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Multiple Solutions for Nonlinear Eigenvalue Problem [1]

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Abstract: We obtain a sequence of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for which there exist countably many positive solutions of a fourth-order, singular boundary value problem. These positive eigenfunctions correspond to the sequence of eigenvalues $\lambda \geq 1$ (counting multiplicity). The methods involve application of the fixed point arguments of the Krasnosel'skii cone expansion-compression type and Holder's inequality.

Keywords: Singular boundary-value problem, Eigenvalue, Positive Solution, Holder's inequality, Multiple solution.

AMS(MOS) subject classification: 34B15, 34B16, 74K10.

Introduction

[1]This work was done while the author was visiting Matematiska institutionen of Stockholms Universitet, S-106 91, Sweden. He extends his thanks to all members of the Department for their warm hospitality during his stay.

In this work, we are concerned with the determination of a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{j-1} < \lambda_j \cdots$ (counting multiplicity) for which the following singular boundary value problem which describes deformation of simply supported beam loaded uniformly in the transverse direction has countably many positive solutions:

$$u^{(4)}(t) = \lambda a(t) f(t, u(t), u''(t)), \;\; t \in (0,1) \qquad (1.1)$$
$$u(0) = u(1) = 0, \; u''(0) = u''(1) = 0 \qquad (1.2).$$

where

$(\sum_1)$ $f \in C[(0,1) \times (0,\infty) \times (0,\infty), [0,\infty)]$ satisfies certain oscillatory growth condition

$(\sum_2)$ $a(t) \in L^p[(0,1)]$ for some $p \geq 1$ has countably many non-isolated singularities of the form $|\frac{1}{2} - t|^{-\epsilon}$, $\epsilon \geq 0$ in $[0, \frac{1}{2})$ which include the sequence $\{t_j\}_{j=1}^{\infty}$ such that $\frac{1}{2} > t_1 > t_j \to t_0 \geq 0$, $t_0 \in [0, \frac{1}{2})$ with $a(t_0) > 0$ and $\lim_{t \to t_0} a(t) < \infty \; \forall \; j = i, 2 \cdots$

$(\sum_3)$ $a(t) \geq c > 0 \;\; \forall \;\; t \in [t_0, 1 - t_0]$

and λ is a positive parameter whose values depend on t_j $(j = 1, 2, \cdots)$.

Singular boundary value problems describe many phenomena in applied sciences,particularly a vast class of elastic deflections. Equally numerous is the nature of techniques applied in the analysis of such problems by investigators. These techniques are dictated by the conditions imposed on the inhomogeneous term in the equation as well as the boundary conditions. The ancillary conditions in turn are prescribed by the mechanics of the problem. The literature reveals several forms of the equation studied. Amongst these is the existence and uniqueness result by Usmani (see [14]) for the solution of the boundary value problem of the type:

$$u^{(4)}(t) = f(t)u(t) + g(t), \;\; t \in (0,1) \qquad (1.3)$$
$$u(0) = u_0, \; u(1) = u_1, \; u''(0) = \bar{u}_0, \; u''(0) = \bar{u}_1 \qquad (1.4)$$

under the restriction $\sup |f(t)| < \pi^4$. In his own case Afterbizadeh [1] proved an existence result under the sharp condition that f is a bounded function. The study has progressed to include the works of Del Pino and Manasevich [4], Yang [16], Hao [5], Ma and Wang [10], Agrawal and O'Regan [2] and other outstanding authors. Most of the contributions extended earlier works by imposing stricter conditions on the inhomogeneous term in the equations. Various methods employed include monotone [13], lower and upper solution techniques (11, [12]), operator approximation and degree theory ([6], [15]) and fixed point arguments of Leray-Schauder and Krasnosel'skii [8].

Recently there appears to be a fast growing interest in the investigation of existence result of countably many positive solutions of boundary value problems (see [5], [9], [13] and references therein). Therefore our motivation for this paper is in the direction of this trend. We obtain a sequence of eigenvalues for countably many positive solutions of (1.1)-(1.2) using Krasnosel'skii's cone type fixed point theorem and Holder's inequality. This work complements and extends known results. The outline of the paper continues in Section 2 with a presentation of preliminary notes involving the Green's function of (1.1)-(1.2). We also state a fixed point theorem due to Krasnosel'skii [7] which will be used to yield the multiple solutions of (1.1)-(1.2). We furnish an appropriate Banach space framework of cones in order to apply the fixed point arguments. Additional assumptions on f and standard results using Holder's inequality are also stated here. In the concluding Section, we state and prove the main result of the paper as well as furnish an example of functions $a(t)$ that satisfies conditions $(\Sigma_1) - (\Sigma_3)$

2. Preliminary notes

We begin this Section by defining a positive solution of (1.1) as a function $u(t) \in C^2[0,1] \cap C^4(0,1)$ which satisfies (1.l)-(1.2) and $u(t) \geq 0$ for $t \in (0,1)$. We assume additional conditions on $a(t)$ and f as follows:
$(\sum_4)$ $\exists\ \alpha_i(t) \in C[(0,1),[0,\infty)]$ $(i = 1,2)$ and $h(u) \in C[[0,\infty),[0,\infty)]$ such that $\alpha_i(t)a(t) \geq \max_{t\in[0,1]} \beta_i(t)$ for $\beta_i(t) \in L^p,\ p \geq 1$ $(i = 1,2)$ with
$\beta_1(t)h(u) \leq f(t,u(t),u''(t)) \leq \beta_2(t)h(u), \quad u(t),u''(t) : [0,\frac{1}{2}) \to [0,\infty)$
The Green's function $G(t,s)$ for

$$u^{(4)}(t) = 0,\ t \in (0,1) \qquad (2.1)$$
$$u(0) = u(1),\ u''(0) = u''(1) = 0 \qquad (2.2)$$

is

$$G(t,s) = \begin{cases} s(1-t),\ 0 \leq s \leq t \leq 1 \\ t(1-s),\ 0 \leq t \leq s \leq 1 \end{cases}$$

satisfying the following property:

$(\sum_5)$ $G(t,s) \leq G(s,s), \;\; 0 \leq s,t \leq 1$
It is well known that the solutions of (1.1)-(1.2) are fixed points of the integral operator:

$$Nu(t) = \int_0^1 \int_0^1 G(t,s)G(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau ds, \quad t \in [0,1] \tag{2.3}$$

Then the following statements are consequences of $(\sum_1) - (\sum_5)$:
$(\sum_6)$ $0 < \int_0^1 \int_0^1 G(s,s)G(s,\tau)\beta_1(\tau)d\tau ds, \int_0^1 \int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)d\tau ds < \infty$
$(\sum_7)$ $\exists\ t_0 \in (0,1)$ such that $\beta_i(t_0) > 0$ $(i = 1,2)$
$(\sum_8)$ $\min_{t\in[0,1]} G(t,s) \geq t_0\, G(s,s) \geq t_0\, G(t^0,s) \geq 0, \;\; t^0 \in [0,\frac{1}{2}]$.
It can than be seen that establishing the existence of positive solutions of (1.1) - (1.2) is equivalent to proving the existence of fixed points of the operator equation:

$$u(t) = \lambda N u(t) \tag{2.4}$$

We apply the following fixed point theorem to obtain positive solutions of (2.1)-(2.2) for certain values of λ :
Theorem 1: (Krasnosel'skii) Let K be a cone in a Banach space E. Suppose Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \in \bar{\Omega}_1 \subset \Omega_2$ and

$$N : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \longrightarrow K$$

is a completely continuous operator such that, either
(i) $\|Nu\| \leq \|u\|, \; u \in K \cap \delta\Omega_1$, and $\|Nu\| \geq \|u\|, \; u \in K \cap \delta\Omega_2$, or
(ii) $\|Nu\| \geq \|u\|, \; u \in K \cap \delta\Omega_1$, and $\|Nu\| \leq \|u\|, \; u \in K \cap \delta\Omega_2$.
Then N has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$
Using $(\sum_8)$, we define a set of cones on the Banach space $E = C^2[0,1]$ endowed with the norm $\|u\| = \max_{t\in[0,1]}(|u(t)|, |u'(t)|, |u''(t)|)$. This is done by fixing $t_0 \in [0,\frac{1}{2})$ and then defining $K_{t_0} \subset E$ by

$$K_{t_0} = \{u(t) \in E : u(t) \geq 0, \; t \in [0,1], \; \min_{t\in[0,1]} u(t) \geq \; t_0\|u\|\}$$

In the sequel, we claim in Lemmas 2 and 3 that the integral operator N is completely continuous and cone preserving i.e.

$$N : K_{t_0} \longrightarrow K_{t_0}, \;\; t_0 \in [0, \frac{1}{2})$$

is continuous and compact.

Lemma 2. The integral operator $N : K_{t_0} \longrightarrow K_{t_0}$ is continuous for every $t_0 \in [0, \frac{1}{2})$.

Proof Assume $u \in K_{t_0}$ and a fixed value of $t_0 \in [0, \frac{1}{2}.)$

Let $\alpha_i(s)a(s)f(s, u(s), u''(s)) \geq \max_{s\in[0,1]} \beta_i(s)f(s, u(s), u''(s)) \geq 0 \ (i = 1, 2)$ and $G(t, s) \geq 0 \ \forall \ t, s \in [0, 1]$, then using $(\sum_6)$ we have,

$Nu(t) \geq 0 \ \ \forall \ \ t \in [0, 1], \ \ u \in K_{t_0}$

By (2.3) and $(\sum_8)$ and for $u \in K_{t_0}$ the following inequalities hold:

$$\begin{aligned}
\min_{t\in[0,1]} \lambda Nu(t) \ &\geq \ \min_{t\in[0,1]} \lambda \int_0^1 \int_0^1 G(t, s)G(s, \tau)\beta_2(\tau)h(u)(\tau)d\tau ds && (2.5)\\
&\geq \ \min_{t\in[0,1]} \lambda \int_0^1 \int_0^1 G(t, s)G(s, \tau)\beta_1(\tau)h(u)(\tau)d\tau ds\\
&\geq \ \lambda \textstyle\int_0^1 \int_0^1 G(t, s)G(s, \tau)\beta_1(\tau)h(u)(\tau)d\tau ds\\
&\geq \ \lambda t_0 \textstyle\int_0^1 \int_0^1 G(t^0, s)G(s, \tau)\beta_1(\tau)h(u)(\tau)d\tau ds\\
&= \ \lambda t_0 Nu(t^0) \ \forall \ t^0 \in [0, 1] \ \ t_0 \neq t^0 && (2.6)
\end{aligned}$$

Therefore, $\min\limits_{t\in[0,1]} \lambda Nu(t) \geq \lambda t_0 \|Nu\|$. It follows that $\lambda Nu : K_{t_0} \longrightarrow K_{t_0}$ is continuous and this concludes the proof of the Lemma. □

Lemma 3. The operator $N : K_{t_0} \longrightarrow K_{t_0}$ is compact for each $t_0 \in [0, \frac{1}{2})$

Proof: We prove this assertion by taking any family of bounded sets of E and show that $N(K_{t_0})$ is uniformly bounded and equicontinuous. Without loss of generality, we assume that $R > 0$ with $\|u\| \leq R \ \ \forall \ \ u \in K'_{t_0} \subset E$. Now set $m = \max_{u\in[0,R]} |h(u)|$ for each $u \in K'_{t_0}$ and we have

$$\|Nu\| \leq |\int_0^1 G(s, s)\beta_2(s)h(u)(s))ds| \leq m \int_0^1 G(s, s)G(s, \tau)d\tau$$

This means $N(K_{t_0} \supset N(K'_{t_0})$ is uniformly bounded. Next we have that $\forall \ u \in K'_{t_0}$,

$$\begin{aligned}|(Nu)'(t)| &= |\int_0^t\int_0^1 sG(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau)d\tau ds \\ &\quad -\int_t^1\int_0^1(1-s)G(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau)d\tau ds| \qquad (2.7)\\ &\le |\int_t^1\int_0^1 sG(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau)d\tau ds \\ &\quad +\int_0^t\int_0^1(1-s)G(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau ds| \\ &\le |\int_t^1\int_0^1 sG(s,\tau)\beta_2(\tau)h(u(\tau))d\tau ds \\ &\quad +\int_t^1\int_0^1(1-s)G(s,\tau)\beta_2(\tau)h(u(\tau))d\tau ds| \\ &\le \max_{u\in[0,R]} h(u(\tau))|\int_0^t\int_0^1 sG(s,\tau)\beta_2(\tau)d\tau ds \\ &\quad +\int_t^1\int_0^1(1-s)G(s,\tau)\beta_2(\tau)d\tau ds| \qquad (2.8)\end{aligned}$$

Now set

$$\Phi(t)=\int_0^1\int_0^1 sG(s,\tau)\beta_2(\tau)d\tau ds+\int_t^1\int_0^1(1-s)G(s,\tau)\beta_2(\tau)d\tau ds$$

Integrating the last expression and reversing the order of integration in the resulting expression we obtain:

$$\begin{aligned}\int_0^1 |\Phi(t)|dt \ &\leq \ \int_0^1\int_s^1\int_0^1 G(s,\tau)\beta_2(\tau)d\tau ds dt \\ &+\int_0^1 ds\int_0^s\int_0^1 G(s,\tau)\beta_2(\tau)d\tau dt \\ &\leq \ 2\int_0^1 s(1-s)\int_0^1 G(s,\tau)\beta_2(\tau)d\tau ds \\ &\leq \ 2\int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)d\tau ds < \infty \qquad (2.9)\end{aligned}$$

From (2.6) and (2.7), we have $0 \leq \int_0^1 |(Nu)'(t)| < \infty$. This shows that the last integral is absolutely continuous and for $0 \leq t_1 \leq t_2 \leq 1,\ u \in K'_{t_0} \subset E$ we have

$$|N(u)(t_1) - N(u)(t_2)| = |\int_{t_1}^{t_2} N(u)'(t)dt| \leq \int_{t_1}^{t_2} |(N(u)'(t)|dt.$$

Thus $N(K_{t_0}) \supset N(K'_{t_0})$ is equicontinuous. An appeal to the Arzela-Ascoli theorem then results in N being compact as claimed. $\square$.

In the main result, we will require the notion of the Holder's inequality so as to establish certain norm inequalities in our proof. The reader is referred to the details of the theory in Deimling [3]. Some useful properties of the Green's function in $L^q[0,1]$ is stated hereunder.

Lemma 4 Given $q > 0,\ (q^{-1} + p^{-1} = 1)$ with $G(t,.) \in L^q$, then

$$\max_{t\in[0,1]} \|G(t,.)\|_q = \frac{1}{4}\left(\frac{1}{1+q}\right)^{\frac{1}{q}} \qquad (2.10)$$

and

$$\max_{t\in[0,1]} \|G(t,.)\|_\infty = \frac{1}{4} \qquad (2.11)$$

Proof Assume that $t_0 \in [0, \frac{1}{2})$. This yields

$$\int_{t_0}^{1-t_0} (G(t,s))^q ds = \left(\frac{1}{1+q}\right)(t^q(1-t)^q) - (t_0)^{(q+1)}((1-t)^q + t^q)) \qquad (2.12)$$

which on setting $t_0 = 0$, yields

$$\int_0^1 G(t,s))^q ds = \left(\frac{1}{1+q}\right) t^q(1-t)^q \tag{2.13}$$

from whence (2.10) follows. Letting $q \longrightarrow \infty$ in (2.10) yields (2.11) $\square$

3. Main Result

In this section, we state and prove our main result as the following:

Proposition 1. Let $\alpha_i(t)a(t) = \beta_i(t)$ $(i = 1,2)$ satisfy $(\sum_2) - (\sum_4)$ and let the sequence $\{(t_0)_j\}_{j=1}^{\infty}$ be defined by $t_j > (t_0)_j > t_{j+1}, j = 1,2,\cdots$.
Furthermore, suppose the sequences $\{\xi_j\}_{j=1}^{\infty}$, $\{\eta_j\}_{j=1}^{\infty}$ are such that
$\xi_j > D\eta_j > \eta_j > (t_0)_j\eta_j > \xi_{j+1},\ j = 1,2\cdots$ where

$$D = \max\left\{\frac{4}{c_1 t^0(\frac{1}{4}-t_1^2)^2}, \frac{16}{t^0\|\beta_1\|_p}, 4\right\},\quad t^0 \in [0,1] \tag{3.1}$$

and $f(t,u,u'') \le h(u)$ satisfies the following oscillatory growth condition for each $j \in I\!N$:

$(\sum_9)$ $f(t,u,u'') \le h(u) \le M\xi_j\ \forall\ u \in [0,\xi_j,]$ where $M \le \dfrac{16}{t^0\|\beta_1\|_p}$ and

$(\sum_{10})$ $D\eta_j \le f(t,u,u'') \le h(u)\ \forall\ u \in [(t_0)_j\eta_j, \eta_j]$

Then the boundary value problem (1.1) - (1.2) has infinitely many solutions $\{u_j\}_{j=1}^{\infty}$. Moreover, $\eta_j \le \|u_j\| \le \xi_j$ for every $j = 1,2,\cdots$ and

$$\frac{1}{\displaystyle\int_{(t_0)_j}^{1-(t_0)_j}\int_0^1 G(t,s)G(s,\tau)\beta_2(\tau)(h-\delta)(u(\tau))d\tau ds} < \lambda_j$$

$$< \frac{1}{\displaystyle\int_0^1\int_0^1 G(s,s)G(s,\tau)\beta_1(\tau)(h-\delta)(u(\tau))d\tau ds} \in \Lambda. \tag{3.2}$$

where $\Lambda = \{\lambda : 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots\}$ for each t_j.

Proof: The proof of the proposition is in two parts :

(I) existence of countably many positive solutions $\{u_j\}_{j=1}^{\infty}$ and

(II) existence result for infinitely many corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$

Part I

Define the sequences of open subsets of E by

$$\Omega_{1,j} = \{u \in E : \|u\| < \xi_j\}$$
$$\Omega_{2,j} = \{u \in E : \|u\| < \eta_j\}$$

Suppose $\{(t_0)_j\}_{j=1}^{\infty}$ is as in the hypothesis and $\frac{1}{2} > t_j > (t_0)_j > t_{j+1} > t_0$ $\forall\ j \in I\!N$. Next define the cone K_j by

$$K_j = \{u(t) \in E : u(t) \geq 0,\ t \in [0,1), \min_{t\in[(t_0)_j, 1-(t_0)_j]} u(t) \geq (t_0)_j \|u\|\}$$

Keep j fixed and let $u \in K_j \cap \delta\Omega_{2,j}$. For $s \in [(t_0)_j, 1-(t_0)_j]$

$$\eta_j = \|u\| \geq u(s) \geq \min_{t\in[(t_0)_j, 1-(t_0)_j]} u(s) \geq (t_0)_j \|u\| = (t_0)_j b_\eta \tag{3.3}$$

Using $(\sum_{10})$, we have

$$\begin{aligned}
\|\lambda N u\| &= \lambda \max_{t\in[0,1]} \int_0^1 \int_0^1 G(t,s)G(s,\tau)\alpha_i(\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau ds \\
&\geq \lambda \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} \int_0^1 G(t,s)G(s,\tau)\beta_i(\tau)h(u(\tau))d\tau ds \qquad (3.4) \\
&\geq \lambda \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} \int_0^1 G(t,s)G(s,\tau)d\tau ds \beta_2(\tau) D\eta_j \\
&\geq \lambda \int_{(t_0)_j}^{1-(t_0)_j} \max_{t\in[0,1]} \beta_2(\tau) D\eta_j d\tau \int_0^1 G(t,s)G(s,s)ds \\
&= \lambda \int_0^1 \max_{t\in[0,1]} \beta_2 D\eta_j d\tau \int_{(t_0)_j}^{1-(t_0)_j} G(t,s)G(s,s)ds \\
&\geq \lambda \max_{t\in[0,1]} \beta_2(\tau) D\eta_j \int_{(t_0)_j}^{1-(t_0)_j} G(s,t)d\tau \int_0^1 t_0 G(t^0,s)ds \\
&\geq \lambda D\eta_j c_1 \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} G(t,s)G(t^0,s)ds \int_0^1 t^0 d\tau \qquad (3.5)
\end{aligned}$$

Using $(\sum_4)$, (2.12) with $q=1$, we have

$$\begin{aligned}\lambda\|Nu\| &= \lambda \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} G(t,s)t^0G(t^0,s)ds \int_0^1 D\eta_j c_1 d\tau \\ &= \lambda \max_{t\in[0,1]} \frac{1}{4}[t(1-t)-((t_0)_j)^2][t^0(1-t^0)-((t_0)_j)^2]D\eta_j c_1 \int_0^1 t^0 d\tau \\ &= \lambda \tfrac{1}{4}\left(\tfrac{1}{4}-((t_0)_j)^2\right) D\eta_j c_1 t^0 \qquad (3.6)\end{aligned}$$

But $(t_0)_j < t_1 < \frac{1}{2}$, and as such $(\frac{1}{4}-((t_0))_j)^2) \geq (\frac{1}{4}-(t_1)^2) > 0$.
Since we have $D \geq \left\{\dfrac{4}{(c_1t^0(\frac{1}{4}-(t_1)^2)}\right\}$, it therefore follows that

$$\lambda\|Nu\|\| \geq \frac{1}{4}c_1t^0\left(\frac{1}{4}-(t_1)^2\right)D\eta_j \geq \eta_j = \|u\| \qquad (3.7)$$

The next step proceeds as follows:
Let $u \in K_j \cap \delta\Omega_{1,j}$ then $u(s) \leq \|u\| = \xi_j \ \ \forall \ \ s \in [0,1]$. Using $(\sum_9)$, we have that

$$\begin{aligned}\lambda\|Nu\| &= \lambda \max_{t\in[0,1]} \int_0^1\int_0^1 G(t,s)G(s,\tau)\alpha_i(\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau ds \\ &\leq \lambda \max_{t\in[0,1]} \int_{(t_0))_j}^{1-(t_0)_j}\int_0^1 G(t,s)G(s,\tau)\beta_1(\tau)h(u(\tau))d\tau ds \\ &\leq \lambda \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} G(t,s)G(s,\tau)ds\int_0^1 \beta_1(\tau)d\tau M\xi_j \\ &\leq \lambda\, M\xi_j \max_{t\in[0,1]} \int_{(t_0)_j}^{1-(t_0)_j} G(t,s)G(t^0,s)ds\int_0^1 \beta_1(\tau)t^0 d\tau \qquad (3.8)\end{aligned}$$

There are two cases to consider:
Case 1, $p \neq 1$
Let $p \neq 1$, so that $q = \frac{p}{p-1}$. Using Lemma 4, we have

$$\lambda\|Nu\| \leq \lambda \max_{t\in[0,1]} \|G(t,.)\|_q\|G(t^0,.)\|_q\|\beta_1\|_q\|\beta_1\|_pM\xi_jt^0 \qquad (3.9)$$

Using (2.10) and $(\sum_9)$, we obtain

$$\lambda\|Nu\| \leq \lambda\frac{1}{16}\left(\frac{1}{1+q}\right)^{\frac{2}{q}}\|\beta_1\|_p M\xi_j \leq \lambda\frac{\|\beta_1\|_p}{16}M\xi_j t^0 \leq \lambda\xi_j \tag{3.10}$$

Case 2, $p = 1$.
By (2.11) and $(\sum_9)$, we obtain

$$\begin{aligned}\lambda\|Nu\| = &\leq \lambda\max_{t\in[0,1]}\int_0^1(\beta_1)^{\frac{1}{2}}(\tau)(\beta_1)^{\frac{1}{2}}(\tau)t^0 d\tau\|G(t,.)\|_\infty\|G(t^0,.)\|_\infty\|\frac{1}{16}M\xi_j\\ &\leq \lambda\|\beta_1\|_p t^0\tfrac{M}{16}\xi_j \leq \lambda\xi_j\left(M \leq \tfrac{16}{\|t^0\beta_1\|_p}\right)\end{aligned} \tag{3.11}$$

In either case and because $\|u\| = \xi_j \ \forall \ u \in K_j \cap \Omega_{1,j}$, we have

$$\lambda\|Nu\| \leq \|u\|. \tag{3.12}$$

But $0 \in \Omega_{2,j} \subset \bar{\Omega}_{2,j} \subset \Omega_{1,j}$. Therefore by (3.7), (3.12) and Theorem 1
N has a fixed point $u_j \in K_j \cap (\bar{\Omega}_{1,j}\backslash\Omega_{2,j})$ such that $\eta_j \leq \|u_j\| \leq \xi_j$. But $j \in I\!N$ is arbitrary thus completing the proof of the first part of the Proposition.

Part II.

The arguments in the second part run as follows:
Let λ_j be given as in (3.2) and choose $\delta > 0$ such that

$$\frac{1}{\int_{(t_0)_j}^{1-(t_0)_j}\int_0^1 G(t,s)G(s,\tau)\beta_1(\tau)(h-\delta)d\tau ds} \leq \lambda_j$$

$$\leq \frac{1}{\int_0^1\int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)(h+\delta)d\tau ds}$$

Let N be the cone preserving, continuous and compact operator defined in (2.3). Assume $\exists\ H_1 > 0$ such that $h(u) \leq (h+\delta)u$ whenever $h(u) \in [0,\infty)$ for $u \in [0, H_1]$. We then obtain for $u \in K_{t_0},\ \ \|u\| = H_1,\ \ t_j \in [0,1]$,

$$\begin{aligned} \lambda Nu(t) &\leq \lambda \int_0^1\int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)h(u(\tau))d\tau ds \\ &\leq \lambda \int_0^1\int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)(h+\delta)u(\tau)d\tau ds \\ &\leq \lambda \int_0^1\int_0^1 G(s,s)G(s,\tau)\beta_2(\tau)(h+\delta)d\tau ds\|u\| \leq \|u\| \end{aligned} \tag{3.13}$$

Define $E_1 \subset E = \{u \in E : \|u\| \leq H_1\}$. We have $\lambda\|Nu\| \leq\ \|u\|\ \forall\ \ u \in K_{t_0} \cap \delta E_1$
It remains to consider the case when $\exists\ H_2 > 0$ such that $h(u) \geq (h - \delta)u$, (h is bounded), whenever $h \in [0,\infty)$, for $u \in [0, H_2)$. We have that

$u \in K_{t_0}$, $\|u\| < H_2 \; \forall \; t_j \in [(t_0)_j, 1-(t_0)_j]$ which leads to

$$\begin{aligned}
\lambda Nu(t) &\leq \lambda \int_0^1 \int_0^1 G(t,s)G(s,\tau)\beta_1(\tau)h(u(\tau))d\tau ds \\
&\leq \lambda \int_{(t_0)_j}^{1-(t_0)_j} \int_0^1 G(t,s)G(s,\tau)\beta_1(\tau)h(u(\tau))d\tau ds \\
&\leq \lambda \int_{(t_0)_j}^{1-(t_0)_j} \int_0^1 G(t,s)G(s,\tau)\beta_1(\tau)(h-\delta)u(\tau)d\tau ds \\
&\leq \lambda \int_{(t_0)_j}^{1-(t_0)_j} \int_0^1 G(s,s)G(s,\tau)\beta_1(\tau)(h-\delta)d\tau \|u\| \leq \|u\|. (3.14)
\end{aligned}$$

Define $E_2 \subset E = \{u \in E : \|u\| < H_2\}$, hence by the above inequality, we obtain, $\forall \; t_j \in [(t_0)_j, 1-(t_0)_j] \;\; u \in K_{t_0} \cap \delta\Omega_2$ and $\lambda\|Nu\| \geq \|u\|$ from which the result follows by (3.13) and (3.14). □

Example : As our example, we define a 1-parameter class of functions $a(t;\epsilon) : [0,1] \to I\!\!R^+$ by

$$a(t;\epsilon) = \sum_{j=1}^{\infty} \frac{\kappa[\tau_j, \tau_{j-1}]}{|t-t_j|^{\epsilon}} \text{ where we have set } \tau_j = (t_0)_j$$

and

$$t_j = \frac{1}{3^2} + \frac{1}{j+3^2}, \quad j \geq 1$$

$$\tau_0 = 1 \text{ and } \tau_j = \frac{1}{2}(t_j + t_{j+1}) = \frac{1}{3^2} + \frac{1}{2}(\frac{1}{j+3^2} + \frac{1}{j+1^2+3^2}), \quad j \geq 1$$

We have that

$$a(t:\epsilon) \geq a(1;\epsilon) = (\frac{90}{71})^{\epsilon} > 0 \quad \Longrightarrow t_j \downarrow \frac{1}{9}$$

Now if $0 < \epsilon < \frac{1}{3}$, we show that

$a(t) \in L^1(0,1)$ (and so $\alpha_i a(t) \in L^1(0,1)$ $(i=1,2)$ as follows:

Consider

$$\begin{aligned}
\sum_{j=1}^{\infty}\int_0^1 \frac{\kappa[\tau_j,\tau_{j-1}]}{|t-t_j|^{\epsilon}}dt &= \sum_{j=1}^{\infty}\int_{\tau_j}^{\tau_{j-1}} |t-t_j|^{-\epsilon}dt \\
&= \sum_{j=1}^{\infty}[\int_{\tau_j}^{t_j}(t_j-t)^{-\epsilon}dt+\int_{t_j}^{\tau_{j-1}}(t-t_j)^{-\epsilon}dt] \\
&= \frac{1}{1-\epsilon}[(\frac{21}{220})^{1-\epsilon}+(\frac{71}{90})^{1-\epsilon}] \\
&+ \frac{1}{1-\epsilon}(\frac{1}{2})^{1-\epsilon}\sum_{j=2}^{\infty}[(\frac{1}{j+3^2}-\frac{1}{j+1^2+3^2})^{1-\epsilon} \\
&+ (\tfrac{1}{j+2^2}-\tfrac{1}{j+3^2})^{1-\epsilon}] \\
&\le \frac{2}{1-\epsilon}+\frac{2^{\epsilon}}{2(1-\epsilon)}\sum_{j=2}^{\infty}[\frac{1}{(j+3^2)^{1-\epsilon}} \\
&(\tfrac{1}{(j+1^2+3^2)^{1-\epsilon}}+\tfrac{1}{(j+2^2)^{1-\epsilon}})] \\
&\le \frac{2}{1-\epsilon}+\frac{2^{\epsilon}}{1-\epsilon}\sum_{j=2}^{\infty}\frac{1}{(j+3^2)^{1-\epsilon}}\frac{1}{(j+2^2)^{1-\epsilon}} \\
&\le \frac{2}{1-\epsilon}+\frac{2^{\epsilon}}{1-\epsilon}\sum_{j=2}^{\infty}\frac{1}{j^{\frac{3}{2}(1-\epsilon)}}
\end{aligned}$$

We have that

$$\frac{2^{\epsilon}}{1-\epsilon}\sum_{j=2}^{\infty}\frac{1}{j^{\frac{3}{2}(1-\epsilon)}} \text{ converges since } \frac{3}{2}(1-\epsilon)>1 \text{ for } \epsilon<\frac{1}{3}$$

and it therefore follows that

$$\sum_{j=1}^{\infty}\int_0^1 \frac{\kappa[\tau_j-\tau_{j-1}]}{|t-t_j|^{\epsilon}}dt$$

converges. Thus $\sum_{j=1}^{\infty}\int_0^1 \frac{\kappa[\tau_j-\tau_{j-1}]}{|t-t_j|^{\epsilon}}dt$

$$\begin{aligned}
&= \int_0^1\sum_{j=1}^{\infty}\frac{\kappa[\tau_j-\tau_{j-1}]}{|t-t_j|^{\epsilon}}dt \\
&= \int_0^1 a(t)dt,
\end{aligned}$$

showing that $a(t;\epsilon)\in L^1[0,1]$. By induction on $p\in I\!N$, it can be shown that $a(t;\epsilon)\in L^p[0,1]$ for $\epsilon\in(0,\frac{1}{3}p)$. In this general framework, it is also necessary to relax condition (Σ_3) to read $0 < c \leq a(t)$ on A (of positive measure) $\subset [0,1]$ such that $\frac{1}{3} \in A$. We next define

$$f(t,u,u'') \leq h(u) = \begin{cases} M, \quad u > \xi_1 \\ \\ D\eta_j + \frac{M\xi_j - D\eta_j}{\xi_j - \eta_j}(u - \eta_j), \quad \eta_j \leq u \leq \xi_j,\ j \in I\!N \\ \\ M\xi_{j+1} + \frac{D\eta_j - M\xi_{j+1}}{\tau_j\eta_j - \xi_{j+1}}, \quad \xi_{j+1} < u \leq \tau_j\eta_j,\ j \in I\!N \\ 0, \quad u = 0 \end{cases}$$

where $\xi_1 = 1$, $\xi_{j+1} = \dfrac{1}{3 + (j+1)^2}$, $\eta_j = \dfrac{1}{3^2 + (3^2 + j^2 + 1)}$, $j \in I\!N$ and since the conditions $\xi_j > D\eta_j > \tau_j\eta_j > \xi_{j+1}$ $j \geq 1$ on the sequences $\{\xi_j\}_{j=1}^{\infty}$ and $\{\eta_j\}_{j=1}^{\infty}$ $\Rightarrow \xi_j \downarrow 0$, and $\eta_j \downarrow 0$, we thus have that for $\lambda \geq 1$, the solutions u_j of (1.1) - (1.2) satisfy $||u_j|| \downarrow 0$ and (3.2) holds for each t_j

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