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Differential inclusions

EXISTENCE THEORY FOR RANDOM NON-CONVEX DIFFERENTIAL INCLUSION

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In this paper, the existence of solution for the random boundary value problem of second order non-convex ordinary functional differential inclusion is proved through priori bound method under some monotonicity conditions.
Keywords**:** Random functional differential inclusion, Carathe'odory condition, random solution.

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1. INTRODUCTION

Let *X* be a Banach space and let $\mathrm{P}(X)$ denote the class of all subsets of *X*, called the power set of *X*. Denote

$$\mathrm{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has the property } p\}.$$

Here, *p* may be *p* = closed (in short cl) or *p* = convex (in short cv) or *p* = bounded (in short bd) or *p* = compact (in short cp). Thus $\mathrm{P}_{cl}(X)$, $\mathrm{P}_{cv}(X)$, $\mathrm{P}_{bd}(X)$ and $\mathrm{P}_{cp}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of *X*.

Let $(\Omega, \mathrm{A}, \mu)$ be a complete σ-finite measure space and let R be the real line. Given a closed and bounded interval $J = [a,b]$ in R, consider the functional multi-valued random boundary value problem (in short RBVP) of ordinary second order random differential inclusion

$$\left.\begin{aligned} x''(t,\omega) &\in F\left(s, x(\eta(t),\omega),\omega\right) \quad a.e. \quad t\in J \\ x(a,\omega) &= 0 = x'(b,\omega) \end{aligned}\right\} \tag{1.1}$$

for all $\omega\in\Omega$, where $F: J\times R\times\Omega \to P_p(R)$ and the functions $\eta: J\to J$ is continuous.

By random solution of the multi-valued RBVP (1.1) on $J\times\Omega$, I mean a measurable function $x:\Omega\to AC^1(J,R)$ satisfying for each $\omega\in\Omega$, $x''(t,\omega)=v(t,\omega)$ for some measurable $v:\Omega\to L^1(J,R)$ satisfying $v(t,\omega)\in F\left(t, x(\eta(t),\omega),\omega\right)$ a.e. $t\in J$ and $x(a,\omega)=0=x'(b,\omega)$ where $AC^1(J,R)$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on *J*.

The special case when $F(t,x,\omega)=\{f(t,x,\omega)\}$, the multi-valued RBVP (1.1) reduce to the boundary value problems of functional random differential equation

$$\left.\begin{aligned} x''(t,\omega) &= f\left(t, x(\eta(t),\omega),\omega\right) \quad a.e. \quad t\in J \\ x(a,\omega) &= 0 = x'(b,\omega) \end{aligned}\right\} \tag{1.2}$$

for all $\omega\in\Omega$ where $f: J\times R\times\Omega\to R$. The single-valued RBVP (1.2) have been discussed in Dhage and Kang [2] for various aspects of the solution. The multi-valued RBVP (1.1), this type of existence results are also proved in D. S. Palimkar papers and monograph see references [9-14]. In this paper, I discuss the multi-valued RBVP (1.1) for the existence of random solution for non-convex case under some monotonicity conditions.

2. AUXILIARY RESULTS

Let $\mathrm{M}(\Omega, X)$ and $C(J,R)$ be the space of measurable *X*-valued functions on Ω and continuous real-valued functions on *J* respectively. Let $\beta: J\times R\times\Omega\to P_p(R)$ be a multi-valued mapping. Then for any measurable function $x: J\times\Omega\to R$, let

$$S_\beta(\omega)(x)=\left\{v\in \mathrm{M}\left(\Omega, M(J,R)\right) \middle| v(t,\omega)\in\beta(t,x(t,\omega),\omega) \ \ a.e. \ \ t\in J\right\} \tag{2.1}$$

And

$$S^1_\beta(\omega)(x)=\left\{v\in \mathrm{M}\left(\Omega, L^1(J,R)\right) \middle| v(t,\omega)\in\beta(t,x(t,\omega),\omega) \ \ a.e. \ \ t\in J\right\}. \tag{2.2}$$

This is set of selection functions for β on $J\times R\times\Omega$. When there is no confusion, I denote $S^1_\beta(\omega)(x)=S^1_\beta(\omega)(y)$, where $y(t,\omega)=x(\theta(t),\omega)$ for some continuous function $\theta: J\to J$. The integral of the random multi-valued function β is defined as

$$\int_a^b \beta(s, x(s,\omega),\omega)\,ds=\left\{\int_a^b v(s,\omega)\,ds : v\in S^1_\beta(\omega)(x)\right\}.$$

Furthermore, if the integral $\int_a^b \beta(s,x(s,\omega),\omega)\,ds$ exists for every measurable function $x: J\times\Omega\to R$, then I say the multi-valued mapping β is Lebesgue integrable on $J\times R\times\Omega$.

Lemma 2.1(Hu and Papageorgiou [4]) *Let E be a Banach space. If* $\beta : J \times E \to \mathrm{P}_{cp}(E)$ *is strong Carathe'odory, then the multi-valued map* $(t,x) \mapsto \beta(t,x(t))$ *is jointly measurable for every measurable function* $x : J \to E$.

3. EXISTENCE RESULTS

It is more convenient to deal with the integrals than derivatives, I shall rewrite the FRBVPs into the random integral inclusion; the kernels of the random integral equations are the appropriate Green's function associated with the random boundary conditions. This transformation may be done in several ways. One-way of doing it is via Green's function.

Now, the multi-valued RBVP (1.1) is equivalent to the functional random integral inclusion (FRII)

$$x(t,\omega) \in \int_a^b H(t,s) F(s, x(\eta(t),\omega),\omega)\, ds, \quad \text{if} \quad t \in J, \tag{3.1}$$

where $H(t,s)$ is a Green's function associated with the homogeneous linear FRBVP

$$\left.\begin{array}{c} x''(t,\omega) = 0, \quad a.e. \quad t \in J \\ x(a,\omega) = 0 = x'(b,\omega) \end{array}\right\} \tag{3.2}$$

for a fixed $\omega \in \Omega$ and is given by

$$H(t,s) = \begin{cases} t-a, & \text{if } a \le t \le s \le b \\ s-a, & \text{if } a \le s \le t \le b. \end{cases} \tag{3.3}$$

It is known that the Green's function $H(t,s)$ is continuous and non-negative real-valued function on $J \times J$ satisfying

$$0 \le H(t,s) \le b-a \tag{3.4}$$

And
$$\int_a^b H(t,s)\,ds \le \frac{(a-b)^2}{2}. \tag{3.5}$$

The key result in formulating random fixed point theorems concerning the existence of measurable selector for a multi-valued mapping is the following:

Theorem3.1 (Kuratowskii and Ryll-Nardzewski [6]) *If the multi-valued operator* $Q : \Omega \times X \to \mathrm{P}_p(X)$ *is measurable with closed values, then Q has a measurable selector.*

Remark 3.1 Note that if $Q : \Omega \times X \to \mathrm{P}_{cl}(X)$ is a multi-valued random operator, then the set $S_Q(\omega)(x)$ is non-empty for each $x \in X$.

A random fixed point theorem for the right monotone increasing multi-valued random operators on separable ordered Banach spaces is

Theorem 3.2 (Dhage [1]) *Let* (Ω, A) *be a measurable space and let* $[\alpha, \beta]$ *be a random interval in a separable Banach space X. If* $Q:\Omega\times[\alpha,\beta]\to \mathrm{P}_{cl}([\alpha,\beta])$ *is a compact, upper semi-continuous right monotone increasing multi-valued random operator and the cone K in X is normal, then* $Q(\omega)$ *has a random fixed point in* $[a,b]$.

I also need the following definitions in the sequel.

Definition 3.1 A multi-valued random operator $Q:\Omega\times X\to \mathrm{P}_{cl}(X)$ is called right monotone increasing if for each $\omega\in\Omega$ we have that $S_Q(\omega)(x)\overset{i}{\leq} S_Q(\omega)(y)$ for all $x, y\in X$ for which $x\leq y$.

Definition 3.2 A multi-valued random operator $Q:\Omega\times X\to \mathrm{P}_p(X)$ is called strict monotone increasing if for each $\omega\in\Omega$, $Q(\omega)x\leq Q(\omega)y$ for all $x, y\in X$ for which $x<y$. Similarly, the multi-valued random operator $Q(\omega)$ is called monotone decreasing if for each $\Omega\in\omega, Q(\omega)x\geq Q(\omega)y$ for all $x, y\in X$ for which $x<y$. Finally, $Q(\omega)$ is called monotone if it is a either monotone increasing or monotone decreasing multi-valued random operator on *X*.

Definition 3.3 A multi-valued mapping $F: J\times R\times\Omega\to \mathrm{P}_{cp}(R)$ is called Carathe'odory, if for each $\omega\in\Omega$

(i) $(t,\omega)\mapsto F(t,x,\omega)$ is jointly measurable for each $x\in R$, and

(ii) $x\mapsto F(t,x,\omega)$ is an upper semi-continuous almost everywhere for $t\in J$.

Again, a Carathe'odory multi-valued function *F* is called L^1-Carathe'odory if

(iii) for each real number *r*> 0 there exists a measurable function $h_r:\Omega\to L^1(J,R)$ such that for each $\omega\in\Omega$

$$\|F(t,x,\omega)\|_{\mathrm{P}} = \sup\{|u| : u\in F(t,x,\omega)\}\leq h_r(t,\omega) \quad a.e. \quad t\in J$$

for all $x\in R$ with $|x|\leq r$.

Furthermore, a Carathe'odory multi-valued function *F* is called L^1_R-Carathe'odory if

(iv) there exists a measurable function $h:\Omega\to L^1(J,R)$ such that

$$\|F(t,x,\omega)\|_{\mathrm{P}} \leq h(t,\omega) \quad \text{a.e.} \quad t\in J$$

for all $x\in R$, and the function *h* is called a growth function of *F* on

$J \times R \times \Omega$.

Definition 3.4 A measurable function $\alpha : \Omega \to C(J, R)$ is called a strict lower random solution for the multi-valued RBVP (1.1) if for all $v \in S_F^1(\omega)(\alpha)$, we have

$$\alpha''(t,\omega) \le v(t,\omega) \ , \ \alpha(a,\omega) \le 0 \ge \alpha'(b,\omega)$$

for all $t \in J$ and $\omega \in \Omega$. Similarly, a strict upper random solution β for the multi-valued RBVP (1.1) on $J \times \Omega$ is defined.

A random fixed-point theorem for strict monotone increasing multi-valued random operators on separable ordered Banach spaces is

Theorem 3.3 (Dhage [1]) *Let* (Ω, A) *be a measurable space and let* $[\alpha, \beta]$ *be a random order interval in a separable Banach space X. If* $Q : \Omega \times [\alpha, \beta] \to \mathrm{P}_{cl}([\alpha, \beta])$ *is a strict monotone increasing completely continuous multi-valued random operator and the cone K in X is normal, then* $Q(\omega)$ *has the least random fixed point* $x_*(\omega)$ *and the greatest random fixed point* $y*(\omega)$ *in* $[\alpha, \beta]$. *Moreover, the sequences* $\{x_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *defined by*

$$x_0(\omega) = \alpha(\omega), x_{n+1}(\omega) \in Q(\omega) x_n, n = 0,1,2,...,$$

and

$$y_0(\omega) = \beta(\omega), y_{n+1}(\omega) \in Q(\omega) y_{n,} n = 0,1,2,...,$$

converge to $x_*(\omega)$ *and* $y*(\omega)$ *respectively.*

I use the Theorems 3.2 and 3.3 for proving the main existence results of this paper.

Then, I have also quoted the following lemmas which are used for proving main result.

Lemma 3.1 (Lasota and Opial [7])*Let E be a Banach space. If* $\dim(E) < \infty$ *and* $\beta : J \times E \times \Omega \to \mathrm{P}_{cp}(E)$ *is strong* L^1*-Carathe'odory, then* $S_\beta^1(\omega)(x) \ne \emptyset$ *for each* $x \in E$.

Lemma 3.2(Lasota and Opial [7]) *Let E be a Banach space, F a Carathe'odory multi-valued operator with* $S_\beta^1(\omega) \ne \emptyset$ *and Let* $\mathrm{L} : L^1(J, E) \to C(J, E)$ *be a continuous linear mapping. Then the composite operator*

$$\mathrm{L} \circ S_\beta^1 : C(J, E) \to \mathrm{P}_{bd,cl}(C(J, E))$$

has closed graph on $(C(J, E)) \times (C(J, E))$.

Consider the following set of assumptions

(H_0) $F(t,x,\omega)$ is closed and bounded subset of R for each $(t,x,\omega)\in J\times R\times\Omega$.

(H_1) The multi-valued map $(t,\omega)\mapsto F(t,x(t),\omega)$ is jointly measurable for each measurable function $x: J\times\Omega\to R$.

(H_2) F is L^1-Carathe'odory.

(H_3) The multi-valued map $x\mapsto S_F^1(\omega)(x)$ is right monotone increasing in $x\in C(J,R)$ almost everywhere for $t\in J$.

(H_4) The multi-valued RBVP (1.1) has a strict lower random solution α and a strict upper random solution β with $\alpha\le\beta$ on $J\times\Omega$.

Hypotheses $(H_0)-(H_2)$ are common in the literature. Some nice sufficient conditions for guarantying $S_F^1(\omega)\ne\emptyset$ appear in Lasota and Opial [7]. A mild hypothesis of (H_4) has been used in Hu and Papageorgiou [4]. Hypothesis (H_3) relatively new to the literature, but the special forms have been appeared in the works of several authors.

4. MAIN EXISTENCE RESULT

Theorem 4.1*Assume that the hypotheses $(H_0)-(H_4)$ hold. Then the multi-valued RBVP (1.1) has a random solution in $[\alpha,\beta]$ defined on $J\times\Omega$.*

Proof Let $X=C(J,R)$. Define a random order interval $[\alpha,\beta]$ in X which is well defined in view of hypothesis (H_4). Now the multi-valued RBVP (1.1) is equivalent to the random integral inclusion

$$x(t,\omega)\in\int_a^b H(t,s)F\big(s,x(\eta(t),\omega),\omega\big)ds,\quad t\in J. \tag{4.1}$$

for all $\omega\in\Omega$. Define a multi-valued operator $Q:\Omega\times[\alpha,\beta]\to P_{cl}(X)$ by

$$Q(\omega)x=\left\{u\in M(\Omega,X)\,|\,u(t,\omega)=\int_a^b H(t,s)v(s,\omega)ds,\ v\in S_F^1(\omega)(x)\right\} \tag{4.2}$$

$$=\big(K\circ S_F^1(\omega)\big)(x)$$

where $K: M\left(\Omega,L^1(J,R)\right)\to M\left(\Omega,C(J,R)\right)$ is a continuous operator defined by

$$K(t,\omega)=\int_a^b H(t,s)v(s,\omega)ds \tag{4.3}$$

Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (H_2). I shall show that $Q(\omega)$ satisfies all the conditions of Theorem 3.2.

Step I: First, I show that Q is closed valued multi-valued random operator on $\Omega\times[\alpha,\beta]$. Observe that the operator $Q(\omega)$ is equivalent to the composition $\mathrm{K}\circ S_F^1(\omega)$ of two operators on $L^1(J,R)$, where $\mathrm{K}:\mathrm{M}\left(\Omega,L^1(J,R)\right)\to \mathrm{M}\left(\Omega,X\right)$ is the continuous operators defined by 4.3. To show $Q(\omega)$ has closed values, it suffices to prove that the composition operator $\mathrm{K}\circ S_F^1(\omega)$ has closed values on $[\alpha,\beta]$. Let $x\in[\alpha,\beta]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(\omega)(x)$ converging to v in measure. Then, by the definition of $S_F^1(\omega), v_n(t,\omega)\in F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ a.e. for $t\in J$. Since $F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ is closed, $v\left(t,\omega\right)\in F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ a.e. for $t\in J$. Hence, $v\in S_F^1(\omega)(x)$. As a result, $S_F^1(\omega)(x)$ is closed set in $L^1(J,R)$ for each $\omega\in\Omega$. From the continuity of K, it follows that $\left(\mathrm{K}\circ S_F^1(\omega)\right)(x)$ is a closed set in X. Therefore, $Q(\omega)$ is a closed-valued multi-valued operator on $[\alpha,\beta]$ for each $\omega\in\Omega$.

Next, I show that $Q(\omega)$ is a multi-valued random operator on $[\alpha,\beta]$. First, I show that the multi-valued map $(\omega,x)\mapsto S_F^1(\omega)(x)$ is measurable. Let $f\in\mathrm{M}\left(\Omega,L^1(J,R)\right)$ be arbitrary. Then

$$d\left(f,S_F^1(\omega)(x)\right)=\inf\left\{\|f(\omega)-h(\omega)\|_{L^1}:h\in S_{F(\omega)}(x)\right\}$$

$$=\int_a^b\inf\left\{|f(t,\omega)-z|:z\in F\left(t,x\left(\eta(t),\omega\right),\omega\right)\right\}dt$$

$$=\int_a^b d\left(f(t,\omega)\right),F\left(t,x\left(\eta(t),\omega\right),\omega\right)dt.$$

But by hypothesis (H_1), $F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ is jointly measurable and it is known that the multi-valued map $z\to d\left(z,F\left(t,x,\omega\right)\right)$ is continuous. Hence the multi-valued mapping $d\left(f(t,\omega)\right)$, $F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ is jointly measurable from $J\times X\times\Omega\times L^1(J,R)$ in to R^+. Now the integral is the limit of the finite sum of measurable functions, and so, $d\left(f,S_F^1(\omega)(x)\right)$ is measurable. As a result, the multi-map $(\cdot,\cdot)\to S_F^1(\cdot)(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J\times X\times\Omega$ by

$$\phi(t,x,\omega)=\left(\mathrm{K}\circ S_F^1(\omega)\right)(x)(t)=\int_a^b H(t,s)F\left(s,x\left(\eta(s),\omega\right),\omega\right)ds.$$

I shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on R. Let $\{t_n\}$ be a sequence in J converging to $t\in J$. Then

$$
\begin{aligned}
& d_H\left(\phi(t_n,x,\omega),\phi(t,x,\omega)\right) \\
& \quad = d_H\left(\int_a^b H(t_n,s)F\left(s,x(\eta(s),\omega),\omega\right)ds, \int_a^b H(t,s)F\left(s,x(\eta(s),\omega),\omega\right)ds\right) \\
& \quad = \int_a^b |H(t_n,s)-H(t,s)| \left\| F\left(s,x(\eta(s),\omega),\omega\right)\right\|_{\mathrm{P}} ds \\
& \quad = \int_a^b |H(t_n,s)-H(t,s)| h_r(s,\omega)\, ds \\
& \quad \to 0 \quad as \quad n\to\infty.
\end{aligned}
$$

Thus the multi-valued map $t\mapsto \phi(t,x,\omega)$ is continuous and hence, by Lemma 2.1, the map $(t,x,\omega)\mapsto \int_a^b H(t,s)F\left(s,x(\eta(s),\omega),\omega\right)ds$ is jointly measurable. Consequently, $Q(\omega)$ is a random operator on $[\alpha,\beta]$.

Step II: Secondly, I show that $Q(\omega)$ is right monotone increasing and multi-valued random operator on $[\alpha,\beta]$ into itself for each $\omega\in\Omega$. Let $x,y\in[\alpha,\beta]$ be such that $x\le y$. Since (H_2) holds, I have that $S_F^1(\omega)(x)\overset{i}{\le} S_F^1(\omega)(y)$. Hence $Q(\omega)(x)\overset{i}{\le} Q(\omega)(y)$. From (H_3) it follows that $\alpha\le Q(\omega)\alpha$ and $Q(\omega)\beta\le\beta$ for all $\omega\in\Omega$. Now $Q(\omega)$ is right monotone increasing on X, so we have for each $\omega\in\Omega$

$$
\alpha\le Q(\omega)\alpha \overset{i}{\le} Q(\omega)x \overset{i}{\le} Q(\omega)\beta\le\beta
$$

for all $x\in[\alpha,\beta]$. Hence Q defines a right monotone increasing multi-valued random operator $Q:\Omega\times[\alpha,\beta]\to \mathrm{P}_{cl}\left([\alpha,\beta]\right)$.

Step III: Next, I show that $Q(\omega)$ is completely continuous for each $\omega\in\Omega$. First, I show that $Q(\omega)\left([\alpha,\beta]\right)$ is compact for each $\omega\in\Omega$. Let $\{y_n(\omega)\}$ be a sequence in $Q(\omega)\left([\alpha,\beta]\right)$ for some $\omega\in\Omega$. I will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in *X*.

Case I: First, I show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, I have a $v_n(\omega)\in S_F^1(\omega)(x_n)$ for some $x_n\in[\alpha,\beta]$ such that

$$
y_n(t,\omega)=\int_a^b H(t,s)v_n(s,\omega)ds, \quad t\in J.
$$

Therefore,
$$|y_n(t,\omega)| \le \int_a^b H(t,s)|v_n(s,\omega)|\,ds$$
$$\le \int_a^b H(t,s)\|F(s,x_n(\eta(s),\omega),\omega)\|_{\mathrm{P}}\,ds$$
$$\le \int_a^b (b-a)h_r(s,\omega)ds$$
$$\le (b-a)\|h_r(\omega)\|_{L^1}$$

for all $t \in J$, where $r = \|\alpha(\omega)\| + \|\beta(\omega)\|$. Taking the supremum over t in the above inequality yields

$$\|y_n(\omega)\| \le (b-a)\|h_r(\omega)\|_{L^1},$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)([\alpha,\beta])$.

Next, I show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([\alpha,\beta])$. Let $t,\tau \in J$. Then

$$|y_n(t,\omega) - y_n(\tau,\omega)| \le \left|\int_a^b H(t,s)v_n(s,\omega)ds - \int_a^b H(\tau,s)v_n(s,\omega)ds\right|$$
$$\le \left|\int_a^b [H(t,s) - H(\tau,s)]v_n(s,\omega)ds\right|$$
$$\le \left|\int_a^b |H(t,s) - H(\tau,s)|h_r(s,\omega)ds\right|.$$

From the above inequality, it follows that

$$|y_n(t,\omega) - y_n(\tau,\omega)| \to 0 \text{ as } t \to \tau.$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([\alpha,\beta])$. Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega \in \Omega$, so it has a cluster point in view of Arzela-Ascoli theorem. As a result, $Q(\omega)$ is a compact multi-valued random operator on $[\alpha,\beta]$.

Case II: Next, I show that $Q(\omega)$ is an upper semi-continuous multi-valued random operator on $[\alpha,\beta]$. Let $\{x_n(\omega)\}$ be a sequence in *X* such that $x_n(\omega) \to x_*(\omega)$. Let $\{y_n(\omega)\}$ be a sequence such that $y_n(\omega) \in Q(\omega)x_n$, and $y_n(\omega) \to y_*(\omega)$. I shall show that $y_*(\omega) \in Q(\omega)x_*$. since $y_n(\omega) \in Q(\omega)x_n$, there exists a $v_n(\omega) \in S^1_{F(\omega)}(x_n)$ such that

$$y_n(t,\omega)=\int_a^b H(t,s)v_n(s,\omega)ds, \quad t\in J.$$

I must prove that there is a $v_*(\omega)\in S_F^1(\omega)(x_*)$ such that

$$y_*(t,\omega)=\int_a^b H(t,s)v_*(s,\omega)ds, \quad t\in J.$$

Consider the continuous linear operator $\mathrm{L}:\mathrm{M}\left(\Omega,L^1(J,R)\right)\to \mathrm{M}\left(\Omega,C(J,R)\right)$ defined by

$$\mathrm{L}v(t,\omega)=\int_a^b H(t,s)v(s,\omega)ds, \quad t\in J.$$

Now $\qquad \|y_n(\omega)-y_*(\omega)\|\to 0$ as $n\to\infty$.

From Lemma 3.2, it follows that $\mathrm{L}\circ S^1_{F(\omega)}$ is a closed graph operator. Also from the definition of L I have

$$y_n(t,\omega)\in\left(\mathrm{L}\circ S^1_{F(\omega)}\right)(x_n).$$

Since $y_n(\omega)\to y_*(\omega)$, there is a point $v_*(\omega)\in S^1_{F(\omega)}(x_*)$ such that

$$y_*(t,\omega)=\int_a^b H(t,s)v_*(s,\omega)ds, \quad t\in J$$

This show that $Q(\omega)$ is a upper semi-continuous multi-valued random operator on $[\alpha,\beta]$. Thus, $Q(\omega)$ is an upper semi-continuous and compact and hence completely continuous multi-valued random operator on $[\alpha,\beta]$. Now an application of Theorem 3.2 yields that $Q(\omega)$ has a random fixed point, which further implies that the multi-valued RBVP (1.1) has a random solution on $J\times\Omega$. This completes the proof.

5. EXAMPLE

One of the most important examples of differential inclusions comes from control theory. Consider the control system $x'=f(x,u), u\epsilon U,$ where u is a control parameter.

It appears that the control system and differential inclusion $x'\in f(x,u)=\bigcup_{u\in U} f(x,u)$ have the some trajectories of the sets control depending on x i.e. $U=U(x)$. Then we obtain the differential inclusion $x'\in f(x,u(x))$.

The equivalence between the control system and the corresponding differential inclusion is the central idea used to prove the existence theorem in optional control theory.

Since the dynamics of economical, social and biological macro systems is multi-valued, differential inclusions serve as natural models in macro system dynamics. Differential inclusions are also used to describe some systems with hysteresis.

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