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The Proof of Convergence with Probability 1 in the Method of Expansion of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series

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Abstract. The article is devoted to the formulation and proof of the theorem on convergence with probability 1 of expansion of iterated Itô stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the sense of norm in Hilbert space. The cases of multiple Fourier–Legendre series and multiple trigonomertic Fourier series are considered in detail. The proof of the mentioned theorem is based on the general properties of multiple Fourier series as well as on the estimate for the fourth moment of approximation error in the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series.

Key words: Iterated Itô stochastic integral, generalized multiple Fourier series, multiple Fourier–Legendre series, multiple trigonometric Fourier series, Parseval equality, Legendre polynomials, convergence with probability 1, mean-square convergence, convergence in the mean of arbitrary degree, expansion, approximation.

1 Introduction

The beginning of an intensive study of the problem of mean-square approximation of iterated Itô and Stratonovich stochastic integrals in the context of the numerical solution of Itô stochastic differential equations dates back to the 1980s–1990s. To date, there are many publications on the mentioned problem [1]-[36] (also see bibliographic references in these works). There are various approaches to solving the problem of the mean-square approximation of iterated stochastic integrals. Among them, we note the approach based on the Karhunen–Loeve expansion of the Brownian bridge process [1]-[4], [13], [18], [21], approach based on the expansion of the Wiener process using various basis systems of functions [6], [10], [30], [31], approach based on the conditional joint characteristic function of a stochastic integral of multiplicity 2 [11], [12] as well as an approach based on multiple integral sums [1], [19].

The use of multiple and iterated generalized Fourier series by various complete orthonormal systems of functions in the space $L_2([t,T])$ for the expansion of iterated Itô and Stratonovich stochastic integrals was reflected in a number of author's works [7]-[9], [14]-[17], [20], [22]-[29], [35]. The mentioned results based on generalized multiple and iterated Fourier series are systematized in the monograph [36] (2020).

The idea of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series is as follows: the iterated Itô stochastic integral of multiplicity k ($k \in \mathbb{N}$) is represented as a multiple stochastic integral from the certain nonrandom discontinuous function of k variables, defined on the hypercube $[t,T]^k$, where $[t,T]$ is an interval of integration of the iterated Itô stochastic integral. Then, the indicated nonrandom function is expanded into the generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come [14] (2006) to the mean-square converging expansion of the iterated Itô stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of multiplicity k of the iterated Itô stochastic integral.

In a lot of author's publications the convergence of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series has been considered in different probability meanings. For example, the mean-square convergence [14]-[17], [20], [22]-[29], [35], [36] and convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [15]-[17], [20], [22], [23], [36] have been proved. On the examples of specific iterated Itô stochastic integrals of mutiplicities 1 and 2 the convergence with probability 1 also has been considered [15]-[17], [20], [22], [23]. This article is devoted to the development of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series. Namely, we formulate and prove the theorem on convergence with probability 1 of the mentioned method for an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Itô stochastic integrals. Moreover, the cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series are considered in detail.

2 Method of Expansion of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbb{N}$) Based on Generalized Multiple Fourier Series

Let (Ω, F, P) be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{w}_t^{(i)}$ ($i = 1,\ldots,m$) of this process are independent.

Let us consider an efficient method [14]-[17], [20], [22]-[29], [35], [36] of the expansion and mean-square approximation of iterated Itô stochastic integrals of the form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (1)$$

where $0 \le t < T < \infty$, $\psi_l(\tau)$ ($l = 1,\ldots,k$) are continuous nonrandom functions on the interval $[t,T]$, $\mathbf{w}_\tau^{(i)}$ ($i = 1,\ldots,m$) are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1,\ldots,i_k = 0,1,\ldots,m$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$ and define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ & \\ 0, & \text{otherwise} \end{cases}, \qquad (2)$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous on the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ converges to $K(t_1,\ldots,t_k)$ on the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{3}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{4}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\qquad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\quad\text{if}\quad N\to\infty, \tag{5}$$

where $\Delta\tau_j=\tau_{j+1}-\tau_j$.

Theorem 1 [14] (2006), [15]-[17], [20], [22]-[29], [35], [36]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k},$$

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right), \tag{6}$$

where

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr) \qquad (7)$$

and

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \qquad (8)$$

are independent standard Gaussian random variables for various i *or* j (*if* $i\ne 0$), $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (4), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is the discretization* (5), *estimate* (6) *is valid for* $T-t\in(0,\infty)$ *and* $i_1,\ldots,i_k=1,\ldots,m$ *or* $T-t\in(0,1)$ *and* $i_1,\ldots,i_k=0,1,\ldots,m.$

Note that in [14]-[17], [20], [22], [23], [36] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Some modifications of Theorem 1 for another types of iterated stochastic integrals (including iterated Itô stochastic integrals with respect to the infinite-dimensional Q-Wiener process) as well as for complete orthonormal systems of functions with weight $r(t_1)\ldots r(t_k)\ge 0$ in the space $L_2([t,T]^k)$ can be found in [14]-[17], [20], [22], [23], [29], [36].

Obtain transformed particular cases of Theorem 1 for $k=1,\ldots,5$ [14]-[17], [20], [22]-[29], [35], [36]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \qquad (9)$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \qquad (10)$$

$$
\begin{aligned}
J[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\\
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} &- \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),
\end{aligned}
\qquad (11)
$$

$$
\begin{aligned}
J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(&\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} &- \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} &- \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} &- \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}&\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}&\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}&\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
\end{aligned}
\qquad (12)
$$

$$
\begin{aligned}
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(&\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} &- \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} &- \mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} &- \mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} &- \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} &- \mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} &+ \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} &+ \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} &+ \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} &+ \mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
\end{aligned}
$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \qquad (13)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the generalization of the formulas (9)–(13) for the case of arbitrary k ($k\in\mathbb{N}$).

Theorem 2 [16] (2009), [17], [20], [22], [23], [29], [36]. *In conditions of Theorem* 1 *the following mean-square converging expansion is valid*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \qquad (14)$$

where $[\cdot]$ *is an integer part of a real number,*

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}$$

means the sum with respect to all possible permutations of the set

$$(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}),$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, *braces mean an disordered set, and parentheses mean an ordered set; other notations are the same as in Theorem* 1.

For further consideration, we need the following statement.

Theorem 3 [15] (2007), [16], [17], [20], [22], [23], [36]. *In conditions of Theorem* 1 *the following estimate is valid*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le
$$

$$
\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \times
$$

$$
\times\ \left(\ \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n, \tag{15}
$$

where notations are the same as in Theorem 1.

Since according to the Parseval's equality

$$
\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\ \to\ 0
$$

if $p_1,\ldots,p_k\to\infty$, then the inequality (15) means that the expansions of iterated Itô stochastic integrals in Theorem 1 converge in the mean of degree $2n$ $(n\in\mathbb{N})$.

Let us consider the following iterated Itô stochastic integrals from the Taylor–Itô expansion [3]

$$
J_{(\lambda_1\ldots\lambda_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{16}
$$

where $i_1,\ldots,i_k=0,1,\ldots,m,$ $\lambda_l=1$ if $i_l=1,\ldots,m$ and $\lambda_l=0$ if $i_l=0$ $(l=1,\ldots,k)$. Remind that $\mathbf{w}_\tau^{(i)}$, $i=1,\ldots,m$ are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)}=\tau$.

For example, using Theorem 1 (see (9)-(11)) and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ we obtain the following approximations of iterated Itô stochastic integrals (16) [14]-[17], [20], [22]-[29], [35], [36] (also see early publications [8], [9])

$$
J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{17}
$$

$$
J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{18}
$$

$$
J_{(10)T,t}^{(i_10)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{19}
$$

$$J_{(11)T,t}^{(i_1 i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{20}$$

$$J_{(11)T,t}^{(i_1 i_1)}=\frac{1}{2}(T-t)\left(\left(\zeta_0^{(i_1)}\right)^2-1\right),$$

$$J_{(111)T,t}^{(i_1 i_2 i_3)p}=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{21}$$

$$J_{(111)T,t}^{(i_1 i_1 i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right),$$

where

$$C_{j_3 j_2 j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where the Gaussian random variable $\zeta_j^{(i)}$ (if $i\ne 0$) is defined by (8) and $P_j(x)$ $(j=0,1,2,\ldots)$ is a complete orthonormal system of Legendre polynomials in the space $L_2([-1,1])$ [37].

Note that formula (20) has been obtained for the first time in [8] (1997). For pairwise different $i_1,i_2,i_3=1,\ldots,m$ we have [8], [9], [14]-[17], [20], [22]-[29], [35]

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1 i_2)}-J_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{22}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)}-J_{(111)T,t}^{(i_1 i_2 i_3)p}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^2. \tag{23}$$

The problem of the exact calculation of the mean-square error of approximation in Theorem 1 is solved completely for an arbitrary k $(k\in\mathbb{N})$ and any possible combinations of numbers $i_1,\ldots,i_k=1,\ldots,m$ in [23] (Sect. 6.2.7), [36] (Sect. 1.2).

3 Convergence With Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbb{N})$ in Theorem 1

Let us address now to the convergence with probability 1 (w. p. 1) in Theorem 1. As we mentioned above this question has been studied for simplest iterated Itô stochastic integrals of multiplicities 1 and 2 in [15]-[17], [20], [22], [23], [36].

In this section we formulate and prove the general result on convergence w. p. 1 of expansions of iterated Itô stochastic integrals in Theorem 1 for the case of multiplicity k $(k \in \mathbb{N})$ for these integrals.

Theorem 4. *Let* $\psi_l(\tau)$ $(l = 1, \ldots, k)$ *are continuously differentiable nonrandom functions on the interval* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p \to \infty \quad \text{w. p. } 1,$$

where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ *is defined by* (7) *for the case* $p_1 = \ldots = p_k = p$, *i.e.* (*see Theorem* 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \text{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right),$$

where $i_1, \ldots, i_k = 1, \ldots, m$, *other notations are the same as in Theorem* 1.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k = \lim_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2, \qquad (24)$$

where

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases},$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient.

Taking into account the definitions of $K(t_1,\ldots,t_k)$ and $C_{j_k\ldots j_1}$, we obtain

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k. \tag{25}$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\textsf{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

If $p_1=\ldots=p_k=p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}\stackrel{\textsf{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\textsf{def}}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\textsf{def}}{=}\sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

and so on.

Using the Parseval equality and Lemma 2 (see Appendix) we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}+$$

$$+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\ldots=$$

$$=\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\ldots+\sum_{j_1=0}^{p}\ldots\sum_{j_{k-1}=0}^{p}\sum_{j_k=p+1}^{\infty}C^2_{j_k\ldots j_1}\le$$

$$\le\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\sum_{j_1=0}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+$$

$$+\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}+\ldots+\sum_{j_1=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\sum_{j_k=p+1}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{s=1}^{k}\left(\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\right). \qquad (26)$$

Note that deriving (26) we use the following

$$\sum_{j_1=0}^{p}\ldots\sum_{j_{s-1}=0}^{p}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\le$$

$$\le\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\le$$

$$\le\lim_{m_{s-1}\to\infty}\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$
=\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le\ldots\le
$$

$$
\le\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2,
$$

where $m_1,\ldots,m_{s-1}>p$.

Denote

$$
C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,
$$

where $s=1,\ldots,k-1$.

For $s<k$ due to Lemma 3, Dini Theorem (see Appendix) and Parseval equality we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\int\limits_t^{t_k}\psi_{k-1}^2(\tau)\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau dt_k\le
$$

$$
\le M\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
=M\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=
$$

$$= M\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_{k-2}^2(\theta)\left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le M'\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le \ldots \le$$

$$\le M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\int\limits_t^T\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$= M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \qquad (27)$$

where constants M, M' depend on $T-t$ and constant M_k depends on $T-t$ and k.

Let us explane more precisely how we obtain (27). For any function $g(s)\in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty}\left(\int\limits_t^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s)ds=\int\limits_t^{\tau}g^2(s)ds. \qquad (28)$$

Equality (28) has been applied repeatedly when we obtaining (27).

Using the replacement of integrating order in Riemann integrals, we have

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s=$$

$$=\int\limits_t^{\tau}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{\tau}\phi_{j_2}(t_2)\psi_2(t_2)\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1.$$

For $l=1,\ldots,s$ we will use the following notation

$$
\tilde{C}_{j_s\ldots j_l}(\tau,\theta)=
$$

$$
=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.
$$

Using the Parseval equality and Dini Theorem (see Appendix), from (27) we obtain

$$
\sum_{j_1=0}^{\infty}\cdots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\le
$$

$$
\le M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_{t}^{T}\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau=
$$

$$
=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau=
$$

$$
=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\int\limits_{t_1}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le
$$

$$
\le M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\int\limits_{t}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le
$$

$$
\le M_k^{'}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2d\tau\le\ldots\le
$$

$$
\le M_k^{''}\sum_{j_s=p+1}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_{s-1}^2(t_{s-1})\left(\tilde{C}_{j_s}(\tau,t_{s-1})\right)^2dt_{s-1}d\tau\le
$$

$$
\le\tilde{M}_k\sum_{j_s=p+1}^{\infty}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\left(\int\limits_{u}^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2dud\tau,\qquad(29)
$$

where constants $M_k^{'}$, $M_k^{''}$, and $\tilde{M}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (29). For any function $g(s)\in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty}\left(\int\limits_{\theta}^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_{t}^{T}\mathbf{1}_{\{\theta<s<\tau\}}\phi_j(s)g(s)ds\right)^2=$$

$$=\int\limits_{t}^{T}\left(\mathbf{1}_{\{\theta<s<\tau\}}\right)^2g^2(s)ds=\int\limits_{\theta}^{\tau}g^2(s)ds. \tag{30}$$

Equality (30) has been applied repeatedly when we obtain (29).

Let us estimate the integral

$$\int\limits_{u}^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta \tag{31}$$

from (29) for the cases when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the estimates for the integral

$$\int\limits_{t}^{\tau}\phi_j(\theta)\psi(\theta)d\theta,\quad j\ge p+1, \tag{32}$$

where $\psi(\theta)$ is a continuously differentiable function on the interval $[t,T]$, have been obtained in [20], [22], [23], [36].

Let us estimate the integral (31) using the approach from [20], [22], [23], [36].

First consider the case of Legendre polynomials. Then $\phi_j(\theta)$ looks as follows

$$\phi_j(\theta)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(\theta-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j\ge 0,$$

where $P_j(x)$ $(j=0,1,2\ldots)$ is a complete orthonormal system of Legendre polynomials in the space $L_2([-1,1])$.

Further, we have

$$\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta=\frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy=$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x)) - P_{j-1}(z(x)))\psi(x)-$$

$$-(P_{j+1}(z(v)) - P_{j-1}(z(v)))\psi(v)-$$

$$-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y) - P_{j-1}(y))\psi'(u(y))dy\Biggr), \tag{33}$$

where $x, v \in (t, T)$, $j \ge p+1$, $u(y)$ and $z(x)$ are defined by the following relations

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(x) = \left(x - \frac{T+t}{2}\right)\frac{2}{T-t},$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (33) we used the following well-known property of the Legendre polynomials [37]

$$\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j = 1, 2, \ldots$$

From (33) and the well-known estimate for the Legendre polynomials [41]

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j \in \mathbb{N},$$

where constant K does not depend on y and j, it follows that

$$\left|\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + \frac{1}{(1-(z(v))^2)^{1/4}} + C_1\right), \tag{34}$$

where $z(x), z(v) \in (-1, 1)$, $x, v \in (t, T)$ and constants C, C_1 does not depend on j.

From (34) we obtain

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 < \frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + \frac{1}{(1-(z(v))^2)^{1/2}} + C_3\right), \tag{35}$$

where constants C_2, C_3 does not depend on j.

Let us apply (35) for the estimate of the right-hand side of (29). We have

$$\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/2}}+\int\limits_{-1}^{1}\int\limits_{-1}^{x}\frac{dy}{\left(1-y^2\right)^{1/2}}dx+K_2\right)\le$$

$$\le\frac{K_3}{j_s^2}, \tag{36}$$

where constants K_1, K_2, K_3 are independent of j_s.

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r\end{cases} \tag{37}$$

where $r=1,2,\ldots$

Using the system of functions (37) we have

$$\int\limits_v^x\phi_{2r-1}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x\sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=-\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\cos\frac{2\pi r(x-t)}{T-t}-\psi(v)\cos\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x\cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{38}$$

$$\int\limits_v^x\phi_{2r}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x\cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$
=\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\sin\frac{2\pi r(x-t)}{T-t}-\psi(v)\sin\frac{2\pi r(v-t)}{T-t}-\right.
$$

$$
\left.-\int\limits_v^x\sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{39}
$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (38) and (39) we obtain for the trigonometric case

$$
\left(\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta\right)^2\le\frac{C_4}{j^2}, \tag{40}
$$

where constant C_4 is independent of j.

From (40) we finally have

$$
\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau\le\frac{K_4}{j_s^2}, \tag{41}
$$

where constant K_4 is independent of j_s.

Combining (29), (36) and (41) we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le
$$

$$
\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2}\le L_k\int\limits_p^\infty\frac{dx}{x^2}=\frac{L_k}{p}, \tag{42}
$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (26) and (42) we obtain

$$
\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\le\frac{G_k}{p}, \tag{43}
$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use estimate (15). Using (43) and estimate (15) for the case $p_1 = \ldots = p_k = p$ and $n = 2$ we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le$$

$$\le C_{2,k}\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2\right)^2 \le$$

$$\le \frac{H_{2,k}}{p^2}, \tag{44}$$

where

$$C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

and $H_{2,k} = G_k^2 C_{2,k}$.

Let us consider Lemma 1 (see Appendix) with

$$\xi_p = \left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right| \quad \text{and} \quad \alpha = 4.$$

Then from (44) we get

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty. \tag{45}$$

Using Lemma 1 (see Appendix) and the estimate (45) we obtain

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p\to\infty \quad \text{w. p. 1},$$

where (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -\right.$$

$$\left.-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right) \tag{46}$$

or (see Theorem 2)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \qquad (47)$$

where $i_1,\ldots,i_k = 1,\ldots,m$ in (46) and (47). The proof of Theorem 4 is completed.

4 Appendix

Lemma 1 [38]. *If for the sequence of random variables ξ_p and for some $\alpha > 0$ the number series*

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

Lemma 2. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 \qquad (48)$$

for any permutation $(q_1,\ldots,q_k)$ such that $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$, where $C_{j_k\ldots j_1}$ is defined by (25).

Proof. Let us remind the well-known fact from the mathematical analysis which is connected to existence of iterated limits.

Proposition 1 [39]. *Let $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ be a double sequence and let there exists the limit*

$$\lim_{n,m\to\infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m}<\infty \quad \textit{for all} \;\; m, \qquad \lim_{m\to\infty} x_{n,m}<\infty \quad \textit{for all} \;\; n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \quad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}=\lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}=a.$$

Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \tag{49}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Obviously, (49) is the non-decreasing sequence with respect to p. Moreover

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \le \sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \le$$

$$\le \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2<\infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2=\sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g} C_{j_k\ldots j_1}^2 \le \sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k} C_{j_k\ldots j_1}^2 \le \sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r} C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \tag{50}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{51}$$

and equality of limits (50) and (51).

Consequently,

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{52}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists (see the Parseval equality (24)), then from Proposition 1 we have

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{53}$$

Using (52) and Proposition 1 we have

$$\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{54}$$

Combining (54) and (53) we obtain

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2.$$

Repeating the above steps, we complete the proof of Lemma 2.

Lemma 3. *The following equality takes place*

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}, \qquad (55)$$

where $s=1,\ldots,k$ *and* $C_{j_k\ldots j_1}$ *is defined by* (25).

Proof. Applying the arguments that we used in the proof of Lemma 2, we obtain

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_s=0}^{p}\ \sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C^2_{j_k\ldots j_1} \qquad (56)$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\}=\{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we have

$$\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C^2_{j_k\ldots j_1}=\ldots=$$

$$=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}\sum_{j_s=0}^{p}C^2_{j_k\ldots j_1}. \qquad (57)$$

Using (56), (57) and Lemma 2 we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.$$

Equality (55) is proved.

Theorem (Dini) [40]. *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a, b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a, b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a, b]$.*

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