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# Exponential stabilization of linear systems with interval nondifferentiable time-varying delays in state and control

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**Abstract**

This paper presents new exponentially stabilization conditions for a class of linear systems with time-varying delays in state and control. The time-delays function is assumed to be continuously belonging to a given interval in which the lower bound of delay is not restricted to zero. New delay-dependent, based on the constructing of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula, sufficient conditions for the exponential stabilization via memoryless control are established in terms of LMIs. That allows us to compute simultaneously the two bounds that characterize exponential stability of the solution. Numerical examples are given to demonstrate that the derived conditions are much less conservative than those given in the literature.

**Keywords:** Interval nondifferentiable time-varying delay, Exponentially stabilization, Linear matrix inequalities.
**2000MSC:** 34D05, 34D20, 34K20, 34K35

## 1 Introduction

Time-varying delays in control input are often encountered in many practical systems because of transmission of the measurement information. The existence of these delays may be the source of instability and poor performance of the closed-loop systems. Therefor, the problem of stabilization of control system with input delays has been received considerable attention from many researchers [4, 5, 6, 7, 9, 10, 11]. By using an improved state transformation, Chen and Zheng [3], Yue [19], Yue and Han [20] derived sufficent conditions for the robust stabilization of linear uncertain systems with unknown input delay in terms of LMI's but the system is required to be global controllable. In [8] Hien by using an improved Lyapunov-Krasovskii functional, a delay-dependent conditions for exponential stabilization are derived in terms of LMI. The conditions do not require any assumption about the controllability of the nominal system. However, the stabilization condition reported in this work require the delay functions to be constans.

On the other hand, the stabilization of dynamic system with interval time-varying delays has been a focused topic of theoretical and practical importance [2, 14, 15, 17, 18] in very recent years. Interval time-varying delay is a time delay that varies in an interval which the lower bound is not restricted to be 0. Delay-dependent robust exponential stabilization criteria for interval time-varying delay systems with norm-bounded uncertainties are proposed in [17], by using Lyapunov-Krasovskii functionals combined with the free-weighting matrices but the time-varying delays are required to be differentiable. It is noted that the former has more matrix variables than our sesult, but our sesult has less conservative and matrix variables than [17]. Recent, in [2] T. Botmart, P. Niamsup and V. N. Phat, based on the constructing of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula and the technique of dealing with some integral terms, a delay-dependent sufficient conditions for the exponential stabilization via memoryless control are established in terms of LMIs without introducing any free-weighting matrices. However, in the results, the interval time-varying delayed control input was not considered there. In addition, the approach used in paper [2] can not be applied to the system is studied by us.

In this paper, the problem of exponential stabilization for a class of linear systems with interval time-varying delays in state and control is studied. The time delays is a continuous functions belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is not necessary to be differentiable. Based on the constructing of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula, new delay-dependent sufficient conditions for the exponential stabilization via memoryless control are established in terms of LMIs.

The approach allows us to compute simultaneously the two bounds that characterize the exponential stability of the solution. Numerical examples are given to demonstrate that the derived conditions are much less conservative than those given in the literature.

## 2 Preliminaries

The following notation will be used in this paper. $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the $n-$dimensional space and the vector norm $\| \ . \ \|$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)-$dimensions. $A^T$ denotes the transpose of matrix $A$; $A$ is symmetric if $A = A^T$; $I$ denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of $A$; $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda; \lambda \in \lambda(A)\}$. $x_t := \{x(t+s) : s \in [-h,0]\}, \|x_t\| = \sup_{s\in[-h,0]}\sqrt{\|x(t+s)\|^2 + \|\dot{x}(t+s)\|^2}$; $C^1([0,t],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n-$valued continuously differentiable functions on $[0,t]$; $L_2([0,t],\mathbb{R}^m)$ denotes the set of all the $\mathbb{R}^m-$valued square integrable functions on $[0,t]$. Matrix $A$ is called semi-positive definite $(A \geq 0)$ if $\langle Ax,x\rangle \geq 0$, for all $x \in \mathbb{R}^n$; $A$ is positive definite $(A > 0)$ if $\langle Ax,x\rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

Consider a linear control system with interval time-varying delays in state and control of the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Dx(t-h(t)) + Bu(t) + B_1u(t-d(t)),\\ &x(t) = \phi(t), t \in [-\overline{h},0], \overline{h} = \{h_2,d_2\}, \end{aligned} \tag{1}$$

where the time-varying delays functions $h(t), d(t)$ satisfies

$$0 \leq h_1 \leq h(t) \leq h_2,$$

$$0 \leq d_1 \leq d(t) \leq d_2,$$

$x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the control, $A, D, B, B_1$ are given matrices of appropriate dimensions and $\phi(t) \in C^1([-h_2,0],\mathbb{R}^n)$ is the initial function with the norm

$$\|\phi\| = \sup_{-\bar{h}\leq t\leq 0}\sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

It is worth noting that the time delays are assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not restricted to being zero.

**Definition 2.1.** Given $\alpha > 0$. The system (1), where $u(t) = 0$, is $\alpha-$exponential stable if there exist a positive number $\beta > 0$ such that every solution $x(t, \phi)$ satisfies the following condition:

$$\| x(t, \phi) \| \leq \beta e^{-\alpha t} \| \phi \|, \quad \forall t \geq 0.$$

**Definition 2.2.** Given $\alpha > 0$. The system (1) is $\alpha-$exponential stabilizable if there exists a feedback control $u(t) = Kx(t), K \in \mathbb{R}^{m\times n}$ such that the closed-loop system

$$\begin{aligned} &\dot{x}(t) = [A + BK]x(t) + Dx(t - h(t)) + B_1 Kx(t - d(t)) \\ &x(t) = \phi(t), \quad t \in [-\overline{h}, 0]. \end{aligned} \tag{2}$$

is $\alpha-$exponential stable.

We introduce the following technical well-known propositions, which will be used in the proof of our results.

**Proposition 2.1.** (Gu, [4]) For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds

$$\Big( \int_0^\gamma \omega(s)\, ds \Big)^T M \Big( \int_0^\gamma \omega(s)\, ds \Big) \leq \gamma \Big( \int_0^\gamma \omega^T(s) M \omega(s)\, ds \Big)$$

**Proposition 2.2.** ( Matrix Cauchy inequality) For any symmetric possitive definite matrix $M \in \mathbb{R}^{n\times n}$ and $x, y \in \mathbb{R}^n$, we have

$$\pm 2x^T y \leq x^T M x + y^T M^{-1} y.$$

**Proposition 2.3.** (Schur complement lemma) Given constant symmetric matrices $X, Y, Z$ with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

# 3 Main result

Let us denoted

$$X = T^{-1}, \overline{P} = XPX, \overline{Q} = XQX, \overline{R} = XRX, \overline{S} = XSX, \overline{Z} = XZX,$$

$$\overline{G} = XGX, \overline{U} = XUX,$$

$$\lambda = \lambda_{\min}(\overline{P}),$$

$$\Lambda = \lambda_{\max}(\overline{P}) + h_1\lambda_{\max}(\overline{Q}) + \frac{1}{2}h_2^3\lambda_{\max}(\overline{R}) + \frac{1}{2}(h_2+h_1)(h_2-h_1)^2\lambda_{\max}(\overline{S}),$$

$$+ d_1\lambda_{\max}(\overline{Z}) + \frac{1}{2}d_2^3\lambda_{\max}(\overline{U}) + \frac{1}{2}(d_2+d_1)(d_2-d_1)^2\lambda_{\max}(\overline{G}),$$

$$\Xi_{11} = Q + Z + 2\alpha P - e^{-2\alpha h_2}R - e^{-2\alpha d_2}U + AT + TA^T + BY + Y^TB^T,$$

$$\Xi_{12} = DT + TA^T + Y^TB^T + e^{-2\alpha h_2}R,$$

$$\Xi_{13} = TA^T + Y^TB^T,$$

$$\Xi_{14} = B_1Y + TA^T + Y^TB^T + e^{-2\alpha d_2}U,$$

$$\Xi_{15} = TA^T + Y^TB^T,$$

$$\Xi_{16} = TA^T + Y^TB^T + P - T,$$

$$\Xi_{22} = DT + TD^T - e^{-2\alpha h_2}R - e^{-2\alpha h_2}S,$$

$$\Xi_{23} = e^{-2\alpha h_2}S + TD^T, \Xi_{24} = B_1Y + TD^T, \Xi_{26} = -T + TD^T,$$

$$\Xi_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_2}S,$$

$$\Xi_{44} = -e^{-2\alpha d_2}U - e^{-2\alpha d_2}G + B_1Y + Y^TB_1^T,$$

$$\Xi_{45} = Y^TB_1^T + e^{-2\alpha d_2}G, \Xi_{46} = Y^TB_1^T - T,$$

$$\Xi_{55} = -e^{-2\alpha d_1}Z - e^{-2\alpha d_2}G,$$

$$\Xi_{66} = h_2^2R + d_2^2U + (h_2-h_1)^2S + (d_2-d_1)^2G - 2T,$$

**Theorem 3.1.** *Given $\alpha > 0$. The system (1) is $\alpha-$exponential stabilization if there exists symmetric positive definite matrices $T, P, Q, R, S, Z, U, G$ and matrice $Y$, such that the following LMI holds:*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & TD^T & \Xi_{26} \\ * & * & \Xi_{33} & B_1Y & 0 & -T \\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ * & * & * & * & \Xi_{55} & -T \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0. \quad (3)$$

*The memoryless feedback control is given by*

$$u(t) = YT^{-1}x(t), \quad t \in \mathbb{R}^+$$

*and the solution $x(t, \phi)$ satisfies*

$$\|x(t, \phi)\| \leq \sqrt{\frac{\Lambda}{\lambda}} e^{-\alpha t} \|\phi\|, \quad t \in \mathbb{R}^+.$$

*Proof.* We consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{7} V_i(t, x_t),$$

where

$$V_1 = x^T(t)\overline{P}x(t), \quad V_2 = \int_{t-h_1}^{t} e^{2\alpha(s-t)} x^T(s)\overline{Q}x(s)\, ds,$$
$$V_3 = h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{R}\dot{x}(\tau)\, d\tau\, ds,$$
$$V_4 = (h_2 - h_1) \int_{-h_2}^{-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{S}\dot{x}(\tau)\, d\tau\, ds,$$
$$V_5 = \int_{t-d_1}^{t} e^{2\alpha(s-t)} x^T(s)\overline{Z}x(s)\, ds,$$
$$V_6 = d_2 \int_{-d_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{U}\dot{x}(\tau)\, d\tau\, ds,$$
$$V_7 = (d_2 - d_1) \int_{-d_2}^{-d_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{G}\dot{x}(\tau)\, d\tau\, ds.$$

It is easy to check that

$$\lambda \parallel x(t) \parallel^2 \leq V(t, x_t) \leq \Lambda \parallel x_t \parallel^2, \quad t \in \mathbb{R}^+. \tag{4}$$

Taking the derivative of $V_i, i = 1, \ldots, 7$ along the solution of system (2) we have

$$\begin{aligned}
&\dot V_1 = 2x^T(t)\overline{P}\dot x(t),\\
&\dot V_2 = -2\alpha V_2 + x^T(t)\overline{Q}x(t) - e^{-2\alpha h_1}x^T(t-h_1)\overline{Q}x(t-h_1),\\
&\dot V_3 = -2\alpha V_3 + h_2^2\dot x^T(t)\overline{R}\dot x(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot x^T(s)\overline{R}\dot x(s)\,ds,\\
&\dot V_4 = -2\alpha V_4 + (h_2-h_1)^2\dot x^T(t)\overline{S}\dot x(t) - (h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot x^T(s)\overline{S}\dot x(s)\,ds,\\
&\dot V_5 = -2\alpha V_5 + x^T(t)\overline{Z}x(t) - e^{-2\alpha d_1}x^T(t-d_1)\overline{Z}x(t-d_1),\\
&\dot V_6 = -2\alpha V_6 + d_2^2\dot x^T(t)\overline{U}\dot x(t) - d_2e^{-2\alpha d_2}\int_{t-d_2}^{t}\dot x^T(s)\overline{U}\dot x(s)\,ds,\\
&\dot V_7 = -2\alpha V_7 + (d_2-d_1)^2\dot x^T(t)\overline{G}\dot x(t) - (d_2-d_1)e^{-2\alpha d_2}\int_{t-d_2}^{t-d_1}\dot x^T(s)\overline{G}\dot x(s)\,ds.
\end{aligned} \tag{5}$$

Applying Proposition 2.1 and the Leibniz - Newton formula, we have

$$\begin{aligned}
-h_2\int_{t-h_2}^{t}\dot x^T(s)\overline{R}\dot x(s)\,ds &\le -h(t)\int_{t-h(t)}^{t}\dot x^T(s)\overline{R}\dot x(s)\,ds\\
&\le -\Big[\int_{t-h(t)}^{t}\dot x(s)\,ds\Big]^T\overline{R}\Big[\int_{t-h(t)}^{t}\dot x(s)\,ds\Big]\\
&= -[x(t)-x(t-h(t))]^T\overline{R}[x(t)-x(t-h(t))]\\
&= -x^T(t)\overline{R}x(t) + 2x^T(t)\overline{R}x(t-h(t))\\
&\quad - x^T(t-h(t))\overline{R}x(t-h(t))
\end{aligned} \tag{6}$$

$$\begin{aligned}
-d_2\int_{t-d_2}^{t}\dot x^T(s)\overline{U}\dot x(s)\,ds &\le -d(t)\int_{t-d(t)}^{t}\dot x^T(s)\overline{U}\dot x(s)\,ds\\
&\le -\Big[\int_{t-d(t)}^{t}\dot x(s)\,ds\Big]^T\overline{U}\Big[\int_{t-d(t)}^{t}\dot x(s)\,ds\Big]\\
&= -[x(t)-x(t-d(t))]^T\overline{U}[x(t)-x(t-d(t))]\\
&= -x^T(t)\overline{U}x(t) + 2x^T(t)\overline{U}x(t-d(t))\\
&\quad - x^T(t-d(t))\overline{U}x(t-d(t))
\end{aligned} \tag{7}$$

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}\dot{x}(s)\,ds \le -(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)\overline{S}\dot{x}(s)\,ds\\
&\le -\left[\int_{t-h(t)}^{t-h_1}\dot{x}(s)\,ds\right]^T\overline{S}\left[\int_{t-h(t)}^{t-h_1}\dot{x}(s)\,ds\right]\\
&= -[x(t-h_1)-x(t-h(t))]^T\overline{S}[x(t-h_1)-x(t-h(t))]\\
&= -x^T(t-h_1)\overline{S}x(t-h_1)+2x^T(t-h_1)\overline{S}x(t-h(t))\\
&\quad -x^T(t-h(t))\overline{S}x(t-h(t)),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&-(d_2-d_1)\int_{t-d_2}^{t-d_1}\dot{x}^T(s)\overline{G}\dot{x}(s)\,ds \le -(d(t)-d_1)\int_{t-d(t)}^{t-d_1}\dot{x}^T(s)\overline{G}\dot{x}(s)\,ds\\
&\le -\left[\int_{t-d(t)}^{t-d_1}\dot{x}(s)\,ds\right]^T\overline{G}\left[\int_{t-d(t)}^{t-d_1}\dot{x}(s)\,ds\right]\\
&= -[x(t-d_1)-x(t-d(t))]^T\overline{G}[x(t-d_1)-x(t-d(t))]\\
&= -x^T(t-d_1)\overline{G}x(t-d_1)+2x^T(t-d_1)\overline{G}x(t-d(t))\\
&\quad -x^T(t-d(t))\overline{G}x(t-d(t)).
\end{aligned} \tag{9}$$

Therefor

$$\begin{aligned}
&\dot{V}(t,x_t)+2\alpha V(t,x_t)\\
&\le x^T(t)[\overline{Q}+\overline{Z}+2\alpha\overline{P}-e^{-2\alpha h_2}\overline{R}-e^{-2\alpha d_2}\overline{U}\,]x(t)\\
&+\dot{x}^T(t)[h_2^2\overline{R}+d_2^2\overline{U}+(h_2-h_1)^2\overline{S}+(d_2-d_1)^2\overline{G}\,]\dot{x}(t)\\
&+x^T(t-h_1)[-e^{-2\alpha h_1}\overline{Q}-e^{-2\alpha h_2}\overline{S}\,]x(t-h_1)\\
&+x^T(t-h(t))[-e^{-2\alpha h_2}\overline{R}-e^{-2\alpha h_2}\overline{S}\,]x(t-h(t))\\
&+x^T(t-d_1)[-e^{-2\alpha d_1}\overline{Z}-e^{-2\alpha d_2}\overline{G}\,]x(t-d_1)\\
&+x^T(t-d(t))[-e^{-2\alpha d_2}\overline{U}-e^{-2\alpha d_2}\overline{G}\,]x(t-d(t))\\
&+2x^T(t)\overline{P}\dot{x}(t)+2e^{-2\alpha h_2}x^T(t)\overline{R}x(t-h(t))+2e^{-2\alpha h_2}x^T(t-h_1)\overline{S}x(t-h(t))\\
&+2e^{-2\alpha d_2}x^T(t)\overline{U}x(t-d(t))+2e^{-2\alpha d_2}x^T(t-d_1)\overline{G}x(t-d(t)).
\end{aligned} \tag{10}$$

By using the following identity relation

$$-\dot{x}(t)+[A+BK]x(t)+Dx(t-h(t))+B_1Kx(t-d(t))=0,$$

we obtain

$$2\Big[x^T(t)X + x^T(t-h(t))X + x^T(t-h_1)X + x^T(t-d(t))X + x^T(t-d_1)X +$$
$$\dot{x}^T(t)X\Big] \times \Big[-\dot{x}(t) + [A+BK]x(t) + Dx(t-h(t)) + B_1Kx(t-d(t))\Big] = 0. \tag{11}$$

Adding the zero item of (11) into (10), we have

$$\dot{V}(t,x_t) + 2\alpha V(t,x_t) \le \xi^T(t)\Omega\xi(t), \tag{12}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} \\ * & * & * & \Omega_{44} & \Omega_{45} & \Omega_{46} \\ * & * & * & * & \Omega_{55} & \Omega_{56} \\ * & * & * & * & * & \Omega_{66} \end{bmatrix},$$

$\Omega_{11} = XA + A^TX + XBK + K^TB^TX + \overline{Q} + \overline{Z} + 2\alpha\overline{P} - e^{-2\alpha h_2}\overline{R} - e^{-2\alpha d_2}\overline{U}$,
$\Omega_{12} = XD + A^TX + K^TB^TX + e^{-2\alpha h_2}\overline{R}$,
$\Omega_{13} = A^TX + K^TB^TX$,
$\Omega_{14} = XB_1K + A^TX + K^TB^TX + e^{-2\alpha d_2}\overline{U}$,
$\Omega_{15} = A^TX + K^TB^TX$,
$\Omega_{16} = -X + A^TX + K^TB^TX + \overline{P}$,
$\Omega_{22} = XD + D^TX - e^{-2\alpha h_2}\overline{R} - e^{-2\alpha h_2}\overline{S}$,
$\Omega_{23} = e^{-2\alpha h_2}\overline{S} + D^TX$,
$\Omega_{24} = XB_1K + D^TX, \Omega_{25} = D^TX, \Omega_{26} = -X + D^TX$,
$\Omega_{33} = -e^{-2\alpha h_1}\overline{Q} - e^{-2\alpha h_2}\overline{S}, \quad \Omega_{34} = XB_1K, \Omega_{35} = 0, \Omega_{36} = -X$,
$\Omega_{44} = -e^{-2\alpha d_2}\overline{U} - e^{-2\alpha d_2}\overline{G} + XB_1K + K^TB_1^TX$,
$\Omega_{45} = K^TB_1^TX + e^{-2\alpha d_2}\overline{G}, \Omega_{46} = -X + K^TB_1^TX$,
$\Omega_{55} = -e^{-2\alpha d_1}\overline{Z} - e^{-2\alpha d_2}\overline{G}, \Omega_{56} = -X$,
$\Omega_{66} = h_2^2\overline{R} + d_2^2\overline{U} + (h_2-h_1)^2\overline{S} + (d_2-d_1)^2\overline{G} - 2X$,
$\xi^T(t) = \Big[x^T(t) \;\; x^T(t-h(t)) \;\; x^T(t-h_1) \;\; x^T(t-d(t)) \;\; x^T(t-d_1) \;\; \dot{x}^T(t)\Big]$.

Pre- and post-multiplying both sides of $\Omega$ with

$$\Theta = \text{diag}\{T, T, T, T, T, T\},$$

and using the memoryless feedback control

$$u(t) = Kx(t), K = YT^{-1},$$

we have

$$\Xi = \Theta\Omega\Theta.$$

Note that $\Omega < 0$ if and only if $\Xi < 0$. Therefor, from condition (3), we obtain

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq 0. \tag{13}$$

Integrating both sides of (13) from 0 to $t$, we obtain

$$V(t, x_t) \leq V(0, x_0)e^{-2\alpha t}, \quad \forall t \in \mathbb{R}^+.$$

Furthermore, taking condition (4) into account, we have

$$\lambda \parallel x(t, \phi) \parallel^2 \leq V(t, x_t) \leq V(0, x_0)e^{-2\alpha t} \leq \Lambda e^{-2\alpha t} \parallel \phi \parallel^2,$$

then the solution $x(t, \phi)$ of the system satisfy

$$\parallel x(t, \phi) \parallel \leq \sqrt{\frac{\Lambda}{\lambda}} e^{-\alpha t} \parallel \phi \parallel, \quad \forall t \geq 0,$$

which implies the closed-loop system is $\alpha-$exponential stable. This completes the proof of the theorem. □

**Remark 3.1.** Based on the constructing of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula and the technique of dealing with some integral terms, new delay-dependent sufficient conditions for the exponential stabilization via memoryless control are established in terms of LMIs without introducing any free-weighting matrices. In addition, the time varying functions considered in this paper is not necessary differentiable and interval time-varying delays in state and control. So, the proposed stabilization criterion is independent of the derivative of time delay which can reduce the conservatism. Moreover, our results extend the results of T. Botmart, P. Niamsup and V. N. Phat [2].

Finally, as an application for Theorem 3.1, we give a sufficient condition for exponential stability of the class system without delayed control input was studied in [2].

Consider the control linear system with interval time-varying delays in state

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Dx(t-h(t)) + Bu(t),\\ &x(t) = \phi(t), \quad t \in [-h_2, 0]. \end{aligned} \tag{14}$$

Where the delay function $h(t)$ satisfies

$$0 \le h_1 \le h(t) \le h_2.$$

Let us denote

$$X = T^{-1}, \overline{P} = XPX, \overline{Q} = XQX, \overline{R} = XRX, \overline{S} = XSX,$$

$$\begin{aligned}
&\lambda_1 = \lambda_{\min}(\overline{P}),\\
&\lambda_2 = \lambda_{\max}(\overline{P}) + h_1\lambda_{\max}(\overline{Q}) + \frac{1}{2}h_2^3\lambda_{\max}(\overline{R}) + \frac{1}{2}(h_2+h_1)(h_2-h_1)^2\lambda_{\max}(\overline{S}),\\
&\Gamma_{11} = AT + TA^T + BY + Y^TB^T + Q + 2\alpha P - e^{-2\alpha h_2}R,\\
&\Gamma_{12} = DT + TA^T + Y^TB^T + e^{-2\alpha h_2}R,\\
&\Gamma_{14} = TA^T + Y^TB^T + P - T,\\
&\Gamma_{22} = DT + TD^T - e^{-2\alpha h_2}R - e^{-2\alpha h_2}S,\\
&\Gamma_{23} = e^{-2\alpha h_2}S + TD^T,\\
&\Gamma_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_2}S,\\
&\Gamma_{44} = h_2^2R + (h_2-h_1)^2S - 2T.
\end{aligned}$$

**Corollary 3.1.** *Given $\alpha > 0$. The system (14) is $\alpha-$exponential stabilization if there exists symmetric positive definite matrices $T, P, Q, R, S$ and matrice $Y$, such that the following LMI holds:*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & TA^T + Y^TB^T & \Gamma_{14}\\ * & \Gamma_{22} & \Gamma_{23} & -T + TD^T\\ * & * & \Gamma_{33} & -T\\ * & * & * & \Gamma_{44} \end{bmatrix} < 0 \tag{15}$$

*The memoryless feedback control is given by*

$$u(t) = YT^{-1}x(t), \quad t \in \mathbb{R}^+$$

*and the solution $x(t,\phi)$ satisfies*

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|, \quad t \in \mathbb{R}^+.$$

*Proof.* We consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{4} V_i(t, x_t),$$

for the system (14) where

$$V_1 = x^T(t)\overline{P}x(t), \quad V_2 = \int_{t-h_1}^{t} e^{2\alpha(s-t)} x^T(s)\overline{Q}x(s)\, ds,$$
$$V_3 = h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{R}\dot{x}(\tau)\, d\tau\, ds,$$
$$V_4 = (h_2 - h_1) \int_{-h_2}^{-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau)\overline{S}\dot{x}(\tau)\, d\tau\, ds,$$

By the same technique as in theorem 3.1, it is easy to prove this result. □

# 4 Numerical examples

In this section, we provide numerical examples to show the effectiveness of our result.

**Example 4.1.** Consider the linear system with interval nondifferentiable time-varying delays in state and control (1), where

$$\begin{cases} h(t) = 0.1 + 0.1\cos^2 t \quad if \quad t \in \mathcal{I} = \cup_{k\geq 0}[2k\pi, (2k+1)\pi] \\ h(t) = 0 \qquad\qquad\qquad if \quad t \in R^+ \setminus \mathcal{I}, \end{cases}$$

$$\begin{cases} d(t) = \beta(t), \quad if \quad t \in [0, 1] \\ d(t) = \beta(t-k), \quad if \quad t \in [k, k+1], k = 1, 2, ..., \end{cases}$$

where

$$\beta(t) = \begin{cases} t + 0.2, \quad t \in [0, 0.1] \\ -t + 0.4, \quad t \in (0.1, 0.2] \\ t, \quad t \in (0.2, 0.3] \\ -t + 0.1, \quad t \in (0.3, 0.4] \\ t - 0.2, \quad t \in (0.4, 0.5], \\ -t + 0.8, \quad t \in (0.5, 0.6], \\ t - 0.4, \quad t \in (0.6, 0.7], \\ -t + 0.5, \quad t \in (0.7, 0.8], \\ t - 0.6, \quad t \in (0.8, 0.9], \\ -t + 0.7, \quad t \in (0.9, 1] \end{cases},$$

$$A = \begin{bmatrix} 0.2 & 0 \\ 0 & -1 \end{bmatrix}, D = \begin{bmatrix} -0.5 & 0 \\ 0.3 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t), d(t)$ are non-differentiable. Therefore, the mothods is used in [8, 19, 20, 21] are not applicable to this system. We have

$$0.1 \le h(t) \le 0.2, 0.2 \le d(t) \le 0.3.$$

Given $\alpha = 0.1$. By using LMI toolbox of Matlab, we can verify that, the LMI (3) is satisfied with $h_1 = 0.1, h_2 = 0.2, d_1 = 0.2, d_2 = 0.3$ and

$$T = \begin{bmatrix} 1.4836 & -0.0810 \\ -0.0810 & 1.1414 \end{bmatrix}, P = \begin{bmatrix} 3.8637 & -0.3304 \\ -0.3304 & 3.5628 \end{bmatrix}, Q = Z = \begin{bmatrix} 1.0616 & 0.1174 \\ 0.1174 & 4.3953 \end{bmatrix},$$

$$R = S = U = G = \begin{bmatrix} 635.2305 & 0 \\ 0 & 635.2305 \end{bmatrix}, Y = \begin{bmatrix} -0.4186 & -1.8474 \end{bmatrix},$$

with a memoryless feedback controller

$$u(t) = \begin{bmatrix} -0.3720 & -1.6449 \end{bmatrix} x(t).$$

Thus, the system is $0.1-$exponential stabilization and the value $\sqrt{\frac{\Lambda}{\lambda}} = 2.8890$, so the solution of the closed-loop system satisfies

$$\|x(t, \phi)\| \le 2.8890 e^{-0.1t} \|\phi\|, \quad \forall t \ge 0.$$

**Example 4.2.** Consider the linear system with interval nondifferentiable time-varying delay in state (14), where

$$\begin{cases} h(t) = 0.1 + 0.25\sin^2 t \quad if \quad t \in \mathcal{I} = \cup_{k\geq 0}[2k\pi, (2k+1)\pi] \\ h(t) = 0 \qquad\qquad\qquad if \quad t \in R^+ \setminus \mathcal{I}, \end{cases}$$

$$A = \begin{bmatrix} -0.2 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.2 & -0.1 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t)$ is non-differentiable. Therefore, the mothods is used in [ 3, 6, 12, 13] are not applicable to this system. We have

$$0.1 \leq h(t) \leq 0.35.$$

Given $\alpha = 0.5$. By using LMI toolbox of Matlab, we can verify that, the LMI (15) is satisfied with $h_1 = 0.1, h_2 = 0.35$ and

$$T = \begin{bmatrix} 0.7764 & -0.0784 \\ -0.0784 & 1.3692 \end{bmatrix}, P = \begin{bmatrix} 1.5968 & -0.2093 \\ -0.2093 & 2.6627 \end{bmatrix}, Q = \begin{bmatrix} 0.1767 & -0.0962 \\ -0.0962 & 1.9114 \end{bmatrix},$$

$$R = S = \begin{bmatrix} 82.3741 & 0 \\ 0 & 82.3741 \end{bmatrix}, Y = \begin{bmatrix} 0.2160 & -4.0323 \end{bmatrix},$$

with a memoryless feedback controller

$$u(t) = \begin{bmatrix} -0.0195 & -2.9460 \end{bmatrix} x(t).$$

Thus, the system is 0.5−exponential stabilization and the value $\sqrt{\frac{\lambda_2}{\lambda_1}} = 2.3398$, so the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \leq 2.3398e^{-0.5t}\|\phi\|, \quad \forall t \geq 0.$$

Given $\alpha = 0.5$. When $h_1 = 0.1$, the maximum allowable bounds for $h_2$ obtained from Corrollary 3.1 of the system is 0.405 and from [2] is 0.3. Therefor, the maximum allowable bounds for $h_2$ obtained from Corrollary 3.1 are much better than that obtained in [2]. Moreover, the method is used in [1, 3, 6, 12, 13] are not applicable to this system.

# 5 Conclusions

In this paper, we have investigated the exponential stabilization via memoryless control for a class of linear systems with interval time-varying delays in state and control. The interval time-varying delay function is not necessary to be differentiable which allows time-delay function to be a fast time-varying function. A new class of Lyapunov-Krasovskii functional is constructed to prove delay-dependent sufficient conditions for the exponential stabilization via memoryless control in terms of LMIs. Numerical examples are given to illustrate the effectiveness of the theoretical results.

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# References

[1] I. Amri, D. Soudani and M. Benrejeb, Exponential stability and stabilizaion of linear systems with time varying delays, *6th International Multi-Conference on Systems, Signals and Devices*, (2009), 1-6.

[2] T. Botmart, P. Niamsup and V.N. Phat, Delay-dependent exponential stabilization for uncertain linear systems with interval nondifferentiable time-varying delays, *Applied Mathematics and Computation,* To appear.

[3] W.H. Chen and W.X.Zheng, On improved robust stabilization of uncertain systems with unknown input delay, *Automatica*, 42 (2006), 1072-1076.

[4] K. Gu, V.L. Kharitonov and J.Chen, *Stability of time-delay system*, Boston: Birkhauser; 2003.

[5] J.K. Hale and S.M. Verduyn Lunee, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[6] Q.-L. Han, Robust stability for a class of linear systems with time varying delay and nonlinear perturbations, *Comput. Math. Appl.*, 47 (2004), 1201-1209.

[7] Q.-L. Han and K. Gu, Stability of linear systems with time-varying delay: A generalized discretized Lyapunov functional approach, *Asian J. Control*, 3 (2001), 170-180.

[8] L.V. Hien and H.V. Thi, Exponential stabilization of linear systems with mixed delays in state and control, *Differential equations and Control processes,* 4 (2008), 32-42.

[9] L.V. Hien and V.N. Phat, Exponential stability and stabilizaion of a class of uncertain linear time-delay systems, *6J. Franklin Inst.*, 346 (2009), 611-625.

[10] V.L. Kharitonov and D. Hinrichsen, Exponential estimates for time-delay systems, *Systems and Control Letters*, 53 (2004), 395-405.

[11] V.B. Kolmanovskii, S. Niculescu and J. P. Richard, On the Lyapunov-Krasovskii functionals for stability analysis of linear delay systems, *Int. J. Control,* 72 (1999), 374-384.

[12] O.M. Kwon, and J.H. Park, Delay-range-dependent stabilization of uncertain dynamic systems with interval time-varying delays, *Appl. Math. Comput.,* 208 (2009), 58-68.

[13] X. Li and C.E. de Souza, Delay-Dependent Robust Stability and Stabilization of Uncertain Linear Delay Systems: A Linear Matrix Inequality Approach, *IEEE Trans. Automat. Control,* 42 (1997), 1144-1148.

[14] H.-Y. Shao, New delay-dependent stability criteria for systems with interval delay, *Automatica*, 45 (2009), 744-749.

[15] M. V. Thuan, Novel expoenetial estimate for nonlinear systems with mixed interval time-varying nondifferentiable delays, *African Diaspora Journal of Mathematics,* 11 (2011), 110-123.

[16] Y. Wang, L. Xie and C.E. de SOUZA, Robust control of a class of uncertain nonlinear systems, *Syst. control. Lett.* 19 (1992), 139-149.

[17] D. Wang and W. Wang, Delay-dependent robust exponential stabilization for uncertain systems with interval time-varying delays, *J. Control Theory Appl.,* 7(3) (2009), 257-263.

[18] K.-Y. Yu, C.-H. Lien, Stability criterion for uncertain neutral systems with interval time-varying delays, *Chaos Solitons Fractals*, 38 (2008), 650-657.

[19] D.Yue, Robust stabilization of uncertain systems with unknown input delay, *Automatica,* 40 (2004), 331-336.

[20] D. Yue and Q.L. Han, Delayed feedback control of uncertain systems with time-varying input delay, *Automatica,* 41 (2005), 233-240.

[21] X.-M. Zhanga, M. Wua, J.-H. Shec and Y. He, Delay-dependent stabilization of linear systemswith time-varying state and input delays, *Automatica*, 41 (2005), 1405-1412.