DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 3, 2013
Electronic Journal,
reg. N ΦC77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru

General control problems

Demand and supply dynamics

G.S. Osipenko
Sevastopol Institute of Banking, Ukraine
george.osipenko@mail.ru
E.K. Ershov
Saint-Petersburg State University of Architecture and Civil Engineering, Russia
ershov@ee13858.spb.edu

Abstract

The paper is devoted to the study of a 3rd order discrete dynamical system modelling the dynamics of supply and demand. It was proved that there is a stable balans for small parameter values . The complex bifurcation were investigated and parameter values for which periodical orbits and chaotic regimes exist were obtained.

1 Mathematical model

Demand and supply dynamics will be modeled by a discrete dynamical system. In modelling we will assume that the following requiremnts are satisfied:

1) the supply increases when the demand exceeds the supply or the price of the commodity tends to increase; the supply decreases when the supply exceeds the demand or the price of the commodity tends to decrease;

2) the price of the commodity increases when the demand exceeds the supply and decreases when the demand is less than the supply;

3) the demand decreases when the purchasing expense exceeds the expense intended; the demand increases when the expense intended is greater than the actual cost of the commodity.

Let the variable x signify the volume of the commodity proposed, the variable y signify the price of a commodity unit, and the variable z signify the volume of the commodity demand. Then $\min(x, z)$ specifies the volume of the commodity sold (realized) and $y \min(x, z)$ determines the true cost of the commodity purchase. Let H denote the planned expenditure for commodity purchase. In what follows we denote by $n = 1, 2, \ldots$ a discrete time which stands for the number of a time interval corresponding to the realization of demand, supply and sale of the commodity. For an n-time interval we denote by x_n, y_n and z_n respectively the volume of the supply, the price, and the volume of the demand associated with this time interval.

Let us consider the function

$$f(x, z) = \frac{z - x}{z + x}.$$

Clearly f is a zero-degree homogeneous function, i.e. $f(tx, tz) = t^0 f(x, z)$. It is not difficult to see that the function $f(x, z)$ takes the values in the range from -1 to +1 since the variables $x,\ y,\ z$ are postive. In addition, $f(x, z) < 0$ when $z < x$ and $f(x, z) > 0$ when $z > x$. The function $\exp(a\frac{z-x}{z+x})$, where $a > 0$, takes the values in the range from e^{-a} to e^a. In addition, $\exp(af(x, z)) < 1$, when the supply exceeds the demand $x > z$ and $\exp(af(x, z)) > 1$ when the demand exceeds the supply $z > x$.

The ratio y_n/y_{n-1} governs a tendency for changing the price as one passes from the $n-1$-time interval of sale to the n-time interval of sale. Thus the supply dynamics is given by the mapping

$$x_{n+1} = x_n(\frac{y_n}{y_{n-1}})^r \exp(a\frac{z_n - x_n}{z_n + x_n}),$$

where the factor $\exp(a\frac{z_n-x_n}{z_n+x_n})$ is the coefficient of variation of the supply for $(n + 1)$-time interval depending on the relationship between de of mand and supply over the n-time interval. Notice that the mapping described above determines the volume of the future supply and consequently the volume of commodity production. The parameter $a > 0$ determines the adaptation of the system to the variation of demand and supply. The indicator r determines the rate of the supply dependency on the price variation. If the supply does not depend on the price variation then $r = 0$.

In the same manner one can specify the dependence of the price on demand and supply with some coefficient of adaptation b and bearing in mind that the price on the $(n+1)$-time interval depends on supply and demand on the same $(n+1)$-time interval. Hence, the price dynamics is given by the (implicit) mapping

$$y_{n+1} = y_n \exp(b\frac{z_{n+1} - x_{n+1}}{z_{n+1} + x_{n+1}}),$$

where $b > 0$ is the coefficient of adaptation of the price to the variation of supply and demand. It should be noted that the implicit character of the last mapping (both the left and right sides depend on the $n+1$-time interval) produces certain difficulties in the study of system dynamics.

The demand on the $(n+1)$-time interval depends on the difference between the planned expenditure H and the real expenditure for commodity purchase on the n-time interval. If the planned expenditure H is greater than the real one then the demand increases; otherwise it decreases. Thus the demand dynamics can be represented in the form

$$z_{n+1} = z_n \exp(c\frac{H - y_n \min(x_n, z_n)}{H + y_n \min(x_n, z_n)}),$$

where $c > 0$ is the coefficient of adaptation of the demand to the variation of expenditure.

Thus we obtain the following system of difference equations

$$x_{n+1} = x_n(\frac{y_n}{y_{n-1}})^r \exp(a\frac{z_n - x_n}{z_n + x_n}), \tag{1}$$

$$y_{n+1} = y_n \exp(b\frac{z_{n+1} - x_{n+1}}{z_{n+1} + x_{n+1}}), \tag{2}$$

$$z_{n+1} = z_n \exp(c\frac{H - y_n \min(x_n, z_n)}{H + y_n \min(x_n, z_n)}). \tag{3}$$

Since the second equation contains $(n+1)$-time interval in its both sides then the system determines a discrete dynamical system in the implicit form. To obtain difference equations in the explicit form one needs to substitute x_{n+1} z_{n+1} from the first and third equations to the second one. As a result we obtain a standard discrete dynamical system.

2 Theoretical results

First we consider the case where the supply does not depend on the price variation, i.e. $r = 0$. It is not difficult to see that the system (1,2,3) has a curve filled by the balanced states (fixed points), with the balanced states given by the following equalities

$$\begin{aligned} x &= z \\ H &= y\min(x,z). \end{aligned} \tag{4}$$

The study of the balanced states is complicated by the fact that the function $\min(x, z)$ is not smooth at the balanced states $x = z$. Let us show that for every balanced state there exists a surface which passes through it and consists of full system orbits, i.e. an invariant surface of the system.

To this end consider the function

$$U(x,y,z) = \frac{x^b}{y^a}\exp(ab\frac{z-x}{z+x}).$$

Proposition 1. The function $U(x, y, z)$ is constant on each orbit of the system (1,2,3).

Proof. From the second equation of the system it follows that

$$y_{n+1}\exp(-b\frac{z_{n+1}-x_{n+1}}{z_{n+1}+x_{n+1}}) = y_n.$$

Raising both sides of last equality to the power a, we obtain

$$y_{n+1}^a\exp(-ab\frac{z_{n+1}-x_{n+1}}{z_{n+1}+x_{n+1}}) = y_n^a.$$

Similarly, raising both sides of equation (1) to the power b, we obtain

$$x_{n+1}^b = x_n^b\exp(ab\frac{z_n-x_n}{z_n+x_n}).$$

The last equality divided by the previous one yields

$$\frac{x_{n+1}^b}{y_{n+1}^a}\exp(ab\frac{z_{n+1}-x_{n+1}}{z_{n+1}+x_{n+1}}) = \frac{x_n^b}{y_n^a}\exp(ab\frac{z_n-x_n}{z_n+x_n}). \tag{5}$$

By virtue of definition of the function $U(x, y, z)$, (5) can be written in the form

$$U(x_{n+1}, y_{n+1}, z_{n+1}) = U(x_n, y_n, z_n).$$

This means that the value of the function $U(x, y, z)$ does not vary at one iteration and thus does not vary on the full orbit. Consequently, the level surface $U(x, y, z) = const$ is invariant for the system (1,2,3). This ends the proof.

It is not difficult to see that each level surface intersects the balanced curve at just one point. In fact, it follows from the balanced equations (4) and the equality $U(x, y, z) = h$ that

$$H = xy, \qquad h = \frac{x^b}{y^a}.$$

The coordinates of a fixed point on the level surface $U(x, y, z) = h$ are thus seen to be uniquely determined. By this means the system (1,2,3) has a foliation with invariant layers and each layer contains exactly one balanced state. It can be shown also that the foliation $\{U(x, y, z) = h\}$ has a transverse intersection with the balanced curve (4).

The existence of the function $U(x, y, z)$ with properties stated previously allows to reduce the 3-dimensional system (1,2,3) into a 2-dimensional system by eliminating the variable y. In fact, for the level surface $U(x, y, z) = h$ we have the equality

$$\frac{x^b}{y^a} \exp(ab\frac{z-x}{z+x}) = h,$$

from which it follows that

$$y = (1/h)^{1/a} x^{b/a} \exp(b\frac{z-x}{z+x}).$$

Substitution of this presentation of y to the equation (3) yields the equation

$$z_{n+1} = z_n \exp(c\frac{H - (\frac{1}{h})^{1/a}(x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}{H + (\frac{1}{h})^{1/a}(x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}), \tag{6}$$

which is independent of the price y.

Proposition 2. For every $H > 0$ and $h > 0$ the system (1) and (6) are equivalent to the following system

$$x_{n+1} = x_n \exp(a\frac{z_n - x_n}{z_n + x_n}), \tag{7}$$

$$z_{n+1} = z_n \exp(c\frac{1 - (x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}{1 + (x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}). \tag{8}$$

Proof. The equation (6) can be written in the form

$$z_{n+1} = z_n \exp(c\frac{1 - \frac{1}{H}(\frac{1}{h})^{1/a}(x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}{1 + \frac{1}{H}(\frac{1}{h})^{1/a}(x_n)^{b/a} \exp(b\frac{z_n-x_n}{z_n+x_n}) \min(x_n, z_n)}) \tag{9}$$

In the system (1) and (9) let us perform the following change of variables

$$(x,z) \to (tx, tz),$$

where t is a constant determined later. This change of variables does not affect the equation (1). As for the equation (9), it takes the form

$$z_{n+1} = z_n \exp(c\frac{1 - \frac{1}{H}(\frac{1}{h})^{1/a} t^{(a+b)/a}(x_n)^{b/a} \exp(b\frac{z_n - x_n}{z_n + x_n}) \min(x_n, z_n)}{1 + \frac{1}{H}(\frac{1}{h})^{1/a} t^{(a+b)/a}(x_n)^{b/a} \exp(b\frac{z_n - x_n}{z_n + x_n}) \min(x_n, z_n)}).$$

Choose $t > 0$ such that

$$\frac{1}{H}(\frac{1}{h})^{1/a} t^{(a+b)/a} = 1.$$

It is not difficult to see that such a t exists and is unique. At t given above we obtain the required system. This ends the proof.

Notice that the change of variables in the proof of Poposition 2 is substantially choosing the unit of measurement of demand and supply. Clearly, the system (7,8) has the fixed point (1,1) which is the unique balanced state of the economic model.

3 Price dependence on the supply

Let us consider the system (1,2,3) provided that in the first equation the parameter $r > 0$, i.e. the supply depends on the price. In this case the supply increases provided the price in the previous period has rised and decreases otherwise. It is clear that the parameter $r > 0$ controls the extent to which the supply depends on the price.

Proposition 3. The system (1,2,3) is equivalent to the following system

$$x_{n+1} = x_n \exp((a + rb)\frac{z_n - x_n}{z_n + x_n}), \tag{10}$$

$$y_{n+1} = y_n \exp(b\frac{z_{n+1} - x_{n+1}}{z_{n+1} + x_{n+1}}), \tag{11}$$

$$z_{n+1} = z_n \exp(c\frac{H - y_n \min(x_n, z_n)}{H + y_n \min(x_n, z_n)}). \tag{12}$$

Proof. The equation (1) contains the factor

$$(\frac{y_n}{y_{n-1}})^r.$$

Using the second equation for the n-time interval

$$y_n = y_{n-1}\exp(b\frac{z_n - x_n}{z_n + x_n}),$$

we can represent this factor in the form

$$(\frac{y_n}{y_{n-1}})^r = \exp(rb\frac{z_n - x_n}{z_n + x_n}).$$

Substituting the last equality in the first equation we obtain the equality

$$x_{n+1} = x_n\exp((a + rb)\frac{z_n - x_n}{z_n + x_n}),$$

which proves the proposition.

From Proposition 3 it follows that the price dependence of the supply reduces to the system free from such a dependence provided that the parameter a is replaced by $a + br$. Thus one needs to study only systems where the supply does not depend on the price.

Figure 1: Steady balanced state for $a = 1$, $b = 0.5$, $c = 1$.

4 Numerical results

This section is devoted to the numerical study of system dynamics in relation to the adaptation parameters a, b, and c. In the section we use algorithms and computer softwares described in monographs [4] and [5]. First of all, notice that for small values of a, b c the system (7,8) has the steady balanced state (1,1), see. Fig. 1. As the values of parameters a, b c increase the balanced state (1,1) fails its stability and at the same time the stable 3-periodic orbit

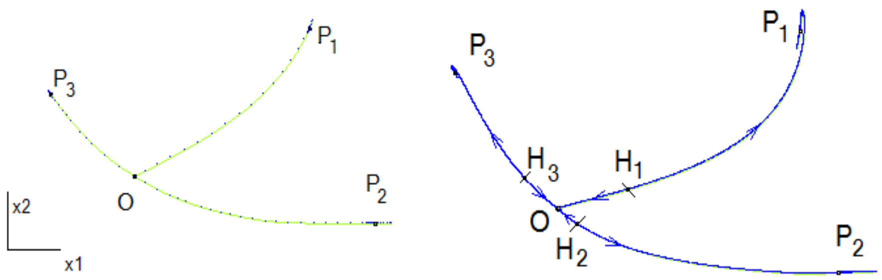

Figure 2: Unsteady balanced state O and 3-periodic stable orbit P for $a = 2.5$, $b = 1.5$, $c = 2.3$. Steady balanced state O, unstable manifold $W^u(H)$ of the 3-periodic hyperbolic orbit H, and 3-periodic stable orbit P for $a = 2.5$, $b = 2$, $c = 2.5$.

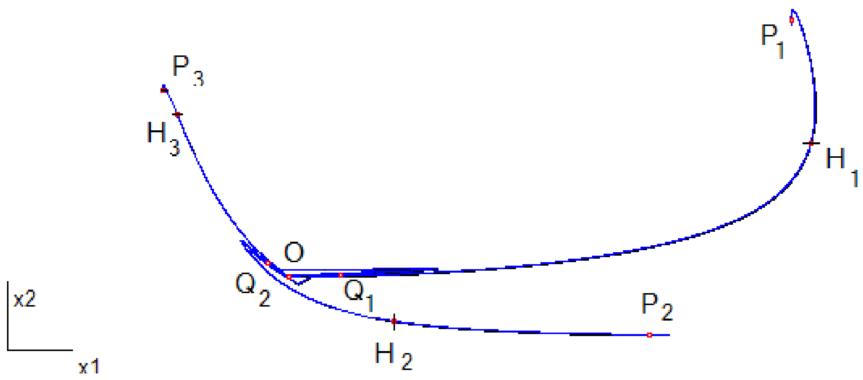

Figure 3: Unsteady balanced state O, stable 2-periodic orbit Q, unstable manifold $W^u(H)$ 3-periodic hyperbolic orbit H and 3-periodic stable orbit P for $a = 3.2$, $b = 2.5$, $c = 2.5$

arises. For example, for $a = 2.5$, $b = 1.5$, and $c = 2.3$ (see the left Fig. 2) the balanced state (1,1) is unstable, while the 3-periodic stable orbit P is generated by iterations of the point (1.7305, 1.8706). Moreover, the 3-periodic regime P attracts every orbit except for the balanced state. This means that from the practical point of view, starting from any initial state with time we will observe only the 3-periodic regime P. A similar dynamics holds for $a = 2.5$, $b = 1.5$, and $c = 2.5$.

As the parameter b increases the following bifurcations occur: from the balanced state O the hyperbolic 3-periodic orbit H is split out, while the balanced state becomes stable, see the right Fig. 2. The global attractor A is formed from the closure of the unstable manifold $W^u(H)$ of the hyperbolic orbit H. In this case the limit dynamic regime is either the stable balanced state O or the stable

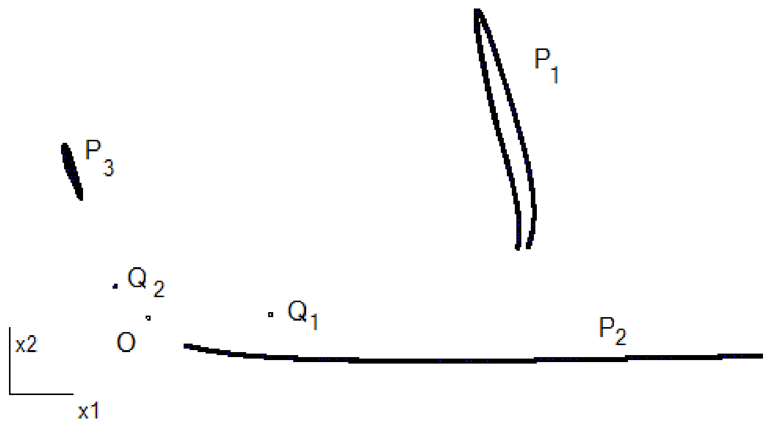

Figure 4: Unstable balanced state O, stable 2-periodic orbit Q, and 3-periodic stable chaotic attractor P for $a = 3.2$, $b = 2.5$, $c = 2.8$

3-periodic regime P depending on the choice of the initial data (x_0, z_0). From the theoretical point of view, there is a possibility to approach the hyperbolic orbit H by choosing an initial point on its stable manifold $W^s(H)$, however, it is not feasible to realize such a choice. For $a = 2.5$, $b = 2$, $c = 2.5$ the hyperbolic orbit H is generated by iterations of the point (1.3741, 1.1532). In the course of a futher increase of parameters new bifurcations arise. For $a = 3.2$, $b = 2.5$, and $c = 2.5$ the balanced state $O(1, 1)$ loses stability and nearby the 2-periodic stable orbit Q of the point $(1.2813, 1.0172)$ occurs , see Fig. 3. There exists the invariant curve $A_0 = \{Q_1 \leftarrow O \rightarrow Q_2\}$ between the point O and the orbit Q by which orbits move from the balanced state O to the stable 2-periodic orbit Q. The curve A_0 is an attractor. There is also the 3-periodic hyperbolic orbit H of the point (3.9442, 2.2020), whose unstable manifold $W^u(H)$ is wound on the attractor A_0 by one end and tends to the stable 3-periodic orbit P of the point (3.8322, 3.3089) by the other. The closure of the unstable manifold $W^u(H)$ yields the global attractor A in the iterior of which the attractor A_0 lies, see. Fig. 3.

The structure of enclosed attractors briefly outlined above

$$((Q \subset A_0) \bigcup P) \subset A$$

generates a filtration [3] preserved under small perturbations of the system. The structure of attractors described holds for a great deal of values of parameters a, b, and c, however their topology can vary. The attractor A_0 may be thought of as originating from the stable 2-periodic orbit and, as a rule, is small-sized. The attractor A is emerging from the unstable manifold of the 3-periodic hyperbolic

Figure 5: Unstable balanced state and global chaotic attractor for $a = 3.5$, $b = 2$, and $c = 3.5$.

orbit. As the parameter c increases up to 2.8, instead of the stable periodic orbit P the 3-periodic stable chaotic attrator A_1 evolves, see. Fig. 4. The attractor A_1 results from the bifurcation of the stable 3-periodic orbit P.

Recall that the entropy E of a dynamical system is a measure of its randomness. It is known that the entropy can be evaluated as a growth indicator of an arc length under iterations [2]. Using this reasoning, we have obtained that for the system on the attractor A_1 the entropy $E = 0.382$.

As the parameters vary, all the attractors outlined merge into one attractor. In order to describe attractors size we will use coordinates of their points most distant from the balanced state (1,1). As for instance, for $a = 3.5$, $b = 2$, and $c = 3.5$, the global chaotic attractor is very big in size, see. Fig. 5. Here, the most distant point of the global (large) attractor is (54.9, 0.85). The estimate of the entropy is $E = 0.269$.

For $a = 3$, $b = 1.8$ and $c = 3.5$ the balanced state (1,1) is unstable and there exists 5-periodic stable orbit P generated by iterations of the point (1.8822, 1.1475). All orbits (except for the balanced state) tend to P and have a sufficiently intricate structure.

The system dynamics is very sensitive to the variation of parameters. As for instance, for $a = 3$, $b = 1.78$, $c = 3.5$ the system has the global (large) chaotic attractor wherein the small attractor is located, see. Fig. 6. The estimate of the entropy is $E = 0.108$. The balanced state is unstable. For $a = 3$, $b = 2$, and $c = 3.5$ the chaotic attractor disappears. However, for $a = 3$, $b = 2$, and $c = 3.6$ the nontrivial global chaotic attractor (wherein the small attractor lies) appears again, see. Fig. 7. The estimate of the entropy on the small and large attractors is $E = 0.04$ and $E = 0.312$, respectively.

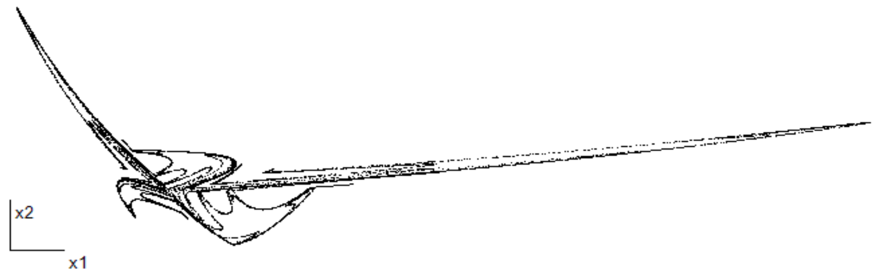

Figure 6: Small chaotic attractor for $a = 3$, $b = 1.78$, $c = 3.5$.

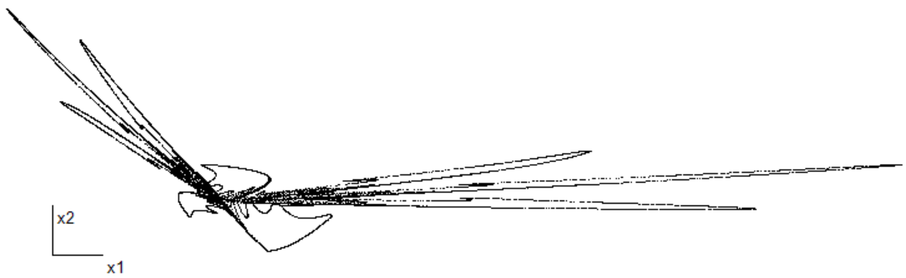

Figure 7: Unstable balanced state and small chaotic attractor for $a = 3$, $b = 2$, and $c = 3.6$.

Figure 8: Medium-sized chaotic attractor $A_1 \subset A$ for $a = 4$, $b = 2$, and $c = 3$.

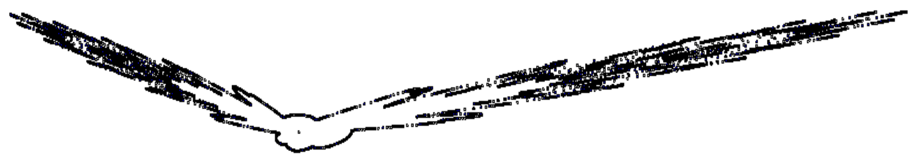

Figure 9: Unstable balanced state and small-sized attractor $A_0 \subset A_1 \subset A$ for $a = 4$, $b = 1.98$, and $c = 3$.

For $a = 4$, $b = 2$, and $c = 3$ the global chaotic attractor A becomes large with the extreme point (28.4, 0.9). The attractor A contains the smaller attractor $A_1 \subset A$, see. Fig. 8 with the extreme point (3.5, 1.5). The estimate of the entropy on the latter is $E = 0.161$. In the interior of this attractor there is one more attractor A_0, see. Fig. 9. It is located very close to the balanced state (1,1) with the extreme point (1.07, 1.02). The estimate of the entropy on the latter is $E = 0.0008$. One can see that the chaotic character of the last attractor is minor and within the accuracy of the analysis.

References

[1] Lebedev V.V., Lebedev K.V. , *Mathematical modeling nonstational ecomical processes,* Moscow, 2011, 336 p.(in Russian).

[2] Newhouse S. and Pignataro T. On the estimation of topological entropy, *Journal of Statistical Physics,* 72, 1993, pp. 1331-1351.

[3] Nitecki Z., Shub M. Filtrations, decompositions, and explosions. *Amer. J. of Math.* vol. 97, 1975, 1029-1047.

[4] Osipenko G.S., Ampilova N.B., *Introduction to symbolic analysis of dynamical systems,* St. Petersburg, 2005 (in Russian).

[5] Osipenko G. *Dynamical systems, Graphs, and Algorithms,* Lectures Notes in Mathematics, vol. 1889, Springer-Verlag, Berlin and New York, 2007.

[6] G. S. Osipenko, T. N. Korzh, E. K. Ershov, Dynamics of price-level, national income and cost of money interaction, *International Conference "Modeling, control, and stability MCS-2012", 10-14 Sept. 2012, Sevastopol, Crimea*, 158-159.