DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
№ 1, 2009
Электронный журнал
reg. № P2375 at 07.03.97
ISSN 1817-2172

http://www.newa.ru/journal
http://www.math.spbu.ru/user/diffjournal
e-mail: jodiff@mail.ru

Applications to physics, electrotechnics, and electronics

Some new properties of the applied-physics related Boubaker polynomials

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Abstract

Some new properties of the Boubaker polynomials are presented in this paper. Among others, it is shown that that all positive zeros of the Boubaker polynomial $B_n(x)$ are in [0,2]. Also, there are only two pure imaginary zeros of $B_n(x)$.

PACS. 02.30.Jr Partial differential equations - 02.30.Sa Functional analysis.
***Mathematics Subject Classification 2000*:** 33E20, 33E30, 35K05, 41A30, 41A55.

1 Introduction

Recently, the use of polynomial expansions took a big part of the most known mathematical expansion schemes and yielded meaningful results to both numerical and analytical analysis [1-8]. In this context, the Boubaker polynomials were established as a guide for solving some applied physics problems [9-22] where appears, i.e. the following equation :

$$\frac{\partial^2 u(x,t)}{\partial x^2} = k\frac{\partial f(x,t)}{\partial t} \tag{1}$$

defined in the domain D:

$$\mathrm{D}:\begin{cases} -H < x < 0 \\ t > 0 \end{cases} \tag{2}$$

In this paper, we intend to to give some new properties of the Boubaker polynomials. We will show among others that all positive zeros of the Boubaker polynomial $B_n(x)$ are in [0,2]. Also, we try to demonstrate that there are only two pure imaginary zeros of $B_n(x)$.

2 History of the Boubaker polynomials

2.1 The Boubaker polynomials

The first monomial definition of the Boubaker polynomials [9-12] appeared in a physical study that yielded an analytical solution to heat equation inside a physical model. This monomial definition is defined by [12-18] :

$$B_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right].(-1)^p .X^{n-2p} \tag{3}$$

where :

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$$

(The symbol: $\lfloor\ \rfloor$ designates the floor function)

The Boubaker polynomials, which are a polynomial sequence with integer coefficients, have the explicit monic expression as follow:

$$B_n(X) = X^n - (n-4).X^{n-2} + \sum_{p=2}^{\xi(n)} \left[\frac{(n-4p)}{p!} \prod_{j=p+1}^{2p-1} (n-j) \right] .(-1)^p .X^{n-2p} \quad (4)$$

The recurrence relation of the Boubaker polynomials is

$$\begin{cases} B_0(X) = 1 \\ B_1(X) = X \\ B_2(X) = X^2 + 2 \\ B_m(X) = X.B_{m-1}(X) - B_{m-2}(X) \quad \text{for} : \mathrm{m} > 2 \end{cases} \quad (5)$$

2.2 The modified Boubaker polynomials (Boubaker-Turki polynomials)

The Boubaker-Turki polynomials or modified Boubaker polynomials [10,17], which are an enhanced form of the formerly defined polynomials, have been established as solutions of the second order differential equation:

$$(X^2 - 1)(3nX^2 + n - 2)\left[\tilde{B}_n(X)\right]'' + P_n(X)\left[\tilde{B}_n(X)\right]' + Q_n(X)\left[\tilde{B}_n(X)\right] = 0 \quad (6)$$

where

$$\begin{cases} P_n(X) = 3X(nX^2 + 3n - 2) \\ Q_n(X) = -n(3n^2X^2 + n^2 - 6n + 8) \end{cases}$$

The modified Boubaker polynomials have a recursive coefficient definition [17] expressed by equation :

$$\begin{cases} \tilde{B}_n(X) = \sum_{j=0}^{\xi(n)} \left[b_{n,j} X^{n-2j} \right]; \xi(n) = \dfrac{2n + ((-1)^n - 1)}{4} \\ \tilde{b}_{n,0} = 2^n; \qquad \tilde{b}_{n,1} = -2^{n-2}(n-4); \\ \tilde{b}_{n,j+1} = \dfrac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \dfrac{(n-4j-4)}{(n-4j)} \times \tilde{b}_{n,j} \\ \tilde{b}_{n,\xi(n)} = \begin{cases} (-1)^{\frac{n}{2}} \times 2 & \text{if n even} \\ 2(-1)^{\frac{n+1}{2}} \times 2(n-2) & \text{if n odd} \end{cases} \end{cases} \tag{7}$$

Both Boubaker and Boubaker-Turki polynomials are the source of several registered integer sequences [12-14].

The ordinary generating function of the Boubaker-Turki polynomials:

$$f_{\tilde{B}}(X,t) = \frac{1+3t^2}{1+t(t-2X)} \tag{8}$$

2.3 The 4q-Boubaker polynomials subsequence

The Boubaker polynomials B_n explicit monomial form evoked, while prospected, some singularities for m=4, 8, 12, etc. In fact for the general case: m=4q the 2q rank monomial term is removed from the explicit form so that the whole expression contains only 2q effective terms. Correspondent 4q-order Boubaker polynomials [11] are presented in equation (9) as a general form and equation (10) as first functions:

$$B_{4q}(X) = 4\sum_{p=0}^{2q} \left[\frac{(q-p)}{(4q-p)} C_{4q-p}^{p} \right] .(-1)^p .X^{2(2q-p)} \tag{9}$$

$$\begin{cases} B_0(X)=1; \\ B_4(X)=X^4-2; \\ B_8(X)=X^8-4X^6+8X^2-2; \\ B_{12}(X)=X^{12}-8X^{10}+18X^8-35X^4+24X^2-2; \\ B_{16}(X)=X^{16}-12X^{14}+52X^{12}-88X^{10}+168X^6-168X^4-48X^2-2; \\ B_{20}(X)=X^{20}-16X^{18}+102X^{16}-320X^{14}+455X^{12}-858X^8+1056X^6-495X^4+80X^2-2; \\ \dots\dots\dots\dots \end{cases} \quad (10)$$

3 The upper bound of the zeros of the Boubaker polynomial

Theorem 3.1 *Let* x_k $(1 \le k \le n)$ *be zeros of the Boubaker polynomial* B_n*, then:*

$$|x_k| < 2, \qquad for \qquad 1 \le k \le n \quad (11)$$

Proof. Making use of the recurrence relation (5) ,we obtain the following relation:

$$[M]\times[B] = x\times[B]+[C] \quad (12)$$

where

$$[M]=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -2 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{n\times n} \quad (13)$$

and

$$[B]=\begin{pmatrix} B_0(x) \\ B_0(x) \\ \vdots \\ B_{n-1}(x) \end{pmatrix}; \qquad [C]=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ B_n(x) \end{pmatrix}$$

Hence, from (12), zeros of the Boubaker polynomial B_n are also eigenvalues of matrix $[M]$. In fact, the eigen polynomial of the matrix is precisely the Boubaker polynomial. We can use the Gerschgorin's theorem [6] to estimate the eigenvalues of $[M]$. By the special structure of $[M]$, it is easy to see that all eigenvalues are in the circle with centre at 0 and radius 2. This means that the result (11) holds.

Theorem 3.2 *There holds the following expression:*

$$B_n(2) = 4n-2; \qquad \text{for n} > 0. \tag{14}$$

Proof. From (5), for m>2, one can get:

$$B_m(2)-B_{m-1}(2) = B_{m-1}(2)-B_{m-2}(2) = \cdots = B_2(2)-B_1(2) \tag{15}$$

Summing the equalities together, gives the desired result as required.

4 Some properties of the Boubaker polynomial B_n

Now let us introduce the m-distribution notion [1]. A nondecreasing bounded function α defined in]-∞, ∞[, is called an m-distribution, if it takes infinitely many distinct values, and its moments, that is, the improper Stieltjes integral:

$$\int_{-\infty}^{\infty} x^n d\alpha(x) = \lim_{\omega_1\to-\infty,\omega_2\to\infty}\left[\int_{\omega_1}^{\omega_2} x^n d\alpha(x)\right] \tag{16}$$

exists and are finite for n=0,1,2,…

Lemma 4.1 *Let* $\{p_n\}_{n=0}^{\infty}$ *be the sequence of the orthogonal polynomials associated with an m-distribution* α *. Then each* p_n *has exactly n simple real zeros lying in the interior of the smallest interval containing supp(*α*).*

Note that $\pm\sqrt{2}i\,(i=\sqrt{-1})$ are two zeros of B_2(x) and $\pm i$ are two zeros of B_3(x).

Theorem 4.1 *The Boubaker polynomial* $B_n(x)$ *does not belong to orthogonal polynomial system associated with any m-distribution.*

The Boubaker polynomials have a similar Christoffel-Darboux formula.

Theorem 4.2 *The following equality holds:*

$$\sum_{k=0}^{n} B_k(x)B_k(y) = 3 + \frac{B_{n+1}(x)B_n(y) - B_n(x)B_{n+1}(y)}{x-y} \tag{17}$$

for all $x \neq y$

Proof. The recurrence relation (5) yields that:

$$B_{k+1}(x)B_k(y) - B_k(x)B_{k+1}(y) = (x-y)B_k(x)B_k(y) - \left[B_{k-1}(x)B_k(y) - B_k(x)B_{k-1}(y)\right] \tag{18}$$

for $k = 2,3,...$, so we have:

$$B_k(x)B_k(y) = \frac{\Delta_k - \Delta_{k-1}}{x-y} \quad \text{for} \quad k = 2,3,... \tag{19}$$

in which : $\Delta_k = B_{k+1}(x)B_k(y) - B_k(x)B_{k+1}(y)$. Summing (19) from 0 to n gives the desired formula.

If $x \to y$ in (17), we obtain the following Corollary

Corollary 4.1 *The following equality is satisfied*

$$\sum_{k=0}^{n} B_k^2(x) = 3 + B'_{n+1}(x)B_n(x) - B'_n(x)B_{n+1}(x) \tag{20}$$

5 Further study on zeros of $B_n(x)$

Lemma 5.1 *Each $B_n(x)$ $(n\geq 1)$ has exactly n simple zeros.*

Theorem 5.1 *The Boubaker polynomial B_n $(n\geq 1)$ has $\left\lfloor \frac{n}{2} \right\rfloor - 1$ zeros for $0 < x < 2$*

Proof. Thanks to the relation

$$B_n(2\cos t) = 4\cos(t)\sin(nt) - 2\cos(nt), \qquad \text{for } n > 1 \tag{21}$$

to find the zeros of $B_n(x)=0$ for $0<x<2$, we set $x=2\cos t$ with $0 < t < \frac{\pi}{2}$ and solve:

$$\tan t = 2\tan(nt) \tag{22}$$

It follows easily that $B_n(x)=0$ has $\left\lfloor \frac{n}{2} \right\rfloor - 1$ zeros for $0 < x < 2$ and the proof is complete.

Remark **5.1** *Now we can count the zeros of B_n as follows:*

1. *when n is even, all of the zeros involve $n/2 - 1$ positive real zeros and $n/2 - 1$ negative real zeros which locate symmetrically in[-2,2] and 2 conjugate pure imaginary zeros.*
2. *when n is odd, all of the zeros involve $(n-1)/2 - 1$ positive real zeros and $(n-1)/2 - 1$ negative real zeros which locate symmetrically in[-2,2] and 2 conjugate pure imaginary zeros.*

The Boubaker polynomial B_n $(n\geq 1)$ has $\left\lfloor \frac{n}{2} \right\rfloor - 1$ zeros for

Corollary 5.1 *The Boubaker polynomial B_n can't have non-simple (or double) zeros)*

In fact, if we suppose that there exists n such that B_n has at least a non-simple (or double) zero , denoted by x_0: In view of Corollary 4.1, we deduce:

$$\sum_{k=0}^{n} B_k^2(x_0) = 3 + B'_{n+1}(x_0)B_n(x_0) - B'_n(x_0)B_{n+1}(x_0) \quad (23)$$

$$= 3.$$

Therefore

$$\sum_{k=0}^{n} B_k^2(x_0) = \sum_{k=0}^{n-1} B_k^2(x_0) + \overbrace{B_n^2(x_0)}^{=0} = \sum_{k=0}^{n-1} B_k^2(x_0) = 3 \quad (24)$$

which is absurd and we are done.

Theorem 5.2 *Let $\pm t_n i$ be the two pure imaginary zeros of the Boubaker polynomial $B_n(x)$ $(n\geq 2)$ then t_n converges to* $2\sqrt{3}/3$

Proof. We also have

$$B_n(2i\sinh t) = \begin{cases} i^n(4\tanh t\sinh(nt) - 2\cosh(nt), & \text{if } n \geq 2 \text{ is even} \\ i^n(4\tanh t\cosh(nt) - 2\sinh(nt), & \text{if } n \geq 1 \text{ is odd} \end{cases} \quad (25)$$

To find solutions of $B_n(x)=0$ for positive imaginary x we set.

$$2\tanh t = \coth(nt) \qquad \text{if } n \text{ is even} \quad (26)$$

and

$$2\tanh t = \tanh(nt) \qquad \text{if } n \text{ is odd} \quad (27)$$

If $n\geq 2$ there is a unique solution $t_n > 0$ Since $\tanh(nt) \to 1$ as $n \to \infty$ for each fixed t>0, we obtain that $2\tanh(t_n) \to 1$. Therefore $2\sinh(t_n) \to 2\sqrt{2/3}$.

Theorem 5.3 *There are only two pure imaginary zeros of the Boubaker polynomial $B_{4q}(x)$ of degree $4q$*

Proof. Let ai (a>0) be a pure imaginary zero of $B_{4q}(x)$.

$$f(a) = B_{4q}(ai)$$

It is an easy task to show that the polynomial $f(a)$ in a of degree 4q has only one positive real zero. From (9), we have:

$$f(a) = \sum_{p=0}^{2q-1}\left[\frac{4(q-p)}{4q-p} C_{4q-p}^{p} a^{4q-2p}\right] - 2 \tag{28}$$

Let us check the sign changes of the coefficients in $f(a)$. Let us denote the ratios of coefficients in $f(a)$ by $\alpha_f[p+1,p]$. Then:

$$\alpha_f[p+1,p] = \frac{(q-p-1)(4q-2p)(4q-2p-1)}{(q-p)(p+1)(4q-p-1)} \tag{29}$$

Hence, the coefficient of a^{4q-2p} is positive if $p<q$, otherwise is negative. By denoting the number of sign changes of coefficients in $f(a)$ by V_f and number of positive real zeros of $f(a)$ by N_f, we can use the Descartes's rule of signs to obtain:

$$N_f = V_f - 2k, \tag{30}$$

where k is an integer.

It is clear that k=0 and then N_f =1, which completes the proof.

6. Conclusion

The upper bound of the zeros of the Boubaker polynomials has been studied. This is of interest not only because of its application to determine new properties of Boubaker polynomials but also because the used method can be applied to solve problems in Physics, Chemistry Biology and Medicine. By means of these polynomials, appropriate mathematical algorithms and computational methods can easily be developed to reveal specific information needed to solve real physiological and pathological problems. For example, using the Boubaker polynomials expansion scheme described here, one can solve the Bloch NMR flow equations for different flow systems. With these possibilities, we can still find new and robust algorithms to solve very old problems. These possibilities will be explored in our next investigation.

References

[1] **P. Borwein, T. Erdelyi**, Polynomial Inequalities, *Sringer-Verlag, New York*, 1995

[2] **R. Okada, N.Nakata, B. F. Spencer, K. Kasai and B.K.Saang**, Rational polynomial approximation modeling for analysis of structures with VE dampers, *Journal of Earthquake Engineering*,10(2006), pp. 97-125.

[3] **Ian H. Sloan and R. S. Womersley**, Good approximation on the sphere, with application to geodesy and the scattering of sound, *Journal of Computational and Applied Mathematics*, 149 (2002), pp. 227-237.

[4] **G. Alvareza,, C. Senb, N. Furukawac, Y. Motomed and E. Dagottoe**, The truncated polynomial expansion Monte Carlo method for fermion systems coupled to classical fields: a model independent implementation, Computer Physics Communications,168(1)(2005), pp. 32-45

[5] **A.N. Philippou, C. Georghiou,** Convolutions of Fibonacci-type polynomials of order k and the negative binomial distributions of the same order, *Fibonacci Quart.* , 27 (1989) pp. 209-216

[6] **J. H. Wilkinson**, *The Algebraic Eigenvalue Problem*, Oxford University Press, 1965

[7] **H. T. Koelink,** The Addition Formula for Continuous q-Legendre Polynomials and Associated Spherical Elements on the SU(2) Quantum Group Related to Askey-Wilson Polynomials, *SIAM Journal on Mathematical Analysis*, 25(1)(1994), pp. 197-217.

[8] **O. B. Awojoyogbe and K. Boubaker**, A solution to Bloch NMR flow equations for the analysis of homodynamic functions of blood flow system using m-Boubaker polynomials, *International Journal of Current Applied Physics*, Elsevier, 9 (2009) pp. 278-288

[9] **A. Chaouachi, K. Boubaker, M. Amlouk and H. Bouzouita**, Enhancement of pyrolysis spray disposal performance using thermal time-response to precursor uniform deposition. *Eur. Phys. J. Appl. Phys*. 37(2007) 105-109.

[10] **K. Boubaker,** On modified Boubaker polynomials: some differential and analytical properties of the new polynomial issued from an attempt for solving bi-varied heat equation, *Trends in Applied Sciences Research*,2(2007) pp. 540-544.

[11] **H. Labiadh and K. Boubaker,** A Sturm-Liouville shaped characteristic differential equation as a guide to establish a quasi-polynomial expression to the Boubaker polynomials, *Differential Equations and Control Processes*, 2 (2007) pp.117-133.

[12] **Roger L. Bagula and Gary Adamson**, Triangle of coefficients of Recursive Polynomials for Boubaker polynomials, OEIS (*Encyclopedia of Integer Sequences*), A137276 (2008).

[13] **The Boubaker polynomials**, Planet-Math Encyclopedia, The Mathematics worldwide Encyclopedia (available also online at : **http://planetmath.org/encyclopedia/BoubakerPolynomials.html**)

[14] **Neil J. A. Sloane**, Triangle read by rows of coefficients of Boubaker polynomial B_n(x) in order of decreasing exponents, *OEIS (Encyclopedia of Integer Sequences)*, A138034 (2008).

[15] **S. Slama, J. Bessrour, B. Karem and M.Bouhafs**, Investigation of A3 point maximal front spatial evolution during resistance spot welding using 4q-Boubaker polynomial sequence, *Proceedings of COTUME 2008*, pp 79-80,(2008)

[16] **H. Labiadh M. Dada, O.B. Awojoyogbe K. B. Ben Mahmoud and A. Bannour,** Establishment of an ordinary generating function and a Christoffel-Darboux type first-order differential equation for the heat equation related Boubaker-Turki polynomials, *Differential Equations and Control Processes*, 1 (2008) pp. 51-66.

[17] **The Boubaker-Tuki polynomials (or Modified Boubaker polynomials),** Planet-Math Encyclopedia, The Mathematics worldwide Encyclopedia (available also online at : http://planetmath.org/encyclopedia/BoubakerTurkiPolynomials.html

[18] **J. Ghanouchi, H. Labiadh and K. Boubaker,** An attempt to solve the heat transfert equation in a model of pyrolysis spray using 4q-order m-Boubaker polynomials *International Journal of Heat and Technology*, 26 (2008) pp. 49-53

[19] **T. Ghrib, K. Boubaker and M. Bouhafs**, Investigation of thermal diffusivity-microhardness correlation extended to surface-nitrured steel using Boubaker polynomials expansion, *Modern Physics Letters B*, 22, (2008) pp. 2893 - 2907

[20] **K. Boubaker,** The Boubaker polynomials, a new function class for solving bi-varied second order differential equations, *F. E. Journal of App. Math.* 31(2008) pp. 299 – 320.

[21] **S. Slama, J. Bessrour, K. Boubaker and M. Bouhafs,** A dynamical model for investigation of A_3 point maximal spatial evolution during resistance spot welding using Boubaker polynomials, *Eur. Phys. J. Appl. Phys.* 44, (2008) pp. 317-322

[22] **S. Slama, M. Bouhafs and K. B. Ben Mahmoud,**A Boubaker Polynomials Solution to Heat Equation for Monitoring A3 Point Evolution During Resistance Spot Welding, *International Journal of Heat and Technology,* 26(2) (2008) pp. 141-146.