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Existence of Periodic Solutions of Fourth Order Functional Differential Equations with $p-$Laplacian [1]

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Abstract. Sufficient conditions for the existence of at least one periodic solution of a nonlinear fourth order functional differential equations with $p-$Laplacian are established. Examples are presented to illustrate the main result.

Keywords. Periodic solution; fourth order functional differential equation with $p-$Laplacian; fixed-point theorem; growth condition

2000 MR subject classification. 34B10, 34B15

1 Introduction

Fourth-order differential equations with or without $p-$Laplacian occur in beam theory [1,3]. The solvability of such equations with different boundary conditions has been studied in papers [4-11,13-30,32-34]. The methods used in above mentioned papers are the fixed point theorems in cones in Banach spaces [1,9,13,16,17,21,19,20,28,30,33], the continuation theorem of coincidence degree [11,26,23,27], the upper and lower solutions methods with the monotone iterative technique [4-6,8,15,32,34] and the Leray-Schauder fixed point theorem [7,13,25,28,33].

The properties of solutions of the fourth order ordinary or functional differential equations are also studied by many authors, for example, Amster and Mariani [2] studied the oscillatory properties of solutions of a fourth order differential equation. In paper [31], Tanigawa established oscillation and non-oscillation theorems for a class of fourth order differential equations with $p-$Laplacian.

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However, the results on the existence of periodic solutions of the fourth order functional differential equations with $p-$Laplacian have not been found in known literature.

To fill this gap, in this paper, we use Mawhin′s continuation theorem of coincidence degree (Theorem IV.13 of [12]) to establish sufficient conditions for the existence of at least one $T-$periodic solution of the following fourth order functional differential equations with $p-$Laplacian

$$[q(t)\phi(x''(t))]'' = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))),\ t \in R, \tag{1-1}$$

where $T > 0$ is a constant, $\tau_k \in C^1(R)$ for all $k = 1, \cdots, m$ are invertible, the inverse function of $\tau = \tau_i(t)$ is denoted by $t = \mu_i(\tau)(i = 1, 2, \cdots, m)$, $f : R \times R^{m+1} \to R$ is a Caratheddory function, i.e., $f : t \to f(t, x_0, x_1, x_2, \cdots, x_m)$ is $T-$periodic and measurable on $[0, T]$, $f : (x_0, x_1, \cdots, x_m) \to f(t, x_0, x_1, \cdots, x_m)$ is continuous and for each $r > 0$ there exists $\phi_r \in L^1[0, T]$ such that $|f(t, x_0, x_1, x_2, \cdots, x_m)| \le \phi_r(t)$ holds for all $t \in [0, T]$ and $|x_i| \le r(i = 0, 1, 2, 3, \cdots, m)$, $q : R \to (0, +\infty)$ is $T-$periodic, $\phi(x) = |x|^{p-2}x$ with $p > 1$, which is called a $p-$Laplacian, and its inverse function is $\phi^{-1}(x) = |x|^{q-2}x$ with $1/q + 1/p = 1$.

The remainder is divided into two sections. In Section 2, we present the main results. In Section 3, we give some examples to illustrate the main theorems.

2 Main Results

Let PC^0 be the set of all continuous $T-$periodic functions on R and $X = PC^0 \times PC^0$, the norm is defined by

$$||(x, y)|| = \max\left\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)|\right\}$$

for $(x, y) \in X$. Then X is a real Banach space.

Let PL^1 be the set of all $T-$periodic functions which are measurable on $[0, T]$ and $Y = PL^1 \times PL^1$, the norm is defined by

$$||(u, v)|| = \max\left\{\int_0^T |u(t)|dt,\ \int_0^T |v(t)|dt\right\}$$

for $(u, v) \in Y$. Then Y is a real Banach space.

We also use the Sobolev spaces

$$PW^{2,1} = \left\{x : R \to R \,\middle|\, \begin{array}{c} x, x' \text{ are absolutely continuous} \\ \text{and } T-\text{periodic on } R \text{ with } x'' \in PL^1 \end{array}\right\}$$

and

$$PW_q^{2,1} = \left\{x : R \to R \,\middle|\, \begin{array}{c} qx, (qx)' \text{ are absolutely continuous} \\ \text{and } T-\text{periodic on } R \text{ with } (qx)'' \in PL^1 \end{array}\right\}.$$

Let $D(L) = PW^{2,1} \times PW_q^{2,1}$. Define the linear operator $L : D(L) \cap X \to Y$ by

$$L\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ (q(t)y(t))'' \end{pmatrix} \text{ for all } (x, y) \in D(L) \cap X. \tag{2-2}$$

Define the nonlinear operator $N : X \to Y$ by

$$N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))) \end{pmatrix} \text{ for all } (x, y) \in X. \qquad (2\text{-}3)$$

Now, we will briefly recall some notation and an abstract existence result. Let X, Y be real Banach spaces, $L : D(L) \bigcap X \to Y$ be a Fredholm map of index 0 and $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$ and $X = \mathrm{Ker}L \bigoplus \mathrm{Ker}P$, $Y = \mathrm{Im}L \bigoplus \mathrm{Im}Q$. It follows that $L_{D(L)\bigcap \mathrm{Ker}P} : D(L)L \bigcap \mathrm{Ker}P \to \mathrm{Im}L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subject of X such that $D(L) \bigcap \Omega \neq \emptyset$, the map $N : X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1.[12] Let X and Y be Banach spaces. Let $L : D(L) \bigcap X \to Y$ be a Fredholm operator of index zero and Ω be an open bounded subset of X with $\Omega \bigcap D(L) \neq \emptyset$. Suppose that $N : X \to Y$ be $L-$compact on $\overline{\Omega}$ and the following conditions are satisfied:

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$;

(iii) $\deg(\wedge QN\big|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, where $\wedge :$ $\mathrm{Ker}L \to Y/\mathrm{Im}L$ is the isomorphism.
Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Let $X = PC^0 \times PC^0$ and $Y = PL^1 \times PL^1$ and L, N be defined by (2) and (3) respectively. It is easy to show the following results. We omit their proofs since the proofs are simple and standard.

(i) $\mathrm{Ker}L = \{(a, b/q(t)) : a, b \in R\}$;

(ii) $\mathrm{Im}L = \{(u, v) \in X : \int_0^T u(s)ds = 0, \int_0^T v(t)dt = 0\}$;

(iii) L is a Fredholm operator of index zero;

(iv) there exist the projectors $P : X \to X$ and $Q : X \to X$ such that $\mathrm{Ker}L = \mathrm{Im}P$ and $\mathrm{Ker}Q = \mathrm{Im}L$. There exists an isomorphism $\wedge : \mathrm{Ker}L \to Y/\mathrm{Im}L$.

(v) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$;

(vi) $(x, y) \in D(L)$ is a solution of the operator equation $L(x, y) = N(x, y)$ implies that x is a $T-$periodic solution of equation (1).

In fact, let $F(t) = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))$. We have, for $a, b \in R$, $(x, y) \in X$ and $(u, v) \in Y$, that

$$
\begin{aligned}
P\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} x(0) \\ q(0)y(0)/q(t) \end{pmatrix}, \\
Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} \frac{1}{T}\int_0^T u(t)dt \\ \frac{1}{T}\int_0^T v(t)dt \end{pmatrix}, \\
K_p\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} \int_0^t (t-s)u(s)ds - \frac{1}{T}\int_0^T (T-s)u(s)ds \\ \frac{1}{q(t)}\left(\int_0^t (t-s)v(s)ds - \frac{1}{T}\int_0^T (T-s)v(s)ds\right) \end{pmatrix}, \\
K_p(I-Q)N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= K_p(I-Q)\begin{pmatrix} \phi^{-1}(y(t)) \\ f(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t))) \end{pmatrix} \\
&= \begin{pmatrix} \int_0^t (t-s)\phi^{-1}(y(s))ds - \frac{1}{T}\int_0^T (T-s)\phi^{-1}(y(s))ds \\ \frac{1}{q(t)}\left(\int_0^t (t-s)F(s)ds - \frac{1}{T}\int_0^T (T-s)F(s)ds\right) \end{pmatrix} \\
&\quad - \begin{pmatrix} \frac{t^2}{2T}\int_0^T \phi^{-1}(y(s))ds - \frac{T}{2}\int_0^T \phi^{-1}(y(s))ds \\ \frac{1}{q(t)}\left(\frac{t^2}{2T}\int_0^T F(s)ds - \frac{T}{2}\int_0^T F(s)ds\right) \end{pmatrix}, \\
\wedge\begin{pmatrix} a \\ b/q(t) \end{pmatrix} &= \begin{pmatrix} b \\ a \end{pmatrix}.
\end{aligned}
$$

Let us list some assumptions:

(B1) there exist the numbers $\beta > 0$, $\theta > 1$, the nonnegative functions $p_i \in PC^0(i = 0, 1, 2, \cdots, m)$, the function $r : R \to R$ with $\int_0^T |r(t)|^{\frac{\theta+1}{\theta}} dt < \infty$, and the Caratheddory functions $g(t, x_0, \cdots, x_m)$, $h(t, x_0, \cdots, x_m)$ such that

$$
f(t, x_0, \cdots, x_m) = g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),
$$
$$
g(t, x_0, x_1, \cdots, x_m)x_0 \le -\beta|x_0|^{\theta+1},
$$

and

$$
|h(t, x_0, \cdots, x_m)| \le \sum_{i=0}^{m} p_i(t)|x_i|^\theta + r(t),
$$

for all $t \in R$, $(x_0, x_1, \cdots, x_m) \in R^{m+1}$.

(B2) there exists a positive constant μ such that $q(t) > \mu$ for all $t \in [0, T]$, and there exist nonnegative constants M_i^0 such that $|\tau_i(T) - \tau_i(0)| \le M_i^0 T$ for $i = 1, \cdots, m$.

Lemma 2.2. Let $\delta_i = \max_{t\in[0,T]} \frac{1}{|\tau_i'(t)|}$, $(i = 1, \cdots, m)$, and $\Omega_1 = \{(x, y) : L(x, y) = \lambda N(x, y),\ ((x, y), \lambda) \in [(D(L) \setminus \mathrm{Ker}L)] \times (0, 1)\}$. Suppose that (B1) and (B2) hold. Then Ω_1 is bounded if

$$
\sup_{t\in[0,T]} |p_0(t)| + \sum_{i=1}^{m} \sup_{t\in[0,T]} |p_i(t)| M_i^0 \delta_i^{\frac{\theta}{\theta+1}} < \beta. \tag{2-4}
$$

Proof. For $(x, y) \in \Omega_1$, we have $L(x, y) = \lambda N(x, y)$, $\lambda \in (0, 1)$, i.e.

$$
\begin{cases} x''(t) = \lambda\phi^{-1}(y(t)), \\ (q(t)y(t))'' = \lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))). \end{cases}
$$

It follows that

$$[q(t)\phi(x''(t))]'' = \phi(\lambda)\lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))). \tag{2-5}$$

Thus

$$[q(t)\phi(x''(t))]''x(t) = \phi(\lambda)\lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t).$$

Integrating it from 0 to T, we get

$$\int_0^T q(t)\phi(x''(t))x''(t)dt = \phi(\lambda)\lambda \int_0^T f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt, \tag{2-6}$$

together with (B1) and $\int_0^T q(t)\phi(x''(t))x''(t)dt \geq 0$, we get that

$$\begin{aligned}
&\beta \int_0^T |x(t)|^{\theta+1}dt \\
&\leq -\int_0^T g(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt \\
&\leq \int_0^T h(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt \\
&\leq \int_0^T |h(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))||x(t)|dt \\
&\leq \int_0^T p_0(t)|x(t)|^{\theta+1}dt + \sum_{i=1}^m \int_0^T p_i(t)|x(\tau_i(t))|^\theta |x(t)|dt + \int_0^T r(t)|x(t)|dt \\
&\leq \max_{t\in[0,T]} |p_0(t)| \int_0^T |x(t)|^{\theta+1}dt + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^m \max_{t\in[0,T]} |p_i(t)| \left[\int_0^T |x(\tau_i(t))|^{1+\theta}dt\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&\leq \max_{t\in[0,T]} |p_0(t)| \int_0^T |x(t)|^{\theta+1}dt + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^m \max_{t\in[0,T]} |p_i(t)| \left[\int_{\tau_i(0)}^{\tau_i(T)} |x(s)|^{1+\theta}d\mu_i(s)\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&\leq \max_{t\in[0,T]} |p_0(t)| \int_0^T |x(t)|^{\theta+1}dt + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^m \max_{t\in[0,T]} |p_i(t)| \left|\int_{\tau_i(0)}^{\tau_i(T)} |x(s)|^{1+\theta}\frac{ds}{|\tau_i'(t)|}\right|^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&\leq \max_{t\in[0,T]} |p_0(t)| \int_0^T |x(t)|^{\theta+1}dt + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}} \left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^m \max_{t\in[0,T]} |p_i(t)|\delta_i^{\frac{\theta}{\theta+1}} M_i^0 \int_0^T |x(t)|^{\theta+1}dt.
\end{aligned}$$

Since

$$\beta > \max_{t\in[0,T]} |p_0(t)| + \sum_{i=1}^m \delta_i^{\frac{\theta}{\theta+1}} M_i^0 \max_{t\in[0,T]} |p_i(t)|, \tag{2-7}$$

there is a constant $M_1 > 0$ such that $\int_0^T |x(t)|^{\theta+1}dt \le M_1$. So there is $\xi \in [0,T]$ such that $|x(\xi)| \le (M_1/T)^{\frac{1}{\theta+1}}$. Further more we have

$$\begin{aligned}
\int_0^T q(t)|x''(t)|^p dt &= \int_0^T q(t)\phi(x''(t))x''(t)dt \\
&= \phi(\lambda)\lambda \int_0^T f(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt \\
&= \phi(\lambda)\lambda \int_0^T g(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt \\
&\quad +\phi(\lambda)\lambda \int_0^T h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt \\
&\le \phi(\lambda)\lambda \int_0^T h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt \\
&\le \int_0^T |h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))||x(t)|dt \\
&\le \max_{t\in[0,T]}|p_0(t)|\int_0^T |x(t)|^{\theta+1}dt + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}}\left[\int_0^T |x(t)|^{\theta+1}dt\right]^{\frac{1}{\theta+1}} \\
&\quad +\sum_{i=1}^m \max_{t\in[0,T]}|p_i(t)|\delta_i^{\frac{\theta}{\theta+1}}M_i^0\int_0^T |x(t)|^{\theta+1}dt \\
&\le ||p_0||M_1 + \left[\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right]^{\frac{\theta}{\theta+1}} M_1^{\frac{1}{\theta+1}} + \sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}}M_i^0 M_1.
\end{aligned}$$

Since there exists $\eta \in [0,T]$ such that $x'(\eta) = 0$, it is easy to see from (B2) that

$$\begin{aligned}
|x(t)| &= \left|x(\xi) + \int_\xi^t x'(s)ds\right| \le (M_1/T)^{\frac{1}{\theta+1}} + \int_0^T |x''(t)|dt \\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(\mu\int_0^T |x''(t)|^p dt\right)^{\frac{1}{p}} \\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(\int_0^T q(t)|x''(t)|^p dt\right)^{\frac{1}{p}} \\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(||p_0||M_1 + T^{\frac{\theta}{\theta+1}}||r||M_1^{\frac{1}{\theta+1}} + \sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}}M_i^0 M_1\right)^{\frac{1}{p}}
\end{aligned}$$

Hence there is a constant $M_2 > 0$ such that $||x|| \le M_2$.

It is easy to show that there exist numbers $\xi, \eta \in [0,T]$ such that $[qy]'(\xi) = 0$ and $[qy](\eta) = 0$. Hence

$$\begin{aligned}
|[q(t)y(t)]'| &= \left|\int_\xi^t [q(t)y(t)]''dt\right| \le \int_0^T |[q(t)y(t)]''|dt \\
&\le \int_0^T |f(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))|dt \\
&\le T \max_{t\in[0,T],|x_i|\le M_2, i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|.
\end{aligned}$$

So

$$|q(t)y(t)| \leq \int_0^T |(q(t)y(t))'|dt \leq T^2 \max_{t\in[0,T],|x_i|\leq M_2, i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|, \ \ t \in [0,T].$$

Then

$$|y(t)| \leq \frac{T^2}{\mu} \max_{t\in[0,T],|x_i|\leq M_2, i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|, \ \ t \in [0,T].$$

This implies that

$$||y|| \leq \frac{T^2}{\mu} \max_{t\in[0,T],|x_i|\leq M_2, i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|, \ \ t \in [0,T].$$

It follows that, for $(x,y) \in \Omega_1$, one has that there is $H > 0$ such that $||(x,y)|| \leq H$. Hence Ω_1 is bounded.

Suppose

(B3) There exists a constant $M > 0$ such that

$$a\int_0^T f(t,a,\cdots,a)dt > 0 \text{ for all } |a| > M.$$

Lemma 2.3. Suppose that (B3) holds. Then $\Omega_2 = \{(x,y) \in \mathrm{Ker}L : \ N(x,y) \in \mathrm{Im}L\}$ is bounded.

Proof. For $(a, b/q(t)) \in \mathrm{Ker}L$, we have $N(a,b) = (\phi^{-1}(b/q(t)), f(t,a,\cdots,a))$. $N(a,b) \in \mathrm{Im}L$ implies that

$$\int_0^T \phi^{-1}(b/q(t))dt = 0, \quad \int_0^T f(t,a,\cdots,a)dt = 0.$$

It follows from condition (B3) that $|a| \leq M$ and $b = 0$. Thus Ω_2 is bounded.

Lemma 2.4. Suppose that (B3) holds. Then $\Omega_3 = \{(x,y) \in \mathrm{Ker}L : \ \lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}$ is bounded, where $\wedge : \mathrm{Ker}L \to Y/\mathrm{Im}L$ defined by $\wedge(a, b/q(t)) = (b,a)$.

Proof. For $(a, b/q(t)) \in \Omega_3$, we have

$$-(1-\lambda)\int_0^T \phi^{-1}(b/q(t))dt = \lambda bT, \ \ -(1-\lambda)\int_0^T f(t,a,\cdots,a)dt = \lambda aT,$$

where $\lambda \in [0,1]$.

If $\lambda = 1$, then $a = b = 0$. If $\lambda \neq 1$, and $|a| > M$, it follows from (B3) that

$$0 \geq -(1-\lambda)a\int_0^T f(t,a,\cdots,a)dt = \lambda a^2 T > 0,$$

a contradiction. So $|a| \leq M$. Similarly, since

$$0 > -(1-\lambda)\int_0^T b\phi^{-1}(b/q(t))dt = \lambda b^2 T \geq 0 \text{ for } \lambda \in [0,1), b \neq 0,$$

we get $b = 0$. Hence Ω_3 is bounded.

Theorem L. Suppose that (B1), (B2) and (B3) hold. Then equation (1) has at least one $T-$periodic solution if (4) holds.

Proof. Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$, where Ω_1 is defined in Lemma 2.2, Ω_2 in Lemma 2.3 and Ω_3 in Lemma 2.4. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus, from Lemma 2.2 and Lemma 2.3, that $L(x,y) \neq \lambda N(x,y)$ for $(x,y) \in D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x,y) \notin \mathrm{Im}L$ for $(x,y) \in \mathrm{Ker}L \cap \partial\Omega$.

We know that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Since (x,y) is a solution of $L(x,y) = N(x,y)$ implies that x is a solution of equation (1). It suffices to get a solution (x,y) of $L(x,y) = N(x,y)$. To apply Lemma 2.1, we prove that (iii) of Lemma 2.1 (Theorem IV.13 of [12]) hold.

In fact, let $H((x,y),\lambda) = \pm\lambda \wedge (x,y) + (1-\lambda)QN(x,y)$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H((x,y),\lambda) \neq 0$ for $(x,y) \in \partial\Omega \cap \mathrm{Ker}L$, thus, from Lemma 2.3, by homotopy property of degree,

$$\begin{aligned} & \deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ = \ & \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0 \text{ since } 0 \in \Omega. \end{aligned}$$

Thus by Lemma 2.1 (Theorem IV.13 of [12]), $L(x,y) = N(x,y)$ has at least one solution in $D(L) \cap \overline{\Omega}$, then x is a $T-$solution of equation (1). The proof is completed.

3 Examples

In this section, we present examples to illustrate the main result in section 2.

Example 3.1. Consider the equation

$$x''''(t) = -\frac{[x(t)]^{\frac{3}{5}}}{1+2[\sin x(t)]^8} + \sum_{i=1}^{m} p_i(t)[x(t-\tau_i)]^{\frac{3}{5}} + r(t), \qquad (3\text{-}8)$$

where $T = 2\pi$, p_i, r are all non-negative continuous $2\pi-$periodic functions, $\tau_i > 0 (i = 1, \cdots, m)$ are constants.

Corresponding to the assumptions of Theorem L, one sees

$$f(t, x_0, x_1, \cdots, x_m) = -\frac{x_0^{\frac{3}{5}}}{1+2(\sin x_0)^8} + \sum_{i=1}^{m} p_i(t) x_i^{\frac{3}{5}} + r(t),$$

we set

$$g(t, x_0, x_1, \cdots, x_m) = -\frac{x_0^{\frac{3}{5}}}{1+2(\sin x_0)^8},$$

and

$$h(t, x_0, \cdots, x_m) = \sum_{i=1}^{m} p_i(t) x_i^{\frac{3}{5}} + r(t)$$

and $\beta = 1/3$, $\theta = 3/5$. It is easy to see that (B1) holds.

Since

$$\begin{aligned} c\int_0^{2\pi} f(t,c,\cdots,c)dt &= \int_0^{2\pi}\left(-\frac{c^{\frac{8}{5}}}{1+2[\sin c]^8}+\sum_{i=1}^m p_i(t)c^{\frac{8}{5}}+cr(t)\right)dt \\ &= -\frac{2\pi c^{\frac{8}{5}}}{1+2[\sin c]^8}+\sum_{i=1}^m\int_0^{2\pi} p_i(t)dtc^{\frac{8}{5}}+c\int_0^T r(t)dt \\ &\geq \left(\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt-2\pi\right)c^{\frac{8}{5}}+c\int_0^{2\pi} r(t)dt \end{aligned}$$

implies that there is $M > 0$ such that $c\int_0^{2\pi} f(t,c,\cdots,c)dt > 0$ for all $|c| > M$ if $\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt-2\pi$. So (B3)holds if $\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt-2\pi$.

It is easy to see that $\mu=1,\delta_i=1,M_i^0=1$, it follows that (B2) holds.

It follows from Theorem L that (8) has at least one $2\pi-$periodic solution if

$$\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt>2\pi,\ \ \sum_{i=1}^m\max_{t\in[0,2\pi]} p_i(t)<\frac{1}{3}.$$

Example 3.2. Consider the equation

$$[((\sin t)^2+2)\phi(x''(t))]''=-\frac{[x(t)]^5}{1+2[\sin x(t)]^8}+\sum_{i=1}^m p_i(t)[x(t-\tau_i)]^5+r(t), \qquad (3\text{-}9)$$

where $T=2\pi$, $\phi(x)=|x|^4x$, $q(t)=(\sin t)^2+2$,p_i,r are all non-negative with $\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt>2\pi$, $\tau_i(i=1,2,\cdots,m)$ are constants.

Corresponding to the assumptions of Theorem L, one sees

$$f(t,x_0,x_1,\cdots,x_m)=-\frac{x_0^5}{1+2(\sin x_0)^8}+\sum_{i=1}^m p_i(t)x_i^5+r(t),$$

we set

$$g(t,x_0,x_1,\cdots,x_m)=-\frac{x_0^5}{1+2(\sin x_0)^8},$$

and

$$h(t,x_0,\cdots,x_m)=\sum_{i=1}^m p_i(t)x_i^5+r(t)$$

and $\beta=1/3$, $\theta=5$. It is easy to see that (B1) holds.

It is easy to see $\mu=2,\delta_i=1,M_i^0=1$, it follows that (B2) holds.

Since

$$\begin{aligned} c\int_0^{2\pi} f(t,c,\cdots,c)dt &= \int_0^{2\pi}\left(-\frac{c^6}{1+2[\sin c]^8}+\sum_{i=1}^m p_i(t)c^6+cr(t)\right)dt \\ &= -\frac{2\pi c^6}{1+2[\sin c]^8}+\sum_{i=1}^m\int_0^{2\pi} p_i(t)dtc^6+c\int_0^T r(t)dt \\ &\geq \left(\sum_{i=1}^m\int_0^{2\pi} p_i(t)dt-2\pi\right)c^6+c\int_0^{2\pi} r(t)dt \end{aligned}$$

implies that there is $M > 0$ such that $c\int_0^{2\pi} f(t,c,\cdots,c)dt > 0$ for all $|c| > M$ if $\sum_{i=1}^m \int_0^{2\pi} p_i(t)dt - 2\pi > 0$. So (B3) holds if $\sum_{i=1}^m \int_0^{2\pi} p_i(t)dt - 2\pi > 0$.

It follows from Theorem L that equation (9) has at least one solution if

$$\sum_{i=1}^m \int_0^{2\pi} p_i(t)dt > 2\pi, \ \ \frac{1}{3} > \sum_{i=0}^m \max_{t\in[0,T]} p_i(t).$$

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