, then if $(x_1, y_1, z_1), (x_1, y_1, z_1)$ are any two solutions of (1.4) then*

$$S(t_2) \leq k_5 S(t_1)\exp\{-k_6(t_2 - t_1)\} \tag{3.19}$$

where

$$S(t) = \{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2\}.$$

Proof Considering the function $W(t)$ defined as

$$W(t) = V(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t)) \tag{3.20}$$

where V is defined as in (3.1), then, we easily have by (3.2) that there exists positive constants D_{13}, D_{14} such that

$$D_{13}S(t) \leq W(t) \leq D_{14}S(t). \tag{3.21}$$

Also in view of , it suffices to prove that

$$W(t_2) \leq W(t_1)\exp\{-D_{13}(t_2 - t_1) + D_{14}\int_{t_1}^{t_2}\phi^{\nu}(\tau)d\tau\}, (\ \text{for} \ \ t_2 \geq t_1).$$

We note that by the earlier calculations on $V(t)$, we can easily have

$$\frac{dW}{dt}|_{1.4} = -W_1(t) + W_2(t)$$

where $W_1(t)$ satisfies

$$W_1(t) \geq D_{15}S(t)$$

and $W_2(t)$ satisfies

$$W_2(t) \leq D_{16}S^{\frac{1}{2}}(t)|\theta|$$

where

$$\theta \equiv r(t, x_1, y_1, z_1 + Q) - r(t, x_2, y_2, z_2 + Q) \tag{3.22}$$

Thus,

$$\frac{dW}{dt}|_{(1.4)} \leq -D_{15}\phi(t))S(t) + D_{16}S^{\frac{1}{2}}(t)|\theta(t)|. \tag{3.23}$$

Let ν be any constant in the range $1 \leq \nu \leq 2$. Set $2\mu = 2-\nu$, so that $0 \leq 2\mu \leq 1$. We rewrite (3.22) in the form

$$\frac{dW}{dt}|_{(1.4)} + D_{15}\phi(t))S(t) \leq D_{16}S^{\mu}W^*.$$

where

$$W^* = (|\theta| - D_{15}D_{16}^{-1}S^{\frac{1}{2}})S^{(\frac{1}{2}-\mu)}.$$

Considering the two cases
(i) $|\theta| \leq D_{15}D_{16}^{-1}S^{\frac{1}{2}}$ and
(ii) $|\theta| > D_{15}D_{16}^{-1}S^{\frac{1}{2}}$
separately, we find that in either case, there exists some constant D_{17} such that $W^* \leq D_{17}|\theta|^{2(1-\mu)}$. Thus,we can rewrite inequality (3.23) as

$$\frac{dW}{dt}|_{(1.4)} + D_{15}S \leq D_{18}S^{\mu}\phi^{2(1-\mu)}S^{(1-\mu)}$$

where $D_{18} \geq 2D_{16}D_{17}$. This immediately gives

$$\frac{dW}{dt}|_{(1.4)} + (D_{19} - D_{20}\phi^{\nu}(t))W \leq 0 \tag{3.24}$$

by (3.21), with D_{19} and D_{20} as some positive constants. On integrating (3.24) from t_1 to t_2 , $(t_2 \geq t_1)$, we obtain

$$W(t_2) \leq W(t_1)\exp\{-D_{19}(t_2 - t_1) + D_{20}\int_{t_1}^{t_2} \phi^{\nu}(\tau)d\tau\}.$$

Again, using (3.21), and since $\int_{t_1}^{t_2} \phi^{\nu}(\tau)d\tau < \infty$, we obtain (3.19). This completes the proof of lemma 3.6.

Proof of Theorem 2.1

Having dealt with the proofs of lemmas 3.1-3.6, the proof of Theorem 2.1 then follows exactly as in the proof of [7, Theorem 1], with the obvious changes as required.

Proof of Theorem 2.2

The method is as in [7] but with certain modifications due to the presence of the perturbation $r(t, x, y, z + Q)$ which is uniformly almost periodic (u.a.p) in t.

Consider the function

$$\Psi(t) = V(x(t+\tau) - x(t), y(t+\tau) - y(t), z(t+\tau) - z(t)) \tag{3.25}$$

where V is the function defined in (3.1) with x, y, z replaced by $x(t+\tau) - x(t), y(t+\tau) - y(t)$ and $z(t+\tau) - z(t)$, respectively. Then, we easily have by (3.2) that there exists positive constants $d_1 > 0, d_2 > 0$ such that

$$d_1 S(t) \leq \Psi(t) \leq d_2 S(t) \tag{3.26}$$

where

$$S(t) = \{[x(t+\tau) - x(t)]^2 + [y(t+\tau) - y(t)]^2 + [z(t+\tau) - z(t)]^2\}.$$

Following the approach of the proof of lemma 3.6, we have that

$$\begin{aligned} \frac{d\Psi}{dt}|_{(1.4)} \ \leq \ & -d_3\{[x(t+\tau) - x(t)]^2 + [y(t+\tau) - y(t)]^2 + [z(t+\tau) - z(t)]^2\} \\ & + d_4\{|x(t+\tau) - x(t)| + |y(t+\tau) - y(t)| + |z(t+\tau) - z(t)|\}|\theta| \end{aligned} \tag{3.27}$$

with $\theta = r(t+\tau, x(t+\tau), y(t+\tau), z(t+\tau) + Q(t+r)) - r(t, x, y, z + Q(t))$ and

d_3, d_4 are some finite positive constants. Now, rewrite (3.27) thus

$$\begin{aligned}\frac{d\Psi}{dt}|_{(1.4)} \leq & -d_3\{[x(t+\tau)-x(t)]^2+[y(t+\tau)-y(t)]^2+[z(t+\tau)-z(t)]^2\} \\ & +d_4\{|x(t+\tau)-x(t)|+|y(t+\tau)-y(t)|+|z(t+\tau)-z(t)|\}|\theta| \\ & +\{|x(t+\tau)-x(t)|+|y(t+\tau)-y(t)|+|z(t+\tau)-z(t)|\}\times \\ & |r(t+\tau,x(t),y(t),z(t)+Q(t+r))-r(t,x(t),y(t),z(t)+Q(t))|\end{aligned} \tag{3.28}$$

Assume now that r is u.a.p in t. Then given arbitrary $\epsilon > 0$, we can find $\tau > 0$ such that

$$|r(t+\tau,x(t),y(t),z(t)+Q(t+r))-r(t,x(t),y(t),z(t)+Q(t))| \leq \ell\epsilon^2, \tag{3.29}$$

where ℓ is a constant whose exact value will be chosen to advantage later. It follows that

$$\frac{d\Psi}{dt}|_{(1.4)} \leq -d_3S(t)+d_5S^{\frac{1}{2}}(t)|\theta|+d_6S^{\frac{1}{2}}(t)\ell\epsilon^2 \tag{3.30}$$

where $d_5 = d_4\sqrt{3}$ and $d_6 = \sqrt{3}$. Since (by Theorem 2.1)

$$\{[x(t+\tau)-x(t)]^2+[y(t+\tau)-y(t)]^2+[z(t+\tau)-z(t)]^2\}^{\frac{1}{2}} \leq \Delta_1, \tag{3.31}$$

then

$$\frac{d\Psi}{dt}|_{(1.4)} + -d_3S(t) \leq \{d_5S^{\frac{1}{2}}(t)|\theta| - d_7S(t)\} + d_6\Delta_1\ell\epsilon^2. \tag{3.32}$$

Let ν be any constant such that $1 \leq \nu \leq 2$ and set $\mu = 1-\frac{1}{2}\nu$, so that $0 \leq \mu \leq 1$. Consider (3.32) in the form

$$\frac{d\Psi}{dt}|_{(1.4)} + d_7S \leq d_5S^{\mu}J^* + d_6\Delta_1\ell\epsilon^2 \tag{3.33}$$

where

$$J^* = S^{\frac{1}{2}-\mu}\left(\|\theta\| - d_5^{-1}d_7S^{\frac{1}{2}}\right).$$

Now, if $\|\theta\| \leq d_5^{-1}d_7S^{\frac{1}{2}}$, we obtain

$$J^* \leq 0;$$

on the other hand, if $\|\theta\| > d_5^{-1}d_7S^{\frac{1}{2}}$, that is,

$$S < (d_5d_7^{-1}\|\theta\|)^2, \quad \text{we obtain}$$

$$J^* < d\|\theta\|^{2(1-\mu)},$$

where $d = (d_5 d_7^{-1})^{2\mu-1}$.
Thus, (3.33) becomes,

$$\frac{d\Psi}{dt}|_{(1.4)} + d_7 S \le dS\phi^{2(1-\mu)} + d_6\Delta_1\ell\epsilon^2.$$

Since from (2.1), θ satisfies $\|\theta\| \le \phi(t)S^{\frac{1}{2}}$, we obtain

$$\dot{\Psi}(t) + \left(d_7 - d\phi^{2(1-\mu)}(t)\right) S \le d_6\Delta_1\ell\epsilon^2.$$

On using inequalities (3.26), we obtain

$$\dot{\Psi}(t) + (d_8 - d\phi^{\nu}(t))\,\Psi \le d_6\Delta_1\ell\epsilon^2 \tag{3.34}$$

for some constant $d_8 > 0$. Following the approach in the proof of Lemma 3.6, there is a $d_9, 0 < d_9 < \infty$ defined by

$$d_9 = \exp\left(d_{10}\int_{-\infty}^{\infty}\phi^{\nu}(t)dt\right).$$

Then, from (3.34), we have that

$$\Psi(t) \le d_9\Psi(t_0)e^{-d_{10}(t-t_0)} + \ell d_{11}\epsilon^2 \tag{3.35}$$

where $d_{11} = d_9 d_6 D_1/d_{10}$. This result holds for arbitrary t_0. In particular, on letting $t_0 \to -\infty$ in (3.35) and noting that $\Psi(t_0)$ is finite, since (3.31) is true, one obtains that

$$W(t) \le \ell d_{11}\epsilon^2$$

for arbitrary t. By (3.26) and by the definition of $W(t)$ this implies that

$$[x(t+\tau) - x(t)]^2 + [y(t+\tau) - y(t)]^2 + [z(t+\tau) - z(t)]^2 \le \ell d_{11}\epsilon^2 d_1^{-1}. \tag{3.36}$$

Suppose now that at the stage (3.29) the constant ℓ had been

$$\ell = d_1 d_{11}^{-1}$$

the result (3.36) would then read

$$[x(t+\tau) - x(t)]^2 + [y(t+\tau) - y(t)]^2 + [z(t+\tau) - z(t)]^2 \le \epsilon^2. \tag{3.37}$$

Since

$$\sqrt{3}\{[x(t+\tau) - x(t)]^2 + [y(t+\tau) - y(t)]^2 + [z(t+\tau) - z(t)]^2\} \le \sqrt{3}\epsilon^2 \le \epsilon,$$

it follows that

$$|x(t+\tau)-x(t)|+|y(t+\tau)-y(t)|+|z(t+\tau)-z(t)| \leq \epsilon \tag{3.38}$$

where τ is chosen to satisfy (3.29) with $\ell = d_1 d_{11}^{-1}$. The set of all τ satisfying (3.29) is relatively dense, and hence (3.38) implies that $(x(t), y(t), z(t))$ is u.a.p. This proves the first part of Theorem 2.2.

To prove the second part of the theorem, assume now that $r(t,x,y,z+Q)$ has the period ω in t and fix the τ in the definition of $\Psi(t)$ equal to ω. The terms on the left hand side of (3.29) is identically zero, and so proceeding as above we shall have, in place of (3.36) that

$$[x(t+w)-x(t)]^2+[y(t+w)-y(t)]^2+[z(t+w)-z(t)]^2 \leq 0.$$

Hence

$$[x(t+w)-x(t)]^2+[y(t+w)-y(t)]^2+[z(t+w)-z(t)]^2 = 0$$

and it is readily seen that

$$x(t+w)=x(t),\ y(t+w)=y(t) \quad \text{and} \quad z(t+w)=z(t)$$

which is the required result. This completes the proof of Theorem 2.2.

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