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Existence of solutions of resonant boundary value problems for fractional differential equations

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Abstract

In this article, we establish existence results for solutions of resonant boundary value problems for nonlinear singular fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_0^{\mu}u(t)) + e(t), \ t \in (0,1), 1 < \alpha < 2, \\ \left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0} = 0, \\ D_{0^+}^{\mu}u(1) = \sum_{i=1}^{m-2}\beta_i D_{0^+}^{\mu}u(\xi_i). \end{cases}$$

Our analysis relies on the well known coincidence degree theory. Here f depends on $D_{0^+}^{\mu}u$ and may be singular at $t = 0$ or $t = 1$. The resonance case is caused by the multi-point boundary conditions.

keywords

Solution, fractional differential equation, coincidence degree theory, resonant boundary value problem, resonant boundary value problem

1 Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering and have been of great interest recently. In its turn, mathematical aspects of studies on fractional differential equations were discussed by many authors, see the text books [1,2], the survey papers [3,4] and papers [5-10] and the references therein.

Recently the coincidence degree theory [11] has been used to study the existence of solutions to boundary value problems for fractional differential equations [12-15].

In [15], the existence of solutions of the resonant boundary value problems for fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_0^{\alpha-1}u(t)) + e(t),\ t \in (0,1), 1 < \alpha < 2, \\ \left[I_{0^+}^{2-\alpha}u(t)\right]\Big|_{t=0} = 0, \\ u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{cases} \tag{1}$$

was studied, where $D_{0^+}^{\alpha}$ (D_0^{α} or D^{α} for short) is the Riemann-Liouville fractional derivative of order α, $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, $\beta_i (i = 1, 2, \cdots, m-2) \in R$ with $\sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} = 1$, $f : [0,1] \times R^2 \to R$ is continuous and $e \in L^1(0,1)$.

We see that f in BVP(1.1) depends on $D_{0^+}^{\alpha-1}x$ and solutions of BVP(1.1) are bounded on $[0,1]$. To our knowledge, there exists no paper concerned with the existence of unbounded solutions of boundary value problems for fractional differential equations.

We find that in all above mentioned papers, the solutions obtained are bounded ones (continuous on $[0,1]$, f depends on either the fractional derivatives lower than $\alpha - 1$ [15,18].

Motivated by this reason, we discuss the boundary value problem of the nonlinear fractional differential equation of the form

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_0^{\mu}u(t)) + e(t),\ t \in (0,1), 1 < \alpha < 2, \\ \left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0} = 0, \\ D_{0^+}^{\mu}u(1) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\mu}u(\xi_i), \end{cases} \tag{2}$$

where $D_{0^+}^{\alpha}$ (D^{α} for short) is the Riemann-Liouville fractional derivative of order α, $0 < \mu \le \alpha - 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $\beta_i (i = 1, 2, \cdots, m-2) \in R$ with $\sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-\mu-2} = 1$, $f : (0,1) \times R^2 \to R$ is continuous and $e \in L^1(0,1)$. f may be singular at $t = 0$ or $t = 1$.

We obtain the existence results for solutions to BVP(1.2) by using the coincidence degree theory. The solutions obtained may be unbounded since there exists the limit $\lim_{t\to 0} t^{2-\alpha}u(t)$.

BVP(1.2) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) = 0,\ t \in (0,1), 1 < \alpha < 2, \\ \left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0} = 0, \\ D_{0^+}^{\mu}u(1) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\mu}u(\xi_i) \end{cases}$$

has $u(t) = ct^{\alpha-2}, c \in R$ as nontrivial unbounded solutions. When $\alpha = 2$ and $\mu = 1$ and $\beta_i = 0$

for all $i = 1, 2, \cdots, m-2$, BVP(1.2) becomes the well known Neumann boundary value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)) + e(t),\ t \in (0,1), \\ u'(0) = 0, \\ u'(1) = 0, \end{cases}$$

which is studied in [16,17]. The result in this paper generalizes those one in [16]. For $1 < \alpha < 2$ and $0 < \mu \le \alpha - 1$, the following BVP

$$\begin{cases} D_{0+}^{\alpha} u(t) = 0,\ t \in (0,1), 1 < \alpha < 2, \\ \left[I_{0+}^{2-\alpha} u(t) \right]' \Big|_{t=0} = 0, \\ D_{0+}^{\mu} u(1) = 0 \end{cases}$$

has a unique solution $u(t) = 0$. Hence the resonance case in BVP(1.2) is caused by the multi-point boundary conditions.

2 Main results

To obtain the main results, we need some notations and an abstract existence theorem by Gaines and Mawhin [11].

Let X and Y be Banach spaces, $L:\ D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P:\ X \to X$, $Q:\ Y \to Y$ be projectors such that

$$\text{Im } P = \text{Ker } L,\ \text{Ker } Q = \text{Im } L,\ \ X = \text{Ker } L \oplus \text{Ker } P,\ \ Y = \text{Im } L \oplus \text{Im } Q.$$

It follows that

$$L|_{D(L) \cap \text{Ker } P} :\ D(L) \cap \text{Ker } P \to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \ne \emptyset$, the map $N:\ X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega} \to X$ is compact.

Lemma 1 *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*

1. *$Lx \ne \lambda Nx$ for every $(x, \lambda) \in [D(L) \setminus KerL) \cap \partial\Omega] \times (0,1)$;*
2. *$Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;*
3. *$deg(\wedge^{-1} QN \big|_{KerL},\ \Omega \cap KerL, 0) \ne 0$, where $\wedge^{-1}:\ Y/ImL \to\ KerL$ is the inverse of the isomorphism $\wedge:\ KerL \to Y/ImL$.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Let $\Gamma(\alpha)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\ \ \mathbf{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

We use the Banach space $L^1[0,1]$ with the norm

$$||u||_1=\int_0^1|u(s)|ds.$$

Define

$$X=\left\{x:(0,1]\to R\ \begin{array}{c} x\in C(0,1],\\ D^{\mu}_{0+}x\in C(0,1],\\ \text{there exist the limits}\\ \lim_{t\to0}t^{2-\alpha}x(t),\\ \lim_{t\to0}t^{\mu+2-\alpha}x(t)\end{array}\right\}.$$

For $u\in X$, define the norm

$$||u||=\max\left\{\sup_{t\in(0,1]}t^{2-\alpha}|u(t)|,\ \sup_{t\in(0,1]}t^{\mu+2-\alpha}|D^{\mu}_{0+}u(t)|\right\}.$$

By means of the linear functional analysis theory, we can prove that X is a Banach space. Choose $Y=L^1[0,1]$.

Define L to be the linear operator from $D(L)\bigcap X$ to Y with

$$D(L)=\left\{u\in X:\ \begin{array}{c} D^{\alpha}_{0+}u\in L^1(0,1),\\ \left[I^{2-\alpha}_{0+}u(t)\right]'\Big|_{t=0}=0,\\ D^{\mu}_0u(1)=\sum_{i=1}^{m-2}\beta_iD^{\mu}_0u(\xi_i)\end{array}\right\}$$

and

$$(Lu)(t)=D^{\alpha}_{0+}u(t),\ u\in D(L).$$

Define $N:X\to Y$ by

$$(Nu)(t)=f\left(t,u(t),D^{\mu}_{0+}u(t)\right)+e(t),\ u\in X.$$

Then BVP(1.2) can be written as

$$Lu=Nu,\ \ u\in D(L).$$

Lemma 2 *It holds that*

1. $KerL=\{ct^{\alpha-2},\ c\in R\}$;
2. $ImL=\left\{v\in Y,\ \begin{array}{c}\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds\\ =\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\mu-1}v(s)ds\end{array}\right\}$;

3. L is a Fredholm operator of index zero.

Proof

(1) One sees that $D_{0^+}^{\alpha}u(t)=0$ has solutions

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}, t\in(0,1)$$

for some $c_i\in R,\quad i=1,2$. We get

$$I_0^{2-\alpha}u(t)=c_1\Gamma(\alpha)t+c_2\Gamma(\alpha-1)$$

and

$$[I_{0^+}^{2-\alpha}u(t)]'=c_1\Gamma(\alpha),$$

and

$$D_0^{\mu}u(t)=c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t^{\alpha-\mu-1}+c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}.$$

Since

$$[I_{0^+}^{2-\alpha}u(t)]'\big|_{t=0}=0,$$

we get $c_1=0$. Since $\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-\mu-2}=1$, we find that $c_2\in R$. Thus $\mathrm{Ker}L=\{ct^{\alpha-2},\ c\in R\}$.

(2) We see that $v\in\ \mathrm{Im}L$ if and only if there exists a function $u\in D(L)$ such that

$$\begin{cases} D_{0^+}^{\alpha}u(t)=v(t), \quad t\in(0,1), 1<\alpha\le 2,\\ \left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0}=0,\\ D_0^{\mu}u(1)=\sum_{i=1}^{m-2}\beta_iD_0^{\mu}u(\xi_i). \end{cases}$$

Then

$$u(t)=I_{0^+}^{\alpha}v(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}, t\in(0,1).$$

It follows that

$$D_0^{\mu}u(t)=I_{0^+}^{\alpha-\mu}v(t)+c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t^{\alpha-\mu-1}+c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2},$$

$$I_{0^+}^{2-\alpha}u(t)=\int_0^t(t-s)v(s)ds+c_1\Gamma(\alpha)t+c_2\Gamma(\alpha-1)$$

and

$$[I_{0^+}^{2-\alpha}u(t)]'=\int_0^t v(s)ds+c_1\Gamma(\alpha). \tag{3}$$

From $\left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0}=0$, we get $c_1=0$. Now $D_0^{\mu}u(1)=\sum_{i=1}^{m-2}\beta_iD_0^{\mu}u(\xi_i)$ implies that

$$\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds=\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\mu-1}v(s)ds. \tag{4}$$

On the other hand, suppose $v \in Y$ and satisfies (2.2). Choose

$$u(t) = I_{0+}^{\alpha} v(t).$$

One sees by computation that

$$D_{0+}^{\alpha} u(t) = v(t), t \in (0,1)$$

and

$$I_{0^+}^{2-\alpha} u(t) = \int_0^t (t-s) v(s) ds, \quad D_0^{\mu} u(t) = I_{0^+}^{\alpha-\mu} v(t).$$

So

$$\left[I_{0^+}^{2-\alpha} u(t) \right]' = \int_0^t v(s) ds, \quad D_0^{\mu} u(t) = \frac{1}{\Gamma(\alpha-\mu)} \int_0^t (t-s)^{\alpha-\mu-1} v(s) ds.$$

One has

$$\left[I_{0^+}^{2-\alpha} u(t) \right]' \Big|_{t=0} = 0, \quad D_0^{\mu} u(1) = \sum_{i=1}^{m-2} \beta_i D_0^{\mu} u(\xi_i).$$

Furthermore, we know that $t^{2-\alpha} u \in C(0,1]$, $t^{\mu+2-\alpha} D_{0+}^{\mu} u \in C(0,1]$ and there exist the limits $\lim_{t\to 0} t^{2-\alpha} u(t)$ and $\lim_{t\to 0} t^{\mu+2-\alpha} D_{0+}^{\mu} u(t)$. Hence $u \in D(L)$ and $Lu = v$. So $v \in \mathrm{Im} L$. Then **(2)** follows.

To prove (**3**), we first claim that there exists $k \in \{0, 1, 2, \cdots, m-2\}$ such that $\sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3} \neq 1$. In fact, suppose it is true, we have

$$\begin{pmatrix} \xi_1^{3\alpha-\mu-3} & \xi_2^{3\alpha-\mu-3} & \cdots & \xi_{m-2}^{3\alpha-\mu-3} \\ \xi_1^{3\alpha-\mu} & \xi_2^{3\alpha-\mu} & \cdots & \xi_{m-2}^{3\alpha-\mu} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \xi_1^{3\alpha+3(m-2)-\mu-3} & \xi_2^{3\alpha+3(m-2)-\mu-3} & \cdots & \xi_{m-2}^{3\alpha+3(m-2)-\mu-3} \end{pmatrix}$$
$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

It is equal to

$$\begin{pmatrix} \xi_1^{3\alpha-\mu-3} & \xi_2^{3\alpha-\mu-3} & \cdots & \xi_{m-2}^{3\alpha-\mu-3} & 1 \\ \xi_1^{3\alpha-\mu} & \xi_2^{3\alpha-\mu} & \cdots & \xi_{m-2}^{3\alpha-\mu} & 1 \\ . & . & \cdots & . & 1 \\ . & . & \cdots & . & 1 \\ . & . & \cdots & . & 1 \\ \xi_1^{3\alpha+3(m-2)-\mu-3} & \xi_2^{3\alpha+3(m-2)-\mu-3} & \cdots & \xi_{m-2}^{3\alpha+3(m-2)-\mu-3} & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_{m-2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}.$$

However, it is well known that the Vandermont Determinant is not equal to zero, so there is a contradiction.

(3) It follows from (1) that dim Ker$L = 1$. Let k satisfy

$$\sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3} \neq 1.$$

Define the projectors $Q: \ Y \to Y$ and $P: \ X \to X$ by

$$(Qv)(t) = \frac{\int_0^1 (1-s)^{\alpha-\mu-1} v(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi-s)^{\alpha-\mu-1} v(s)ds}{\mathbf{B}(\alpha+k, \alpha-\mu)\left(1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3}\right)} t^{\alpha+k-1}$$

for $v \in Y$ and

$$(Pu)(t) = \frac{\left[I_{0^+}^{2-\alpha} u(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} \ \text{ for } \ u \in X,$$

respectively. It is easy to prove that

$$\text{Im } P = \text{Ker } L, \ \text{Ker } Q = \text{Im } L. \tag{5}$$

Furthermore, for $u \in X$, one sees

$$\frac{\left[I_{0^+}^{2-\alpha} u(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} \in \text{Ker} L.$$

The definition of P implies

$$P\left(u(t) - \frac{\left[I_{0^+}^{2-\alpha} u(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2}\right) = 0.$$

We get

$$u(t) - \frac{\left[I_{0^+}^{2-\alpha} u(t)\right]\Big|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} \in \mathrm{Ker} P.$$

One can see that $\mathrm{Ker} L \bigcap \mathrm{Ker} P = \{0\}$. Then

$$X = \mathrm{Ker}\ L \oplus \mathrm{Ker}\ P. \tag{6}$$

For $v \in Y$, denote

$$V(t) = v(t) - \frac{\int_0^1 (1-s)^{\alpha-\mu-1} v(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi - s)^{\alpha-\mu-1} v(s)ds}{\mathbf{B}(\alpha+k, \alpha-\mu)\left(1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3}\right)} t^{\alpha+k-1}.$$

Since

$$\int_0^1 (1-t)^{\alpha-\mu-1} V(t)dt = \int_0^1 (1-t)^{\alpha-\mu-1} v(t)dt$$

$$- \frac{\int_0^1 (1-s)^{\alpha-\mu-1} v(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi - s)^{\alpha-\mu-1} v(s)ds}{\mathbf{B}(\alpha+k, \alpha-\mu)\left(1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3}\right)} \times$$

$$\int_0^1 (1-t)^{\alpha-\mu-1} t^{\alpha+k-1} dt = \int_0^1 (1-t)^{\alpha-\mu-1} v(t)dt$$

$$- \frac{\int_0^1 (1-s)^{\alpha-\mu-1} v(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\mu-1} v(s)ds}{1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{3\alpha+3k-\mu-3}},$$

implies

$$\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}V(t)dt=\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}v(t)dt$$

$$-\frac{\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi-s)^{\alpha-\mu-1}v(s)ds}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\times$$

$$\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}t^{\alpha+k-1}dt$$

$$=\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}v(t)dt$$

$$-\frac{\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi-s)^{\alpha-\mu-1}v(s)ds}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\times$$

$$\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\xi_i^{\alpha-\mu-1}\left(1-\frac{t}{\xi_i}\right)^{\alpha-\mu-1}t^{\alpha+k-1}dt$$

$$=\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}v(t)dt$$

$$-\frac{\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi-s)^{\alpha-\mu-1}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}}\times$$

$$\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}=\int_0^1(1-t)^{\alpha-\mu-1}V(t)dt,$$

we get

$$v-\frac{\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\mu-1}v(s)ds}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}t^{\alpha+k-1}\in \mathrm{Im}L.$$

Together with

$$\frac{\int_0^1(1-s)^{\alpha-\mu-1}v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\mu-1}v(s)ds}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}t^{\alpha+k-1}\in \mathrm{Im}Q,$$

and $\mathrm{Im}L\bigcap\mathrm{Im}Q=\{0\}$,we get

$$Y=\mathrm{Im}\ L\oplus\mathrm{Im}\ Q,\quad Y/\mathrm{Im}L=\mathrm{Im}Q. \tag{7}$$

So dim Ker$L$ = dim $Y/\mathrm{Im}L=1$. Hence L is a Fredholm operator of index zero. The proofs are completed.

Suppose that

(A) $f(t,\cdot,\cdot):R^2\to R$ is continuous for each $t\in(0,1)$ and $e\in L^1(0,1)$, for each $r>0$ there exists $\phi_r\in L^1(0,1)$ such that

$$|f(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y)|\le\phi_r(t),\ \ t\in(0,1),|x|\le r,|y|\le r.$$

Lemma 3 *Suppose that (A) holds. Let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap D(L)\neq\emptyset$. Then N is $L-$compact on $\overline{\Omega}$.*

Proof Let $P:X\to X$ and $Q:Y\to Y$ be defined in the proof of (3)of Lemma 2. For $v\in \mathrm{Im}L$, let

$$(K_Pv)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds=I_{0+}^\alpha v(t)\quad\text{for}\ \ v\in\mathrm{Im}L.$$

One sees $K_Pv\in D(L)$ and

$$\begin{aligned}&P\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\right)\\ =\ &P(I_{0+}^\alpha v(t))=\frac{\left[I_{0+}^{2-\alpha}I_{0+}^\alpha v(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}\\ =\ &\frac{\left[I_{0+}^{2}v(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}=0.\end{aligned}$$

It follows that $(K_pv)\in\mathrm{Ker}P$. Then $K_P:\ \mathrm{Im}\ L\to D(L)\cap\mathrm{Ker}P$ is well defined.

Furthermore, for $v\in\mathrm{Im}L$, we have

$$(LK_P)(v)=L\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\right)=D_{0+}^\alpha\left(I_{0+}^\alpha v(t)\right)=v(t).$$

On the other hand, for $u\in\mathrm{Ker}\ P\cap D(L)$, we have

$$(Pu)(t)=\frac{[I_{0+}^{2-\alpha}u(t)]|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}=0,t\in(0,1).$$

Suppose $D_{0+}^\alpha u=v$. Then

$$\begin{aligned}u(t)\ &=\ I_{0+}^\alpha v(t)+\frac{[I_{0+}^{2-\alpha}u(t)]'|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}+\frac{I_{0+}^{2-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}\\ &=\ I_{0+}^\alpha v(t).\end{aligned}$$

It follows from the definition of K_P that

$$\begin{aligned}(K_PL)u(t)\ &=\ K_PD_{0+}^\alpha u(t)=K_Pv(t)\\ &=\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\\ &=\ I_{0+}^\alpha v(t)=u(t).\end{aligned}$$

Then K_P is the inverse of $L : D(L)\bigcap \mathrm{Ker}L \to \mathrm{Im}L$. The isomorphism $\wedge :\ \mathrm{Ker}L \to Y/\mathrm{Im}L$ is given by

$$\wedge(at^{\alpha-2}) = at^{\alpha+k-1},\ \ a \in R.$$

Furthermore, one has

$$\begin{aligned}
QNu(t) &= Q\left(f(t,u(t),D^{\mu}_{0+}u(t)) + e(t)\right)\\
&= \frac{t^{\alpha+k-1}\int_0^1 (1-t)^{\alpha-\mu-1}\left(f(t,u(t),D^{\mu}_{0+}u(t)) + e(t)\right)dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\\
&- \frac{t^{\alpha+k-1}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}\left(f(t,u(t),D^{\mu}_{0+}u(t)) + e(t)\right)dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)},
\end{aligned}$$

$$\begin{aligned}
K_p(I-Q)Nx(t) &= K_p(I-Q)\left(f(t,u(t),D^{\alpha-1}_{0+}u(t)) + e(t)\right)\\
&= K_P\left(f(t,u(t),D^{\alpha-1}_{0+}u(t)) + e(t)\right)\\
&- K_PQ\left(f(t,u(t),D^{\alpha-1}_{0+}u(t)) + e(t)\right)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,u(s),D^{\alpha-1}_{0+}u(s)) + e(s)\right)ds\\
&- \frac{1}{\Gamma(\alpha)}\left[\frac{\int_0^1 (1-t)^{\alpha-\mu-1}\left(f(t,u(t),D^{\mu}_{0+}u(t)) + e(t)\right)dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\right.\\
&+ \left.\frac{\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}\left(f(t,u(t),D^{\mu}_{0+}u(t)) + e(t)\right)dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\right]\times\\
&\int_0^t (t-s)^{\alpha-1}s^{\alpha+k-1}ds.
\end{aligned}$$

Let Ω be an open bounded subset in Y. Firstly, we show that both $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded in Y.

Secondly, we show that both $t^{2-\alpha}K_P(I-Q)N(\overline{\Omega})$ and $t^{\mu+2-\alpha}D^{\mu}_{0+}K_P(I-Q)N(\overline{\Omega})$ are equicontinuous on each closed interval $[\alpha,\beta]\subset(0,1]$.

Finally, we show that both $t^{2-\alpha}K_P(I-Q)N(\overline{\Omega})$ and $t^{\mu+2-\alpha}D^{\mu}_{0+}K_P(I-Q)N(\overline{\Omega})$ are equiconvergent at $t=0$.

Hence $K_P(I-Q)N(\overline{\Omega})$ is compact. Then $K_P(I-Q)N$ is completely continuous. So N is $L-$compact on Ω. The proofs are completed.

Theorem 1 *Suppose that (A) holds and*

(B) *there exist nonnegative functions a,b,c,r, and a constant $\theta\in[0,1)$ such that for all $(x,y)\in R^2$, $t\in(0,1)$ either*

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + c(t)|y|^{\theta} + r(t) \tag{8}$$

or else

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + c(t)|x|^\theta + r(t). \tag{9}$$

(C) *there exists a constant* $M > 0$ *such that for all* $u \in D(L)$*, if* $|t^{2-\alpha}u(t)| > M$ *for all* $t \in (0,1)$*, then*

$$\int_0^1 (1-s)^{\alpha-\mu-1}(f(s,u(s),D^\mu_{0+}u(s)) + e(s))ds$$
$$\neq \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\mu-1}(f(s,u(s),D^\mu_{0+}u(s)) + e(s))ds.$$

(D) *there exists a constant* $M^* > 0$*, then either*

$$c\left\{\int_0^1 (1-s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}\right) + e(t)\right]dt\right.$$
$$\left. - \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}\right) + e(t)\right]dt\right\} > 0$$

for all $|c| > M^*$ *or else*

$$c\left\{\int_0^1 (1-s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}\right) + e(t)\right]dt\right.$$
$$\left. - \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}\right) + e(t)\right]dt\right\} < 0$$

for all $|c| > M^*$*.*

(E) $B\left(\int_0^1 \frac{a(s)}{s^{2-\alpha}}ds + \int_0^1 \frac{b(s)}{s^{\mu+2-\alpha}}ds\right) < 1$ *with*

$$B = \max\left\{\frac{1}{\Gamma(\alpha-\mu)}, \frac{1}{(\alpha-1)\Gamma(\alpha-\mu-1)}\right\}.$$

Then for every $e \in L^1[0,1]$ *BVP(1.2) has at least one solution.*

Proof

To apply Lemma 1, we should define an open bounded subset Ω of X centered at zero such that (1), (2) and (3) in Lemma 1 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps. Let

$$A = \max\left\{\frac{1}{\Gamma(\alpha-1)}, \frac{1}{\Gamma(\alpha-\mu-1)}\right\} M\Gamma(\alpha-1).$$

Step 1 Let $\Omega_1 = \{u \in D(L) \setminus \mathrm{Ker}L,\ Lu = \lambda Nu \text{ for some } \lambda \in (0,1)\}$. We prove that Ω_1 is bounded.

For $u \in \Omega_1$, we get $Lu = \lambda Nu$ and $Nu \in \mathrm{Im}L$. Then

$$\int_0^1 (1-s)^{\alpha-\mu-1}(f(s,u(s),D_{0^+}^{\mu}u(s))+e(s))ds$$
$$= \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i}(\xi_i-s)^{\alpha-\mu-1}(f(s,u(s),D_{0^+}^{\mu}u(s))+e(s))ds.$$

From (C), there exists $t_0 \in (0,1)$ such that

$$|t_0^{2-\alpha}u(t_0)| \le M.$$

Denote $\lambda Nu(t) = v(t)$. By Lemma 2.5[4: page 74], we have

$$\begin{aligned} u(t) &= I_{0^+}^{\alpha}v(t) + \frac{\left[I_{0^+}^{2-\alpha}u(t)\right]'\Big|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1} + \frac{I_{0^+}^{2-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2} \\ &= I_{0^+}^{\alpha}v(t) + \frac{I_{0^+}^{2-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}. \end{aligned}$$

So

$$t^{2-\alpha}u(t) = t^{2-\alpha}I_{0^+}^{\alpha}v(t) + \frac{I_{0^+}^{2-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-1)}.$$

Then

$$\begin{aligned} \frac{\left|I_{0^+}^{2-\alpha}u(t)|_{t=0}\right|}{\Gamma(\alpha-1)} &= \left|t_0^{2-\alpha}u(t_0) - t_0^{2-\alpha}I_{0^+}^{\alpha}v(t_0)\right| \\ &\le M + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_0}(t_0-s)^{\alpha-1}v(s)ds\right| \\ &\le M + \frac{1}{\Gamma(\alpha)}||v||_1 \end{aligned}$$

One has

$$Pu(t) = \frac{\left[I_{0^+}^{2-\alpha}u(t)\right]\big|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}$$

and

$$D^{\mu}Pu(t) = \frac{\left[I_{0^+}^{2-\alpha}u(t)\right]\big|_{t=0}}{\Gamma(\alpha-\mu-1)}t^{\alpha-\mu-2}.$$

It follows that

$$\begin{aligned} ||Pu|| &= \max\left\{\sup_{t\in[0,1]}\left|t^{2-\mu}Pu(t)\right|, \sup_{t\in[0,1]} t^{\mu+2-\alpha}|D_{0^+}^{\mu}Pu(t)|\right\} \\ &\le \max\left\{\frac{\left|\left[I_{0^+}^{2-\alpha}u(t)\right]\big|_{t=0}\right|}{\Gamma(\alpha-1)}, \frac{\left|\left[I_{0^+}^{2-\alpha}u(t)\right]\big|_{t=0}\right|}{\Gamma(\alpha-\mu-1)}\right\} \\ &= \max\left\{\frac{1}{\Gamma(\alpha-1)}, \frac{1}{\Gamma(\alpha-\mu-1)}\right\}\left|\left[I_{0^+}^{2-\alpha}u(t)\right]\big|_{t=0}\right| \\ &\le \max\left\{\frac{1}{\Gamma(\alpha-1)}, \frac{1}{\Gamma(\alpha-\mu-1)}\right\}\left(M\Gamma(\alpha-1) + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}||v||_1\right). \end{aligned}$$

On the other hand, we have by the definition of K_P that $K_Pv = I_{0^+}^{\alpha}v$. Then $D^{\mu}K_Pv = I_{0^+}^{\alpha-\mu}v$. Hence

$$\begin{aligned} ||K_Pv|| &= \max\left\{\sup_{t\in[0,1]}|t^{2-\mu}K_Pv(t)|,\ \sup_{t\in[0,1]}t^{\mu+2-\alpha}|D^{\mu}K_Pv(t)|\right\} \\ &= \max\left\{\sup_{t\in[0,1]}\left|\frac{t^{2-\mu}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds\right|,\right. \\ &\qquad \left.\sup_{t\in[0,1]}\frac{t^{\mu+2-\alpha}}{\Gamma(\alpha-\mu)}\left|\int_0^t(t-s)^{\alpha-\mu-1}v(s)ds\right|\right\} \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||v||_1. \end{aligned}$$

Again for $u\in\Omega_1$, we have $u\in D(L)\setminus\mathrm{Ker}L$, then $(I-P)u\in D(L)\bigcap\mathrm{Ker}P$ and $LPu=0$. So

$$\begin{aligned} ||(I-P)u|| &= ||K_PL(I-P)u|| \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||L(I-P)u||_1 \\ &= \max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||Lu||_1 \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||\lambda Nu||_1 \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||v||_1. \end{aligned}$$

Hence

$$\begin{aligned} ||u|| &\leq ||Pu||+||(I-P)u|| \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha-1)},\ \frac{1}{\Gamma(\alpha-\mu-1)}\right\}\left(M\Gamma(\alpha-1)+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}||v||_1\right) \\ &\quad +\max\left\{\frac{1}{\Gamma(\alpha)},\ \frac{1}{\Gamma(\alpha-\mu)}\right\}||Nu||_1 \\ &\leq \max\left\{\frac{1}{\Gamma(\alpha-1)},\ \frac{1}{\Gamma(\alpha-\mu-1)}\right\}M\Gamma(\alpha-1) \\ &\quad +\max\left\{\frac{1}{\Gamma(\alpha-\mu)},\frac{1}{(\alpha-1)\Gamma(\alpha-\mu-1)}\right\}||v||_1. \end{aligned}$$

If (2.6) holds, from $||t^{2-\alpha}u||_\infty \le ||u||$ and $||t^{\mu+2-\alpha}D_{0+}^\mu u||_\infty \le ||u||$, then

$$\begin{aligned}
||u|| &\le A + B\int_0^1 |f(s,u(s),D_{0+}^\mu u(s)) + e(s)|ds \\
&\le A + B\int_0^1 \left[\frac{a(s)}{s^{2-\alpha}}|s^{2-\alpha}u(s)| + \frac{b(s)}{s^{\mu+2-\alpha}}|s^{\mu+2-\alpha}D_{0+}^\mu u(s)|\right. \\
&\quad \left. +\frac{s^{\theta(\mu+2-\alpha)}}{c}(s)|s^{\mu+2-\alpha}D_{0+}^\mu u(s)|^\theta + r(s) + \ |e(s)|\right] ds \\
&\le A + B\int_0^1 \frac{a(s)}{s^{2-\alpha}}ds||t^{2-\alpha}u||_\infty + B\int_0^1 \frac{b(s)}{s^{\mu+2-\alpha}}ds||t^{\mu+2-\alpha}D_{0+}^\mu u||_\infty \\
&\quad +B\int_0^1 \frac{c(s)}{s^{\theta(\mu+2-\alpha)}}ds||t^{\mu+2-\alpha}D_{0+}^\mu u||_\infty^\theta + B\int_0^1 [r(s)+|e(s)|]ds \\
&\le A + B\left(\int_0^1 \frac{a(s)}{s^{2-\alpha}}ds + \int_0^1 \frac{b(s)}{s^{\mu+2-\alpha}}ds\right)||u|| \\
&\quad +B\int_0^1 \frac{c(s)}{s^{\theta(\mu+2-\alpha)}}ds||u||^\theta + B\int_0^1 [r(s)+|e(s)|]ds.
\end{aligned}$$

It follows from (E) and $\theta \in [0,1)$ that there exists $M_1 > 0$ such that

$$||u||| \le M_1. \tag{10}$$

If (2.7) holds, similarly to above discussion, we get that there exists $M_1 > 0$ such that (2.8) holds. It follows that Ω_1 is bounded.

Step 2 Let $\Omega_2 = \{x \in \mathrm{Ker}L : \ Nx \in \mathrm{Im}L\}$. We prove that Ω_2 is bounded.

For $x \in \Omega_2$, then $x(t) = ct^{\alpha-2}$, and

$$Nx(t) = f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right) + e(t).$$

So

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right) + e(t)\right]dt \\
= &\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i - s)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right) + e(t)\right]dt.
\end{aligned}$$

From (D), we get that $|c| \le M^*$. This shows Ω_2 is bounded.

Step 3 We prove that either

$$\Omega_3 = \{x \in \mathrm{Ker}\ L : \ \lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\}$$

or

$$\Omega_3 = \{x \in \mathrm{Ker}\ L : \ -\lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\}$$

is bounded.

If the first inequality in (D) holds for all $|c| > M^*$, let

$$\Omega_3 = \{x \in \text{Ker } L: \ \lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\},$$

where $\wedge$ is the isomorphism given by $\wedge(ct^{\alpha-2}) = ct^{\alpha+k-1}$. We prove that Ω_3 is bounded. For $x(t) = ct^{\alpha-2} \in \text{Ker } L$, one sees that

$$\begin{aligned}&-\lambda ct^{\alpha+k-1}\\ &\quad= \frac{t^{\alpha+k-1}\int_0^1(1-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\\ &\quad- \frac{t^{\alpha+k-1}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}.\end{aligned}$$

Then

$$\begin{aligned}-\lambda c &= \frac{\int_0^1(1-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\\ &\quad- \frac{\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}.\end{aligned}$$

If $\lambda = 1$, then $c = 0$. If $\lambda \in [0,1)$, and $|c| > M^*$, we get

$$\begin{aligned}&0 \geq -\lambda c^2\\ &\quad= \frac{c\int_0^1(1-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\\ &\quad- \frac{c\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-t)^{\alpha-\mu-1}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1}t^{\alpha-2-\mu}\right)+e(t)\right]dt}{\mathbf{B}(\alpha+k,\alpha-\mu)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{3\alpha+3k-\mu-3}\right)}\\ &\quad> 0,\end{aligned}$$

a contradiction. Hence $|c| \leq M^*$. Then Ω_3 is bounded.

If the second inequality in (D) holds for all $|c| > M^*$, let

$$\Omega_3 = \{x \in \text{Ker } L: \ \lambda \wedge x - (1-\lambda)QNx = 0, \ \lambda \in [0,1]\},$$

where $\wedge$ is the isomorphism given by $\wedge(ct^{\alpha-2}) = ct^{\alpha+k-1}$. We prove that Ω_3 is bounded.

Step 4 We shall show that all conditions of Lemma 2.1 are satisfied.

Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^3\overline{\Omega_i}$. By Lemma 2.2 and Lemma 2.3, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(a) $Lx \neq \lambda Nx$ for $x \in (D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b) $Nx \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

(c) $\deg(QN|_{\mathrm{Ker}L},\ \Omega \cap \mathrm{Ker}L, 0) \neq 0$. In fact, let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Lemma 1, $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$, which is a solution of BVP(1.2). The proof is complete.

3 An example

Now, we present an example, which can not be covered by known results, to illustrate Theorem 2.1.

Example 1 *Consider the boundary value problem for fractional differential equation*

$$\begin{cases} D_{0^+}^{\frac{3}{2}}(t) = \frac{t^{2-\alpha}}{24}x(t) + \frac{1}{24}\sin\left(t^{\frac{3}{2}}D_{0^+}^{\frac{1}{4}}x(t)\right) \\ \qquad\qquad +3\sin\left(t^{\frac{3}{2}}D_{0^+}^{\frac{1}{4}}x(t)\right)^{\frac{1}{3}} + 1 + \cos^2 t, \\ \left[I_{0^+}^{\frac{1}{2}}x(t)\right]'\Big|_{t=0} = 0, \\ D_{0^+}^{\frac{1}{4}}x(t)\Big|_{t=1} = \beta_1\, D_{0^+}^{\frac{1}{4}}x(t)\Big|_{t=\xi_1} + \beta_2\, D_{0^+}^{\frac{1}{4}}x(t)\Big|_{t=\xi_2}. \end{cases} \tag{11}$$

Proof

Corresponding to BVP(1.2), $\alpha = \frac{3}{2}$, $\mu = \frac{1}{4}$ and

$$0 < \xi_1 < \xi_2 < 1,\ \beta_1 \geq 0,\ \beta_2 \geq 0 \text{ such that } \beta_i\xi_1^{-\frac{3}{2}} + \beta_2\xi_2^{-\frac{3}{2}} = 1$$

$$f(t,x,y) = \frac{t^{2-\alpha}}{24}x + \frac{1}{24}\sin(t^{\frac{3}{2}}y) + 3\sin(t^{\frac{3}{2}}y)^{\frac{1}{3}}, \quad e(t) = 1 + \cos^2 t.$$

It is easy to see that (A) holds.

(B) choose $a(t) = \frac{t^{2-\alpha}}{24}$, $b(t) = \frac{t^{\frac{3}{2}}}{24}$, $c(t) = 3t^{\frac{3}{2}}$, $r(t) = 0$, $\theta = \frac{1}{3}$, then for all $(x,y) \in R^2$, $t \in (0,1)$, then

$$|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)|y|^{\theta} + r(t). \tag{12}$$

(C) choose $M = 122$, it is easy to find that

$$f(t,x,y)+e(t) \geq \frac{t^{2-\alpha}}{24}x - \frac{1}{24} - 3 + 1 = \frac{x-49}{24} > \frac{1}{24} \text{ if } t^{2-\alpha}x > M,$$

then

$$\begin{aligned}
& \int_0^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& -\beta_1 \int_0^{\xi_1} (\xi_1 - s)^{\frac{1}{4}} (f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& -\beta_2 \int_0^{\xi_2} (\xi_2 - s)^{\frac{1}{4}} (f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
\geq \ & \beta_1 \int_{\xi_1}^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& +\beta_2 \int_{\xi_2}^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
> \ & 0.
\end{aligned}$$

It is easy to find that

$$f(t,x,y)+e(t) \leq \frac{t^{2-\alpha}}{24}x + \frac{1}{24} + 3 + 2 = \frac{x+120}{24} < \frac{-2}{24} \text{ if } t^{2-\alpha}x < -M,$$

then

$$\begin{aligned}
& \int_0^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& -\beta_1 \int_0^{\xi_1} (\xi_1 - s)^{\frac{1}{4}} (f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& -\beta_2 \int_0^{\xi_2} (\xi_2 - s)^{\frac{1}{4}} (f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
\leq \ & \beta_1 \int_{\xi_1}^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
& +\beta_2 \int_{\xi_2}^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\mu}u(s))+e(s))ds \\
< \ & 0.
\end{aligned}$$

Then $u \in D(L)$, if $|t^{2-\alpha}u(t)| > M$ for all $t \in (0,1)$, it holds that

$$\begin{aligned}
& \int_0^1 (1-s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\alpha-1}u(s))+e(s))ds \\
\neq \ & \beta_1 \int_0^{\xi_1} (\xi_1 - s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\alpha-1}u(s))+e(s))ds \\
& +\beta_2 \int_0^{\xi_2} (\xi_2 - s)^{\frac{1}{4}}(f(s,u(s),D_{0+}^{\alpha-1}u(s))+e(s))ds.
\end{aligned}$$

(D) since

$$\begin{aligned}
& f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t) \\
= & \frac{1}{24}c+\frac{1}{24}\sin\left(c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right) \\
& +3\sin\left(c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)^{\frac{1}{3}}+1+\cos^2 t \\
& \begin{cases} \geq \frac{c}{24}-\frac{49}{24} \\ \leq \frac{c}{24}+\frac{121}{24}, \end{cases}
\end{aligned}$$

we get for $c > 122$ that

$$\begin{aligned}
N = & c\left\{\int_0^1 (1-s)^{\frac{1}{4}}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t)\right]dt\right. \\
& -\beta_1\int_0^{\xi_1}(\xi_1-s)^{\frac{1}{4}}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t)\right]dt \\
& \left.-\beta_2\int_0^{\xi_2}(\xi_2-s)^{\frac{1}{4}}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t)\right]dt\right\} \\
\geq & c\left\{\beta_1\int_{\xi_1}^1 (1-s)^{\frac{1}{4}}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t)\right]dt\right. \\
& \left.+\beta_2\int_{\xi_2}^1 (1-s)^{\frac{1}{4}}\left[f\left(t, ct^{\alpha-2}, c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}t^{\alpha-2-\mu}\right)+e(t)\right]dt\right\} \\
\geq & c\left\{\beta_1\int_{\xi_1}^1 (1-s)^{\frac{1}{4}}\left[\frac{c}{24}-\frac{49}{24}\right]dt+\beta_2\int_{\xi_2}^1 (1-s)^{\frac{1}{4}}\left[\frac{c}{24}-\frac{49}{24}\right]dt\right\}.
\end{aligned}$$

It is easy to see that there exists $M_1^* > 0$ such that $N > 0$ for all $c > M_1^*$. Similarly we can prove that there exists $M_2^* > 0$ such that $N > 0$ for all $c > M_2^*$. Hence there exists $M^* > 0$ such that $N > 0$ for all $c > M^*$.

We see that

$$B = \max\left\{\frac{1}{\Gamma(\alpha-\mu)}, \frac{1}{(\alpha-1)\Gamma(\alpha-\mu-1)}\right\} = \max\left\{\frac{1}{\Gamma(5/4)}, \frac{2}{\Gamma(1/4)}\right\}.$$

(E)

$$\max\left\{\frac{1}{\Gamma(5/4)}, \frac{2}{\Gamma(1/4)}\right\}\left(\int_0^1 \frac{a(s)}{s^{2-\alpha}}ds+\int_0^1 \frac{b(s)}{s^{\mu+2-\alpha}}ds\right) = \frac{1}{12}\max\left\{\frac{1}{\Gamma(5/4)}, \frac{2}{\Gamma(1/4)}\right\} < 1.$$

It follows from Theorem 2.1 that BVP(3.1) has at least one solution x.

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