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Ordinary differential equations

On the existence of periodic solutions to nonlinear differential equations of second order

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Abstract

In this paper, we are concerned with the existence of periodic solutions to second order nonlinear differential equations of the form

$$x'' + c(t)x' + f(t,x) = p(t,x,x').$$

By using Leary-Schauder principle, we establish some new sufficient conditions which insure that this equation has a periodic solution.

Keywords: Periodic solutions, differential equation of second order.
AMS (MOS) Subject Classifications: 34C25, 65L06.

1. Introduction

It is well-known from qualitative theory of ordinary differential equations that the problem of existence of periodic solutions for various nonlinear differential equations of higher order continues to attract the attentions of many specialists despite its long history. In many works, the authors dealt with the problems by using Lyapunov's or Green's functions, and they obtained criteria for the existence of periodic solutions. In particular, for the use of Green's functions, ones can refer to [1-10] and the references registered in that sources. Meanwhile, it is worth mentioning that in 1998, Mehri and Shadman [3] discussed the existence of periodic solutions to second order nonlinear differential equations of the form

$$x'' + c(t)x' + f(t,x) = e(t)\,.$$

In [3], the authors used Leray-Schauder principle to show the existence of periodic solutions to this equation.

Now, consider the real second order nonlinear differential equations of the type:

$$x'' + c(t)x' + f(t,x) = p(t,x,x')\,, \qquad \textbf{(1)}$$

in which $p(t,x,x')$, $f(t,x)$ and $c(t)$ are continuous functions in their respective domains $[0,\omega)\times\Re^2$, $[0,\omega)\times\Re$ and $[0,\omega)$, respectively. Further, it is assumed that all initial value problems corresponding to equation (1) can be extended to $[0,\omega)$.

2. Main result

The following result is established.

Theorem: We assume that the following conditions hold:

(i) $|f(t,x)| \le \gamma\,|x| + \beta$ for all $t\in[0,\omega]$ and $|x| < \infty$,

where γ and β are some non-negative constants.

(ii) $|p(t,x,x')| \le |e(t)|$ for all t, x and x', and $e(t)$ is a continuous function for all $t\in[0,\omega]$.

(iii) $\gamma\left(\dfrac{\omega}{\pi}\right)^2 + \gamma_1\left(\dfrac{\omega}{\pi}\right) < 1$, $\gamma_1 = \max|c(t)|$.

Then equation (1) possesses a solution satisfying

$$x^{(i)}(0) + x^{(i)}(\omega) = 0\,,\ (i = 0,\ 1). \qquad \textbf{(2)}$$

Proof: First, we show an estimate on the magnitude of the solutions of problem:

$$x'' + c(t)x' = \mu\left[p(t,x,x') - f(t,x)\right],\ \mu\in[0,1],$$

$$x^{(i)}(0) + x^{(i)}(\omega) = 0\,,\ (i = 0,\ 1). \qquad \textbf{(3)}$$

We assume that $x(t)$ is a function of class $C^{n-1}[0,\omega]$ such that $x(t+\omega)+x(t)=0$ for all t, and we use Wirtinger's inequalities in the following from:

$$\left\|x^{(i-1)}(t)\right\|_2 \le \left(\frac{\omega}{\pi}\right)^{n-i+1}\left\|x^{(n)}(t)\right\|_2\ ,\ (i = 1,\ 2,\ldots,\ n), \qquad \textbf{(4)}$$

$$\|\cdot\|_2 = [\int_0^\omega |\cdot|^2 \, dt]^{\frac{1}{2}}.$$

Now, we suppose that $x(t)$ is a solution of the problem given by (3). In view of the assumptions of the theorem, it is easily followed from (3) that

$$|x''(t)| \le \gamma_1 |x'| + \mu \left[|e(t)| + \gamma |x(t)| + \beta \right].$$

Hence, by using the Minkowski's inequality

$$\|x''(t)\|_2 \le \gamma_1 \|x'(t)\|_2 + \mu \left\{ \|e(t)\|_2 + \gamma \|x(t)\|_2 + \beta\sqrt{\omega} \right\},$$

it can be seen from Wirtinger's inequality that

$$\|x''(t)\|_2 \le \gamma_1 \left(\frac{\omega}{\pi}\right) \|x'(t)\|_2 + \mu \left\{ \|e(t)\|_2 + \gamma \left(\frac{\omega}{\pi}\right)^2 \|x''(t)\|_2 + \beta\sqrt{\omega} \right\},$$

where

$$\left[1 - \gamma_1 \left(\frac{\omega}{\pi}\right) - \mu\gamma \left(\frac{\omega}{\pi}\right)^2 \right] \|x''(t)\|_2 \le \mu \left\{ \|e(t)\|_2 + \beta\sqrt{\omega} \right\}.$$

Making use of assumption (iii) of the theorem and in view of the fact $0 \le \mu \le 1$, we obtain

$$\|x''(t)\|_2 \le \frac{\|e(t)\|_2 + \beta\omega^{\frac{1}{2}}}{1 - \gamma_1 \left(\frac{\omega}{\pi}\right) - \gamma \left(\frac{\omega}{\pi}\right)^2}. \qquad \textbf{(5)}$$

Now, we write

$$x^{(i-1)}(t) = x^{(i-1)}(0) + \int_0^t x^{(i)}(\tau) d\tau, \ (i = 1, 2).$$

For $t = \omega$, we have

$$x^{(i-1)}(\omega) = x^{(i-1)}(0) + \int_0^\omega x^{(i)}(\tau) d\tau, \ (i = 1, 2). \qquad \textbf{(6)}$$

By noting the equality $x^{(i)}(0) + x^{(i)}(\omega) = 0$, we get

$$x^{(i-1)}(0) = -x^{(i-1)}(\omega) = -\frac{1}{2}\int_0^\omega x^{(i)}(\tau) d\tau, \ \ (i = 1, 2).$$

Hence

$$x^{(i-1)}(t) = -\frac{1}{2}\int_0^\omega x^{(i)}(\tau) d\tau + \int_0^t x^{(i)}(\tau) d\tau = \frac{1}{2}\int_0^t x^{(i)}(\tau) d\tau \qquad \textbf{(7)}$$

by (6).
Clearly, (7) implies that

$$\left|x^{(i-1)}(t)\right| \le \frac{1}{2}\int_0^{\omega}\left|x^{(i)}(\tau)\right|d\tau . \qquad \textbf{(8)}$$

Now, it follows from (8) that

$$\left|x^{(i-1)}(t)\right| \le \frac{1}{2}\sqrt{\omega}\,\left\|x^{(i)}(t)\right\|_2 ,\ (i=1,2). \qquad \textbf{(9)}$$

Combining the inequality (9) with Wirtinger's inequality and (5), we obtain

$$\left|x^{(i-1)}(t)\right| \le \frac{\mu}{2}\sqrt{\omega}\left(\frac{\omega}{\pi}\right)^{3-i}\frac{\left\|e(t)\right\|_2+\beta\ \sqrt{\omega}}{1-\gamma_1\left(\frac{\omega}{\pi}\right)-\gamma\left(\frac{\omega}{\pi}\right)^2} . \qquad \textbf{(10)}$$

For $\mu = 0$, it is clear from (10) that

$$x^{(i-1)}(t)=0,\ t\in[0,\omega],\ (i=1,2).$$

That is, $x(t)=0$ and $x'(t)=0$. In this case, it follows that

$$x''(t)+c(t)x'(t)=0,$$

$$x^{(i)}(0)+x^{(i)}(\omega)=0,\ (i=1,2)$$

has only a trivial solution which satisfies the boundary conditions (2),

$$x^{(i)}(0)+x^{(i)}(\omega)=0,\ (i=1,2).$$

This guarantees the existence of a Green's function $g(t,s)$ for the problem (3). Clearly, the problem (3) is equivalent to

$$x(t)=\mu\int_0^{\omega} g(t,s)\left[p(s,x(s),x'(s))-f(s,x(s))\right]ds . \qquad \textbf{(11)}$$

Now, we consider the space $C^2[0,\omega]$ normed by

$$\|x\|_{c^2}=\max\left|x^{(i-1)}(t)\right|,\ t\in[0,\omega],\ (i=1,2).$$

Let B_ρ be the space

$$B_\rho=\left\{x(t)\in C^2[a,b]:\|x\|_{c^2}\le\rho\right\},$$

where

$$\rho = \max\left\{ \frac{1}{2}\sqrt{\omega}\left(\frac{\omega}{\pi}\right)^{3-i} \left(\|e(t)\|_2 + \beta\ \sqrt{\omega}\right)\left(1 - \gamma_1\left(\frac{\omega}{\pi}\right) - \gamma\left(\frac{\omega}{\pi}\right)^2\right)^{-1}\right\},\ (i = 1, 2).$$

In view of the space

$$S_R = \left\{x(t) \in C^2[0,\omega] : \|x\|_{c^2} = R\right\},$$

it follows for arbitrary $R > \rho$ that equation (11) has no solution on S_R. Now, by noting Leary-Schauder principle and the complete continuity of the operator

$$(Lx)(t) = \mu\int_0^{\omega} g(t,s)\left[p(s,x(s),x'(s)) - f(s,x(s))\right]ds, \qquad \textbf{(12)}$$

ones can conclude that equation (11) has at least a solution in the open sphere $\left\{x : \|x\|_{c^2} < R\right\}$. As a result, there exists a solution in B_ρ. Therefore, this fact implies that the problem (3) has, at the least, a solution for $\mu = 1$.

Remark: When we take $p(t,x,y) = e(t)$ in equation (1), then the conditions of theorem reduce those of Mehri and Shadman [Theorem 1, 3]. It is also clear that our result generalize that of Mehri and Shadman [Theorem 1, 3].

Corollary: Suppose that the assumptions of the theorem and the following conditions are satisfied:
(iv) $c(t)$ is an ω-periodic function, i.e., $c(t+\omega) \equiv c(t)$,
(v) $p(t,x,y)$ is a 2ω-periodic function, i. e., $p(t+2\omega,x,y) \equiv p(t,x,y)$ and $p(t+\omega,x,y) \equiv -p(t,x,y)$,
(vi) $f(t+\omega,x) \equiv f(t,x)$ and also $f(t,-x) \equiv f(t,x)$.
Then, equation (1) has a 2ω-periodic solution.

Proof: Let $\bar{x}(t)$ be a 2ω-periodic extension of $x(t)$ defined by

$$\bar{x}(t) = \begin{cases} x(t), & 0 \le t \le \omega \\ -x(t+\omega), & -\omega \le t \le 0. \end{cases}$$

Clearly, $\bar{x}(t) \in C^2[-\omega,\omega]$. By using the assumptions (iv) and (v), it can be shown that $\bar{x}(t)$ is a solution of equation (1) which satisfies the boundary conditions

$$\bar{x}^{(i)}(\omega) = \bar{x}^{(i)}(-\omega),\ (i = 1, 2).$$

Since

$$\int_0^{2\omega}\bar{x}(t)dt = \int_0^{\omega}\bar{x}(t)dt + \int_{\omega}^{2\omega}\bar{x}(t)dt,$$

we get

$$\int_0^{2\omega}\bar{x}(t)dt = \int_0^{\omega}\bar{x}(t)dt + \int_0^{\omega}\bar{x}(t+\omega)dt = 0\,.$$

This shows that the solution $\bar{x}(t)$ has a zero mean value. This completes the proof of the corollary.

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